RESERCH REPORT
CONFORMAL FIELD THEORY, BRAID GROUPS AND QUANTUM GROUPS

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Contents

Date: November 2005.

Introduction

From my thesis defense (December 1997) to 1999, my research has centred mainly on the study of Conformal Field Theory from the point of view of Operator Algebras initiated by V. Jones and A. Wassermann. More recently, I have become interested in generalised braid groups and quantum groups and have obtained a monodromic interpretation of quantum Weyl group representations in the spirit of the Kohno–Drinfeld theorem. The present report is accordingly divided in two parts, each containing a summary of the results I obtained in the corresponding area and an outline of work in progress and future directions of research.

Part I. **Conformal field theory and operator algebras**

1. The problem of fusion

Let G be a compact, connected and simply–connected Lie group and

$$LG = C^\infty(S^1, G)$$

its loop group. Recall that a *positive energy representation* of LG is a projective unitary representation $\pi : LG \to PU(\mathcal{H})$ extending to the semi–direct product $LG \rtimes \mathrm{Rot}(S^1)$ in such a way that the infinitesimal generator of rotations is bounded below and has finite–dimensional eigenspaces. Thus,

$$\mathcal{H} = \bigoplus_{n \geq 0} \mathcal{H}(n)$$

where $\mathcal{H}(n)$, *the subspace of energy* n, supports a finite–dimensional representation of $G \subset LG$. Positive energy representation are completely reducible and the irreducible ones are classified by a non–negative integer ℓ called the *level* and their zero energy subspace, a simple G-module. For a given ℓ, only finitely many can appear as lowest energy subspaces [PS].

The pioneering works of Belavin–Polyakov–Zamolodchikov [BPZ] and of Knizhnik–Zamolodchikov [KZ] in Conformal Field Theory implicitly suggest the existence of a tensor or *fusion* product on the positive energy representations of a given level. This operation differs from the usual Hilbert space tensor product and its understanding from a rigorous standpoint has given rise to several independent, and still unrelated definitions [B, KL, Se].

The Operator Algebraic approach to the problem of fusion stems from the observation that any positive energy representation may be regarded as a bimodule over the pair $(L_I G, L_{I^c} G)$ of groups of loops supported in an interval $I \subset S^1$ or its complement. A tensor product operation on bimodules over a von Neumann algebra due to Connes [Co, Sa] can then be used to give a definition of fusion. This approach has the advantage of being manifestly unitary and associative. It was successfully applied by Wassermann [J3, Wa1, Wa2, Wa3] to compute the fusion ring of LSU_n and by Loke [L] to compute that of Diff (S^1).

Several motivations underlie this study. We mention a few

- A positive energy representation of LG defines an inclusion of hyperfinite factors of type III_1 which the explicit knowledge of the fusion ring allows to classify. This subfactor should be the tensor product of a hyperfinite subfactor of type III_1 with one of the subfactors defined by Wenzl [We1, We2].
- The positive energy representations of a given level, endowed with the fusion product, become a modular, braided tensor category and therefore define three–manifold invariants [Tu].
- This category constitues one of the very few examples of a quantum field theory satisfying both the Haag–Kastler and the Wightman axioms and which is not a free field theory.

2. Connes fusion and Verlinde formulae for loop groups of type BCD

The main result of my thesis [TL1] is the definition of the fusion ring for the group $G = \mathrm{Spin}_{2n}$ and the computation of fusion with the vector representation of LSpin_{2n}. This shows that positive energy representations define subfactors of finite index and depth and that the one corresponding to the vector representation is the tensor product of the hyperfinite III_1 with one of the subfactors defined by Wenzl [We2].

The computation rests on (1) an explicit construction of the primary fields of the theory, which allows to prove that they define operator–valued distributions, (2) the explicit solution of a family of Knizhnik–Zamolodchikov equations as well as the computation of its monodromy which allows to characterise the braiding of these fields and (3) the use of the algebraic theory of superselection sectors developed by Doplicher–Haag–Roberts [DHR].

Since 1997, I have completed the computation of the fusion ring of $L\,\mathrm{Spin}_{2n}$. The main steps of this computation can in fact be adapted to yield the fusion ring of the orthogonal and symplectic loop groups [TL4].

3. Integrating unitary representations of infinite–dimensional Lie groups

I discovered a method for integrating unitary representations of infinite–dimensional Lie groups [TL2]. This replaces Nelson's classical procedure based on analytic vectors [Ne], which does not work for groups whose exponential map fails to be locally one–to–one, such as for example the diffeomorphism groups of compact manifolds. As a corollary of our method, we obtain a short proof of Goodman and Wallach's results on the integration of positive energy representations of loop groups and of $\mathrm{Diff}(S^1)$ [GoWa].

4. Loop groups of non simply–connected groups

We have classified positive energy representations of the loop group of a compact, connected and simple Lie group G [TL3], thus generalising the classification outlined in §1, valid when G is simply–connected. They consist of orbits of positive energy representations of the loop group of the universal cover $\widetilde{G}$ of G under an explicit action of $\pi_1(G)$.

5. Loop groups of other Lie groups

Somewhat amazingly, the Fuchsian equation which governes fusion with the defining representation of the orthosymplectic groups also underlies fusion with similar 'vector' representations for all other groups except E_8. Thus, what is a key step in the computation of the fusion ring can be obtained in a uniform way for classical and (most of the) exceptional Lie algebras. This is related to the similarities which the decomposition of the iterated tensor products of these corresponding finite–dimensional representations exhibit. In collaboration with A. Wassermann, we are devising a scheme based upon this fact to compute the fusion rings of the loop groups of all Lie groups except E_8.

6. Connes fusion for non–simply connected loop groups

A natural problem stemming from [TL3] is to define and compute Connes fusion for the loop group of non–simply connected group G. This would give rise to interesting subfactors which, for $G = U_n/\mathbb{T}$ should coincide with the tensor product of the hyperfinite factor of type III_1 with the orbifold subfactors of Evans–Kawahigashi.

The bimodule picture still permits the definition of a suitable tensor product. Passing from the loop group of a simply–connected group to that of a quotient is an instance of the Conformal Field Theory notion of a *simple current extension* (SCI). Fuchs, Schellekens and Schweigert have given a conjectural form of the S–matrices of such extensions [FSS] which has been proved in various axiomatic frameworks [Mu]. I intend to prove this conjecture for the SCI corresponding to non–simply connected loop groups.

Part II. Braid groups and quantum groups

7. The Casimir connection

Let $\mathfrak{g}$ be a complex, semi–simple Lie algebra, $\mathfrak{h} \subset \mathfrak{g}$ Cartan subalgebra and $W \subset GL(\mathfrak{h})$ the Weyl group of $\mathfrak{g}$. I discovered, jointly with J. Millson, a flat, W–equivariant connection on $\mathfrak{h}$ with logarithmic singularities on the root hyperplanes and values in any $\mathfrak{g}$–module V [MTL, TL5, TL6]. This *Casimir connection* is explicitly given by

$$\nabla_C = d - \hbar \sum_{\alpha} \frac{d\alpha}{\alpha} \cdot C_\alpha \tag{7.1}$$

where $\hbar \in \mathbb{C}$ is a deformation parameter, the sum ranges over the positive roots of $\mathfrak{g}$ and C_α is the Casimir operator of the $\mathfrak{sl}_2$–subalgebra of $\mathfrak{g}$ corresponding to α. It was discovered independently by De Concini around 1995 (unpublished) and by Felder *et al* [FMTV].

8. Knizhnik–Zamolodchikov equations

As explained below, the Casimir connection for $\mathfrak{g} = \mathfrak{sl}_n$ coincides, via $(\mathfrak{gl}_k, \mathfrak{gl}_n)$ duality with the Knizhnik–Zamolodchikov (KZ) equations for the Lie algebra $\mathfrak{sl}_k$. It can therefore be regarded as a generalisation of these equations to configuration spaces of types other than A_{n-1}.

Let $\mathfrak{r}$ be a Lie algebra endowed with a non–degenerate, ad–invariant bilinear form $(\cdot,\cdot)$. Let V be a finite–dimensional $\mathfrak{r}$–module, $n \in \mathbb{N}$ and set, for any $1 \leq i < j \leq n$,

$$\Omega_{ij} = \sum_{a=1}^{\dim \mathfrak{r}} 1^{\otimes(i-1)} \otimes T_a \otimes 1^{\otimes(j-i-1)} \otimes T^a \otimes 1^{\otimes(n-j)} \in \mathrm{End}(V^{\otimes n})$$

where $\{T_a\}, \{T^a\}$ are dual bases of $\mathfrak{r}$ $(\cdot,\cdot)$. Let

$$X_n = \{(z_1, \dots, z_n) \in \mathbb{C}^n | \, z_i \neq z_j, \quad \forall i \neq j\}$$

be the configuration space of n ordered points in $\mathbb{C}$.

Proposition 8.1 (Knizhnik–Zamolodchikov)**.** *The 1–form*

$$\nabla_{\mathrm{KZ}} = d - \hbar \sum_{1 \leq i < j \leq n} \frac{d(z_i - z_j)}{z_i - z_j} \Omega_{ij}$$

defines a flat connection on the trivial vector bundle over X_n *with fibre* $V^{\otimes n}$.

Consider now the action of $U\mathfrak{gl}_k \otimes U\mathfrak{gl}_n$ on the space $S^*(\mathcal{M}_{k,n})$ of polynomial functions on $k \times n$ matrices. When $k \geq n$, this allows one to identify the subspace $V_\lambda[\mu]$ of weight $\mu \in \mathbb{N}^n$ of the simple $\mathfrak{gl}_n$–module with highest weight $\lambda \in \mathbb{N}^n$ with the space $\mathrm{sing}_\lambda(\mathbf{S}^\mu \mathbb{C}^k)$ of vectors of highest weight λ with respect to the action of $\mathfrak{gl}_k$ on the tensor product

$$\mathbf{S}^\mu \mathbb{C}^k = S^{\mu_1} \mathbb{C}^k \otimes \cdots \otimes S^{\mu_n} \mathbb{C}^k$$

where $S^m\mathbb{C}^k$ is the mth symmetric power of the vector representation on $\mathfrak{gl}_k$ [Ho].

Theorem 8.2 ([TL5]). *The isomorphism*

$$V_\lambda[\mu] \cong \operatorname{sing}_\lambda(S^{\mu_1}\mathbb{C}^k \otimes \cdots \otimes S^{\mu_n}\mathbb{C}^k)$$

identifies the Casimir connection for $\mathfrak{sl}_n$ *with values in* $V_\lambda[\mu]$ *with the Knizhnik–Zamolodchikov connection for* $\mathfrak{r} = \mathfrak{sl}_k$ *with values in* $\operatorname{sing}_\lambda(\mathbf{S}^\mu)$.

9. A Kohno–Drinfeld theorem for quantum Weyl groups

Set

$$\mathfrak{h}_{\text{reg}} = \mathfrak{h} \setminus \bigcup_\alpha \operatorname{Ker}(\alpha) \qquad \text{and let} \qquad B_W = \pi_1(\mathfrak{h}_{\text{reg}}/W)$$

be the generalised braid group of type W. The monodromy of the connection ∇_C yields a one–parameter family of representations of B_W which, for $\hbar = 0$, factors through the action of (the Tits extension of) W on V. It is an interesting problem to compute this monodromy explicitly.

Recall first that if $\mathfrak{r}$ is a semi–simple Lie algebra, the Kohno–Drinfeld theorem states that the monodromy of the KZ equations described in §8 is given by the R–matrix representation coming from the quantum group $U_\hbar\mathfrak{r}$ [Dr4]. By analogy with this result, I formulated in [TL5, TL6] the following

Conjecture 9.1. *The monodromy representation of* ∇_C *is equivalent to the quantum Weyl group action of* B_W *on* V.

The latter action is defined by Lusztig, Kirillov–Reshetikhin and Soibelman by regarding V as a module over $U_\hbar\mathfrak{g}$ [Lu, KR, So]. The above conjecture was formulated independently by De Concini around 1995 (unpublished). The following is an immediate consequence of conjecture 9.1 and the fact that quantum Weyl group operators are defined over $\mathbb{Q}[\![\hbar]\!]$.

Corollary 9.2. *The monodromy of* ∇_C *is defined over* $\mathbb{Q}[\![\hbar]\!]$.

I checked conjecture 9.1 for a number of pairs $(\mathfrak{g}, V)$ including vector and spin representations of classical Lie algebras and the adjoint representation of any simple Lie algebra [TL6].

10. Dual pairs and the Kohno–Drinfeld theorem

I also proved conjecture 9.1 for all representations of $\mathfrak{g} = \mathfrak{sl}_n$ [TL5]. The proof relies on the use of the Kohno–Drinfeld theorem via theorem 8.2 and its q–version (thm. 10.1 below).

One can define an action of $U_\hbar\mathfrak{gl}_k \otimes U_\hbar\mathfrak{gl}_n$ on the algebra $S^*_\hbar(\mathcal{M}_{k,n})$ of functions on quantum $k \times n$ matrices, which is a deformation of $S^*(\mathcal{M}_{k,n})$ [Ba, Ga, TL5]. This identifies the subspace $\mathcal{V}_\lambda[\mu]$ of weight $\mu \in \mathbb{N}^n$ of the indecomposable $U_\hbar\mathfrak{gl}_n$–module with highest weight $\lambda \in \mathbb{N}^n$ to the space

$\mathrm{sing}_\lambda(\mathbf{S}^\mu_\hbar\mathbb{C}^k)$ of vectors of highest weight λ with respect to the action of $U_\hbar\mathfrak{gl}_k$ on the tensor product

$$\mathbf{S}^\mu_\hbar\mathbb{C}^k = S^{\mu_1}_\hbar\mathbb{C}^k\otimes\cdots\otimes S^{\mu_n}_\hbar\mathbb{C}^k$$

where $S^m_\hbar\mathbb{C}^k$ is the q–deformation of the mth symmetric power of the vector representation of $\mathfrak{gl}_k$.

Theorem 10.1 ([TL5])**.** *The isomorphism*

$$\bigoplus_{\nu\in\mathfrak{S}_n\mu}\mathcal{V}_\lambda[\nu]\cong\bigoplus_{\nu\in\mathfrak{S}_n\mu}\mathrm{sing}_\lambda(S^{\nu_1}_\hbar\mathbb{C}^k\otimes\cdots\otimes S^{\nu_n}_\hbar\mathbb{C}^k)$$

identifies the quantum Weyl group action of the braid group $B_n = B_{\mathfrak{S}_n}$ with its R–matrix coming from the quantum group $U_\hbar\mathfrak{gl}_k$.

Remark 10.2. Although there exist other dual pairs than $(\mathfrak{gl}_k,\mathfrak{gl}_n)$, the proof of conjecture 9.1 sketched above relies on the 'coincidence' between the set of regular elements $\mathfrak{h}_{\mathrm{reg}}$ for $\mathfrak{g}=\mathfrak{gl}_n$ and the configuration space X_n of n ordered points in $\mathbb{C}$ on which the KZ connection lives. It cannot therefore extend to semi–simple Lie algebras other than $\mathfrak{sl}_n$.

11. 1st version of the universal monodromy conjecture

It is natural to formulate a version of conjecture 9.1 which does not invoke any specific $\mathfrak{g}$–module V. Recall first if the base point $x_0\in\mathfrak{h}_{\mathrm{reg}}$ is chosen in the fundamental Weyl chamber $\mathcal{C}$, the braid group $B_W=\pi_1(\mathfrak{h}_{\mathrm{reg}}/W;x_0)$ has a presentation on $n=\dim(\mathfrak{h})$ generators $S_1,\ldots,S_n$ labelled by the orthogonal reflections $s_1,\ldots,s_n\in W$ with respect to the walls of $\mathcal{C}$. The corresponding quantum Weyl group elements $S^\hbar_1,\ldots,S^\hbar_n$ live in the completion $\widehat{U_\hbar\mathfrak{g}}$ of $U_\hbar\mathfrak{g}$ with respect to its finite–dimensional representations.

Note next that the monodromy action of a loop $b\in B_W$ on the $\mathfrak{g}$–module (V,ρ) is equal to $\rho(\mu_{x_0}(b))$ where $\mu_{x_0}(b)$ is the monodromy of the connection ∇_C taken with coefficients in $U\mathfrak{g}$ and lives in the completion $\widehat{U\mathfrak{g}}[\![\hbar]\!]$ of $U\mathfrak{g}[\![\hbar]\!]$ with respect to its finite–dimensional representations.

Conjecture 11.1. *For any $x_0\in\mathcal{C}$, there exists an algebra isomorphism $\Psi_{x_0}: U_\hbar\mathfrak{g}\to U\mathfrak{g}[\![\hbar]\!]$ such that*

- (i) *the restriction of Ψ_{x_0} to $U\mathfrak{h}[\![\hbar]\!]\subset U_\hbar\mathfrak{g}$ is the identity.*
- (ii) *$\Psi_{x_0}(S^\hbar_i)=\mu_{x_0}(S_i)$ for any $i=1,\ldots,n$.*

It is well–known that $U_\hbar\mathfrak{g}$ and $U\mathfrak{g}[\![\hbar]\!]$ are abstractly isomorphic and that an isomorphism $\Psi: U_\hbar\mathfrak{g}\to U\mathfrak{g}[\![\hbar]\!]$ is unique up to an automorphism of $U\mathfrak{g}[\![\hbar]\!]$ of the form $\mathrm{Ad}(p)$ with $p\in 1+\hbar U\mathfrak{g}[\![\hbar]\!]$ [Dr2]. The following result considerably reduces this lack of uniqueness.

Proposition 11.2. *Two isomorphisms $\Psi^i_{x_0}: U_\hbar\mathfrak{g}\to U\mathfrak{g}[\![\hbar]\!]$, $i=1,2$, satisfying conjecture 11.1 differ by an automorphism of $U\mathfrak{g}[\![\hbar]\!]$ of the form*

$$Ad(p)\quad where\quad p\in 1+\hbar U\mathfrak{h}[\![\hbar]\!]^W$$

In particular, $\Psi^1_{x_0}$ *and* $\Psi^2_{x_0}$ *have the same restriction to the subalgebra* $U_\hbar\mathfrak{g}^\mathfrak{h} \subset U_\hbar\mathfrak{g}$ *of* $\mathfrak{h}$*–invariants.*

Proposition 11.2 is a straighforward consequence of the following result

Theorem 11.3 (Etingof [MTL])**.** *The centraliser in* $U\mathfrak{g}$ *of the algebra generated by* $\mathfrak{h}$ *and the Casimirs* C_α *is generated by* $\mathfrak{h}$ *and the centre of* $U\mathfrak{g}$*.*

Note that if $\mathfrak{g}$ is of rank higher or equal to 2, no explicit isomorphism $U_\hbar\mathfrak{g} \cong U\mathfrak{g}[\![\hbar]\!]$ is known [CP].

12. 2ND VERSION OF THE UNIVERSAL MONODROMY CONJECTURE

It is often impractical to compute the monodromy of a connection with respect to a fundamental solution G normalised by $G(x_0) = 1$. De Concini and Procesi constructed a compactification Y of $\mathfrak{h}_{\text{reg}}$ where the hyperplane arrangement $\mathcal{D} = \bigcup_\alpha \text{Ker}(\alpha)$ is replaced by a normal crossings divisor. Their construction also yields fundamental solutions which are suitably normalised at points at infinity in Y [DCP1, DCP2] and generalise those constructed by Drinfeld for KZ equations [Dr3].

Theorem 12.1 (De Concini–Procesi)**.** *There exists a smooth algebraic variety* Y *and a proper morphism* $\pi : Y \to \mathfrak{h}$ *such that*

- (i) π *restricts to an isomorphism* $Y \setminus \pi^{-1}(\mathcal{D}) \longrightarrow \mathfrak{h}_{\text{reg}}$.
- (ii) *The divisor* $\widetilde{\mathcal{D}} = \pi^{-1}(\mathcal{D})$ *has normal crossings.*
- (iii) *The irreducible components of* $\widetilde{\mathcal{D}}$ *are labelled by the irreducible root subsystems* Φ *of the root system of* $\mathfrak{g}$ *with* $\widetilde{\mathcal{D}}_\Phi$ *the unique component such that* $\pi(\widetilde{\mathcal{D}}_\Phi)$ *is the annihilator of all* $\alpha \in \Phi$*.*

It is easier to describe the intersections of the irreducible components of $\widetilde{\mathcal{D}}$ corresponding to the root subsystems generated by subsets of simple roots and therefore to connected subdiagrams B of the Dynkin diagram $D_\mathfrak{g}$ of $\mathfrak{g}$.

Theorem 12.2 (De Concini–Procesi)**.** *An intersection* $\bigcap_B \widetilde{\mathcal{D}}_B$ *is non–empty if, and only if the corresponding diagrams are pairwise compatible, that is such that*

$$B_1 \subset B_2, \quad B_2 \subset B_1 \quad \text{or} \quad B_1 \perp B_2$$

where the last condition means that B_1 *et* B_2 *are not linked by an edge in* $D_\mathfrak{g}$*. When that is the case, the intersection is irreducible and smooth.*

Let $\mathcal{F} = \{B\}$ be a maximal collection of pairwise compatible connected subdiagrams of $D_\mathfrak{g}$ and let

$$y_\mathcal{F} = \bigcap_{B \in \mathcal{F}} \widetilde{\mathcal{D}}_B \in Y$$

the corresponding point at infinity.

Theorem 12.3 (De Concini–Procesi)**.** *Let* $\{u_B = 0\}_{B\in\mathcal{F}}$ *be local equations for the divisors* $\widetilde{\mathcal{D}}_B$ *in the neighborhood of the point* $y_{\mathcal{F}}$*. Then, there exists a unique fundamental solution* $G_{\mathcal{F}}$ *of* $\nabla_C G_{\mathcal{F}} = 0$ *of the form*

$$G_{\mathcal{F}} = H_{\mathcal{F}} \cdot \prod_{B\in\mathcal{F}} u_B^{\hbar C_B}$$

where $H_{\mathcal{F}}$ *is holomorphic in the neighborhood of* $y_{\mathcal{F}}$ *and such that* $H_{\mathcal{F}}(y_{\mathcal{F}}) = 1$*, and* $C_B = \sum_\beta C_\beta$ *where the sum ranges over the roots contained in the root subsystem corresponding to* B*.*

Let U be a small ball centred at $y_{\mathcal{F}}$. The solution $G_{\mathcal{F}}$ allows to write down the monodromy of the local fundamental group $\pi_1(U \setminus \bigcup_{B\in\mathcal{F}} \widetilde{\mathcal{D}}_B; y_{\mathcal{F}}) \cong \mathbb{Z}^n$ in its simplest possible form. Indeed, if γ_B is a small loop around $\widetilde{\mathcal{D}}_B$, then

$$\mu_{G_{\mathcal{F}}}(\gamma_B) = \exp(2\pi i\hbar C_B)$$

Moreover, the image of the restriction of $\mu_{G_{\mathcal{F}}}$ to the subgroup of B_W corresponding to the subdiagram $B \subseteq D_{\mathfrak{g}}$ is contained in the completion $\widehat{U\mathfrak{g}_B}[\![\hbar]\!]$ where $\mathfrak{g}_B \subseteq \mathfrak{g}$ is the simple subalgebra corresponding to B.

Conjecture 12.4. *For any maximal collection* $\mathcal{F}$ *of pairwise compatible connected subdiagrams of* $D_{\mathfrak{g}}$*, there exists an algebra isomorphism* $\Psi_{\mathcal{F}} : U_\hbar\mathfrak{g} \to U\mathfrak{g}[\![\hbar]\!]$ *such that*

(i) *the restriction of* $\Psi_{\mathcal{F}}$ *to* $U\mathfrak{h}[\![\hbar]\!] \subset U_\hbar\mathfrak{g}$ *is the identity.*
(ii) $\Psi_{\mathcal{F}}(S_i^\hbar) = \mu_{\Psi_{\mathcal{F}}}(S_i)$ *for any* $i = 1, \dots, n$*.*
(iii) $\Psi_{\mathcal{F}}(U_\hbar\mathfrak{g}_B) = U\mathfrak{g}_B[\![\hbar]\!]$ *for any* $B \in \mathcal{F}$*.*

13. Semi–classical version of the monodromy conjecture

Boalch proved a result which may be regarded as a semi–classical limit of the monodromy conjecture 11.1 [Bo1, Bo2]. Note first that conjugation by the quantum Weyl group operators $S_i^\hbar$ yields an action of the braid group B_W on $U_\hbar\mathfrak{g}$. Bearing in mind that $U_\hbar\mathfrak{g}$ is a quantisation of the algebra of functions on the dual Poisson–Lie group G^*, this action admits as semi–classical limit a Poisson action of B_W on G^* computed by De Concini–Kac–Procesi [DKP]. In a similar vein, the monodromy of ∇_C based at $x_0 \in \mathfrak{h}_{\mathrm{reg}}$ gives rise to an action of B_W on the enveloping algebra $U\mathfrak{g}[\![\hbar]\!]$. The latter admits as semi–classical limit a Poisson action of B_W on $\mathfrak{g}^*$ given by the monodromy of the connection

$$\nabla_C^{\mathrm{scl}} = d - \sum_\alpha \frac{d\alpha}{\alpha} X_{C_\alpha}$$

where the Casimirs C_α are now regarded as elements of the symmetric algebra $S\mathfrak{g} = \mathrm{Fun}(\mathfrak{g}^*)$ and X_{C_α} are the corresponding Hamiltonian vector fields. Note that the above connection is not linear since the C_α are quadratic functions on $\mathfrak{g}^*$.

Theorem 13.1 (Boalch). *For any* $x_0 \in \mathfrak{h}_{\text{reg}}$*, there exists a Poisson map* $\phi_{x_0} : \mathfrak{g}^* \to G^*$ *which is a local analytic isomorphism and intertwines the above actions of* B_W*.*

It would be interesting to prove a semi–classical analogue of the monodromy conjecture 12.4. A recent result of Alekseev–Meinrenken seems to point in this direction, at least for $\mathfrak{g} = \mathfrak{sl}_n$ [AM].

The following result is also due to Boalch and stems from his interpretation of the semi–classical connection ∇_C^{scl} as that governing isomonodromic deformations of $\mathfrak{g}$–connections on the formal disk with a pole of order 2 at the origin.

Proposition 13.2. *For* $\mathfrak{g} = \mathfrak{sl}_3$*, the map* ϕ_{x_0} *is a solution of the Painlevé VI equation.*

This result underscores the difficulty of constructing explicit isomorphisms betwen $U_\hbar\mathfrak{g}$ and $U\mathfrak{g}[\![\hbar]\!]$ as do Chari–Pressley for $\mathfrak{g} = \mathfrak{sl}_2$ [CP].

14. Quasi–Coxeter algebras and Dynkin diagram cohomology

I recently proved conjecture 12.4 for any semi–simple Lie algebra [TL7, TL8, TL9]. The strategy we follow is very much inspired by Drinfeld's proof of the equivalence of the monodromy of the KZ equations and the R–matrix representation [Dr3, Dr4] coming from $U_\hbar\mathfrak{g}$. It proceeds along the following lines :

(i) Define a suitable category of algebras carrying representations of the generalised braid group B_W on their finite–dimensional modules. We call these algebras *quasi–Coxeter algebras.* They are to B_W what Drinfeld's quasitriangular quasi–Hopf algebras are to the Artin braid groups B_n.
(ii) Show that the monodromy of the Casimir connection and the quantum Weyl group representations arise from quasi–Coxeter algebra structures on $U\mathfrak{g}[\![\hbar]\!]$ and $U_\hbar\mathfrak{g}$ respectively.
(iii) Define an appropriate deformation cohomology for quasi–Coxeter algebra structures (*Dynkin diagram cohomology*) and use it to prove that $U_\hbar\mathfrak{g}$ and $U\mathfrak{g}[\![\hbar]\!]$ are isomorphic as quasi–Coxeter algebras.

15. The De Concini–Procesi associahedron

The definition of quasi–Coxeter algebras is patterned on the work of De Concini–Procesi described in §12. It relies on the combinatorics of *nested sets*, that is collections of pairwise compatible connected subdiagrams of the Dynkin diagram $D_\mathfrak{g}$ of $\mathfrak{g}$. It is easy, and useful, to generalise the combinatorics to the case where $D_\mathfrak{g}$ is replaced by an arbitrary connected graph D. A quasi–Coxeter algebra of type D then defines representations of the Artin groups corresponding to D [BS] on its finite–dimensional modules.

The proof of this result relies on a coherence theorem akin to that of MacLane for monoïdal categories [Mc]. As in that case, cohrence is deduced from the contractibility of a CW–complex $\mathcal{A}_D$ which we call the *De Concini–Procesi associahedron.* This complex generalises Stasheff's associahedron [St], which one gets when D is the Dynkin diagram of type A_n, the Bott–Taubes cyclohedron [BT], which one gets from the affine Dynkin diagram of type $\widetilde{\mathsf{A}}_n$ and the complex constructed by De Concini–Procesi inside their compactification Y of $\mathfrak{h}_{\text{reg}}$ described in §12.

The following result yields a construction of $\mathcal{A}_D$ as a convex polytope, thus settling its existence and contractibility, and producing a new family of such polytopes.

Theorem 15.1 ([TL8]). *There exists a convex, rational, simple polytope $\mathcal{A}_D$ whose face poset is that of nested sets on the connected graph D, ordered by reverse inclusion.*

An independent construction of the polytopes $\mathcal{A}_D$ is given in [CD]. These have been further generalised by Postnikov and Zelevinsky [Po, Ze].

16. Irreducibility of monodromy representations

I obtained, in collaboration with J. Millson, a number of irreducibility results for the monodromy of ∇_C, thus settling a question raised by C. Procesi and a conjecture of Kwon and Lusztig [Kw]. We showed in particular that the action of the pure braid group $P_\mathfrak{g}$ on the weight spaces of a simple $\mathfrak{g}$–module V is irreducible for $\mathfrak{g} = \mathfrak{sl}_2$ and $\mathfrak{sl}_3$, but that, for all other $\mathfrak{g}$'s, this is not the case for most V's [MTL]. It would be desirable to obtain general irreducibility criteria for these representations, extending those of Kwon for quantum Weyl group representations of B_n [Kw]. For $\mathfrak{g} = \mathfrak{sl}_3$, it should be interesting to relate our representations to the classification of low–dimensional ($d \leq 5$) irreducible representations of B_3 obtained by Tuba and Wenzl [TW].

17. Cherednik–Dunkl operators

The Casimir connection ∇_C is closely related to the one defined by Cherednik in [Ch]. The latter is the connection on $\mathfrak{h}$ with values in any W–module U given by

$$\nabla_{\text{Ch}} = d - \sum_\alpha k_\alpha \frac{d\alpha}{\alpha} s_\alpha$$

where $s_\alpha \in W$ is the reflection corresponding to the root α and the $k_\alpha \in \mathbb{C}$ are complex parameters invariant under W. The monodromy of ∇_{Ch} always factors through the Hecke algebra of W, which is not the case of that of the Casimir connection in general. I showed however that the subspace $V[\![0]\!] \subseteq V[0]$ of the zero weight subspace of a $\mathfrak{g}$–module V defined by

$$V[\![0]\!] = \{v \in V[0] | \, e_\alpha^2 v = 0, \, \forall \alpha\}$$

is invariant under W and the Casimir operators C_α that ∇_C and ∇_{Ch}, when taken with values in $V[\![0]\!]$, coincide [TL7]. I also showed that any irreducible representation of the Weyl group $\mathfrak{S}_n$ of $\mathfrak{g} = \mathfrak{sl}_n$ may be realised inside some $V[\![0]\!]$ but that this isn't the case for the orthogonal Weyl groups. This raises the problem of the determination of the Springer parameters of the irreducible representations of W which arise inside some $V[\![0]\!]$. Another interesting question is to understand the link between the Casimir connection and Dunkl operators [He] of which the Cherednik connection is the simplest example.

18. Coxeter–Schlesinger equations

In collaboration with N. Reshetikhin, we have recently considered the semi–classical limit of the connection ∇_C, as he had done for the Knizhnik–Zamolodchikov connection in [Re]. The corresponding non–linear Poisson action of $B_{\mathfrak{g}}$ on $\mathfrak{g}^*$ seems to describe the isomonodromic deformations of certain systems of Fuchsian PDE's on $\mathfrak{h}'/\mathfrak{h}$ where $\mathfrak{h}'$ is the Cartan subalgebra of a complex simple Lie algebra $\mathfrak{g}'$ the Dynkin diagram of which contains that of $\mathfrak{g}$. It would be very interesting to link this interpretation to that obtained recently by P. Boalch [Bo2]. The latter shows that the Casimir connection may be regarded as an isomonodromic Hamiltonian vector field on a moduli space of irregular G–connexions on the unit disk.

19. Elliptic Braid Groups

In collaboration with K. Saito, we are attempting to construct an elliptic analogue of the Casimir connection. This should allow one to define monodromy representations of elliptic braid groups from the representations of elliptic Lie groups that he introduced and studied since 1985 [Sai].

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