

# MOTIVES

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Motives = "building blocks of  
(cohomology groups of)  
algebraic varieties

A. Grothendieck: a motive has:

- a zeta-function
- various realizations



History: 3 stages

(1) Grothendieck (1960's)  $k = \text{field}$

"pure motives over  $k$ "  $\leftrightarrow$  "universal (geometric) cohomology for smooth projective alg. varieties over  $k$ "

Definition (+ properties) conditional on "standard conjectures" on algebraic cycles.

(2) Beilinson, Deligne (1980's)

"mixed motives over  $k$ "  $\leftrightarrow$  univ. (geom.) coh. for all alg. var. /  $k$  (+ their diagrams)

motivic cohomology  $\leftrightarrow$  universal absolute (= arithmetic) cohomology

motivic cohomology was defined  
unconditionally:

- $H_M^*( ) \otimes \mathbb{Q} =$  piece of alg.  
(Beilinson) K-theory
- $H_M^*( ) =$  higher Chow groups  
(S. Bloch) generalization  
of divisor class group

(3) Voevodsky, Morel (1990's)

- motivic homotopy theory
- triangulated category of mixed motives with correct  $H_M^*( )$   
(sheafifying higher Chow grps)

# Examples of pure motives (+ realizations)

$$\underbrace{M \subset H^n(X)}_{\text{"pure of weight } n"}, \quad X \text{ smooth proj.}$$

"pure of weight  $n$ "

Ex 0:  $\mathbb{Z}(0) = h_0(\text{point}) (= h^0(\text{pt})^\vee)$  pure, wt=0

Ex 1:  $\mathbb{Z}(1) = h_1(\mathbb{G}_m)$   $\mathbb{G}_m = \mathbb{P}^1 - \{0, \infty\}$   
the Tate motive (wt=-2) not proj.

$$\mathbb{Z}(1)^\vee = \mathbb{Z}(-1) = h^1(\mathbb{G}_m) = h^2(\mathbb{P}^1) \quad (\text{wt}=2)$$

Betti realization: ( $k \subset \mathbb{C}$ )  $\mathbb{G}_m(\mathbb{C}) = \mathbb{C}^* = \mathbb{C} - \{0\}$

$$\mathbb{Z}(1)_B = H_1(\mathbb{C}_1^*, \mathbb{Z}) = \mathbb{Z} \cdot \gamma$$

$\gamma =$  unit circle



de Rham realization: (rather, its dual)

$$\mathbb{Z}(1)_{dR}^\vee = \mathbb{Z}(-1)_{dR} = \mathbb{Z} \cdot \frac{dz}{z} \quad z = \text{standard coordinate}$$

$$\frac{dz}{z} \in \Omega^1(\mathbb{C}^*), \quad \text{log-singularities at } \infty, 0$$

Étale realizations: (i) over  $\mathbb{C}$ :  $n \geq 1$

$$\mathbb{Z}/n\mathbb{Z}(1) = H_1(\mathbb{C}_1^*, \mathbb{Z}/n\mathbb{Z}) = \mu_n(\mathbb{C}) \quad \begin{array}{l} n\text{-th roots} \\ \text{of unity} \\ \text{in } \mathbb{C} \end{array}$$

(ii) over  $k$  s.t.  $n \cdot 1 \neq 0$  in  $k$ :

$$\mathbb{Z}/n\mathbb{Z}(1) = \mu_n(\bar{k}) \quad \begin{array}{l} n\text{-th roots of unity in } \bar{k} \\ (\bar{k} = \text{sep. closure} \\ \text{of } k) \end{array}$$

This is a Galois representation

(analogue of "monodromy representation")

Recall: Galois gps  $\longleftrightarrow$  covering (= deck) transformations

topology:  $p: Y \rightarrow X$  unram. covering  
 $G = \text{Deck}(Y/X) := \left\{ g: Y \rightarrow Y \text{ cont.} \right\}$   
 $p \circ g = p$

$p$  is regular :=  $G$  acts transitively on  $p^{-1}(x)$

algebra:  $\pi_1(X) = \frac{X = G \backslash Y}{\text{Deck}(\tilde{X}/X)}$   $\tilde{X} = \text{univ. cov.}$   
 $K \subset L$  fields,  $\dim_K L < \infty$

$G = \text{Aut}(L/K) := \left\{ g: L \xrightarrow{\sim} L \text{ preserves } +, \times \right\}$   
 fixes  $K$

$L/K$  is Galois :=  $|G| \geq \dim_K L$

$|G| = \dim_K L$   
 $K = L^G$

Notation:  $G = G(L/K)$

$G_K =$  "limit" of  $G(L/K)$  over all Galois  $L/K = G(\bar{K}/K)$

Back to  $\mu_n = \{n\text{-th roots of } 1\}$ : ( $n \neq 0$ )  
in  $k$

$$G(k(\mu_n)/k) \xrightarrow{\chi_n} \text{Aut}(\mu_n) = GL_1(\mathbb{Z}/n\mathbb{Z})$$

$$(V \zeta \in \mu_n) \quad \sigma(\zeta) = \zeta^a$$

Periods:  $\mathbb{Z}(1)_{\mathbb{B}} \times \mathbb{Z}(1)_{\text{dR}}^V \xrightarrow{\text{integration}} \mathbb{C}$

$$\gamma \quad \left| \quad \frac{dz}{z} \right| \quad \longmapsto \quad \int_{\gamma} \frac{dz}{z} = 2\pi i$$

"Comparison isomorphism" ("de Rham"):

$$\mathbb{Z}(1)_{\mathbb{B}} \otimes \mathbb{C} \xrightarrow{\sim} \mathbb{Z}(1)_{\text{dR}} \otimes \mathbb{C}$$

$$\gamma \otimes 1 \longmapsto \left(\frac{dz}{z}\right)^* \otimes \frac{1}{2\pi i}$$

Hodge types:  $H^2(\mathbb{P}^1(\mathbb{C}), \mathbb{C}) \cong H^1(\mathbb{P}^1(\mathbb{C}), \Omega^1)$

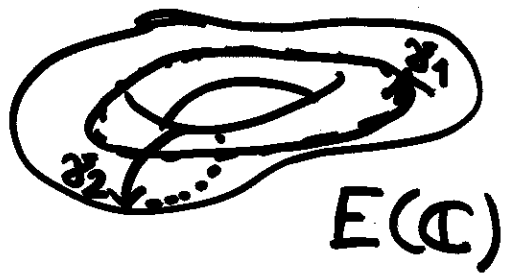
Hodge type (1, 1)

$$\mathbb{Z}(1)_{\mathbb{B}} \quad \text{Hodge type } (-1, -1)$$

Ex 2:  $M = h_1(E)$ ,  $E/k$  elliptic curve

$E =$  smooth proj. curve of genus 1;  $O_E \in E(k)$   
 (pure of wt = -1)  $O_E$  fixed

Betti realization: ( $k \subset \mathbb{C}$ )



$$M_B = H_1(E(\mathbb{C}), \mathbb{Z}) = \mathbb{Z}\alpha_1 \oplus \mathbb{Z}\alpha_2$$

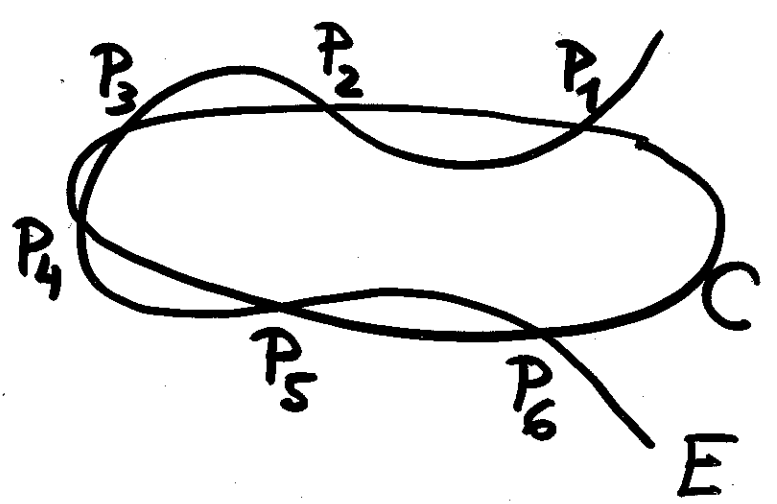
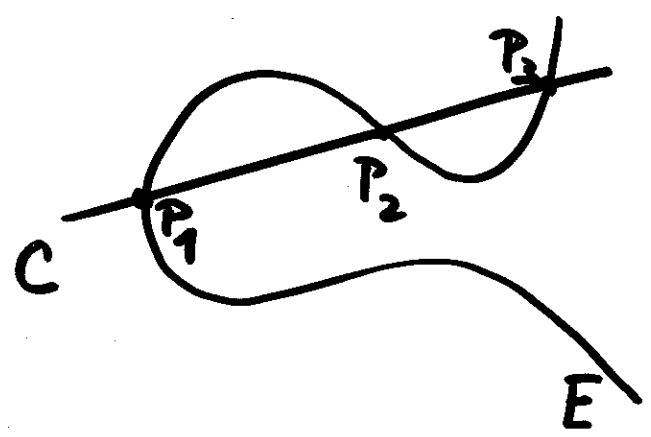
(dual to  $H^1(\dots)$ )

Weierstrass model: (if  $6 \cdot 1 \neq 0$  in  $k$ )

$E: y^2 = f(x)$  (+ one point  $O_E$  at infinity)  
 $\deg(f) = 3$ ,  $\text{disc}(f) \neq 0$  (infinity)  
 $f$  monic

Abelian group law on  $E(K)$ : ( $K \supset k$  field)

$\forall$  proj. curve  $C \subset \mathbb{P}_K^2$   $O_E = \text{zero}$   
 of  $\deg(C) = d$  s.t.  $(C \cap E)(\bar{K})$  is finite  
 $\{P_1, \dots, P_{3d}\}$   
 $P_1 \boxplus \dots \boxplus P_{3d} = O_E$



de Rham realization:

$$M_{dR}^V = \mathbb{Z}\omega \oplus \mathbb{Z}\eta, \quad \omega = \frac{dx}{y} \in \Omega^1(E)$$

periods: ( $k = \mathbb{C}$ )  $\left\{ \begin{array}{l} \eta = \frac{x dx}{y} \in \Omega^1(E - \{O_E\}) \\ \text{res}_{O_E}(\eta) = 0 \end{array} \right.$

$$\omega_j = \int_{\gamma_j} \omega, \quad \eta_j = \int_{\gamma_j} \eta \quad (j=1,2)$$

integration:  $M_B \otimes \mathbb{C} \xrightarrow{\sim} M_{dR} \otimes \mathbb{C}$

has matrix  $\begin{pmatrix} \omega_1 & \eta_1 \\ \omega_2 & \eta_2 \end{pmatrix}$

Legendre:  $\det(\text{---}) = 2\pi i$

Complex uniformization:  $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 \subset \mathbb{C}$

$$\mathbb{C}/L \xrightarrow{\sim} E(\mathbb{C})$$

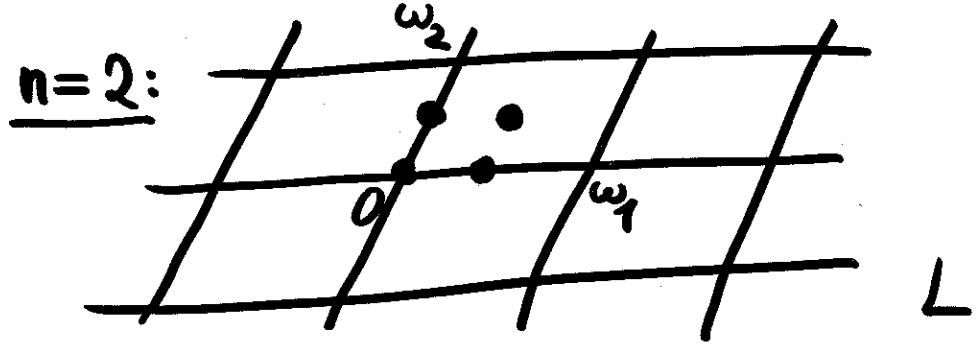
lattice

$$\bar{z} \mapsto \left( \frac{1}{2} \wp(\bar{z}, L), \wp'(\bar{z}, L) \right)$$

Étale realizations:  $n \geq 1, n \cdot 1 \neq 0$  in  $k$

(i)  $k = \mathbb{C}$

$$H_1(E(\mathbb{C}), \mathbb{Z}/n\mathbb{Z}) \simeq \frac{1}{n} L/L \simeq \underbrace{E(\mathbb{C})}_n$$

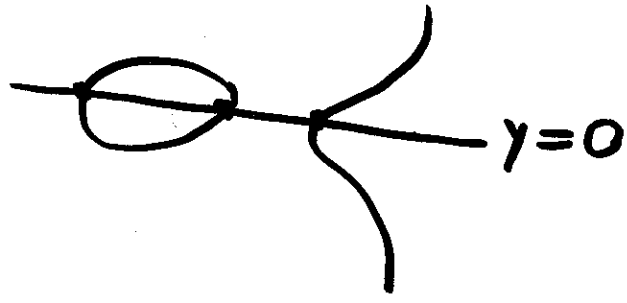


pts of order  $n$

(ii) Over any  $k$ : ( $n \cdot 1 \neq 0$  in  $k$ )

$$E_n := E_n(\bar{k}) \simeq (\mathbb{Z}/n\mathbb{Z})^2$$

$n=2$ :  $E_2 = \{O_E\} \cup (E \cap \{y=0\})$



$n=3$ :  $E_3 = \{\text{inflection pts of } E\}$

$k(E_n)/k$  is a Galois extension

Galois repr.:

$$\rho_{E,n}: G(k(E_n)/k) \hookrightarrow \text{Aut}(E_n) \simeq GL_2(\mathbb{Z}/n\mathbb{Z})$$

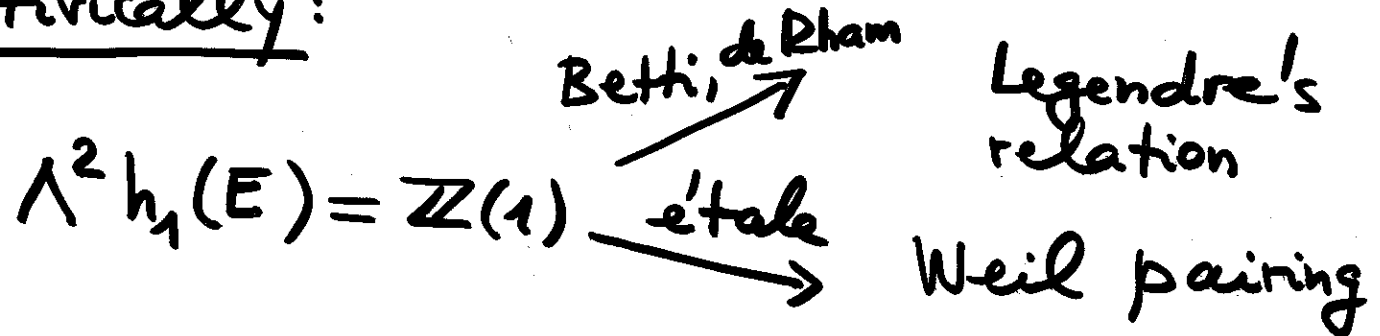
A. Weil:  $\chi_n = \det(\rho_{E,n}) (\Leftrightarrow \wedge^2 E_n \xrightarrow{\sim} \mu_n)$   
 (as repr. of  $G(k(E_n)/k)$ )

pairing  $E_n \times E_n \longrightarrow \mu_n$

$$P, Q \longmapsto f_P(Q)$$

$$\text{div}(f_P) = n(P) - n(O_E) \quad f_P \in k(E)^*$$

Motivically:



# Examples of mixed motives

Ex 1:  $a \in k^* \rightsquigarrow$  Kummer motive  $\mathcal{K}_a$

$\mathcal{K}_a \subset h_1(\mathbb{G}_m, \{1, a\})$  (relative coh.)

consists of paths  $(-)$  s.t.  $\partial(-) \in \mathbb{Z}([a] - [1])$



$$\mathbb{G}_m = \mathbb{P}^1 - \{0, \infty\}$$

Betti realization:  $(k \subset \mathbb{C})$

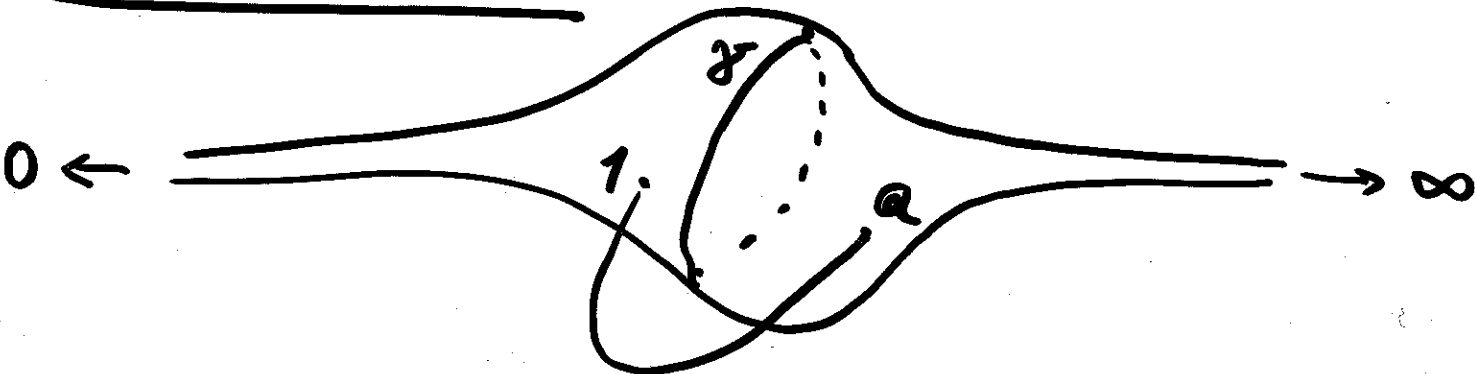
$$\begin{array}{ccccccc}
 0 \rightarrow H_1(\mathbb{C}^*, \mathbb{Z}) & \rightarrow & H_1(\mathbb{C}^*, \{1, a\}, \mathbb{Z}) & \rightarrow & H_0(\{1, a\}, \mathbb{Z}) & \xrightarrow{\Sigma} & H_0(\mathbb{C}^*) \rightarrow 0 \\
 \parallel & & \uparrow & & \underbrace{\mathbb{Z}[a] \oplus \mathbb{Z}[1]} & & \underbrace{\mathbb{Z}} \\
 0 \rightarrow \mathbb{Z} \cdot \gamma & \rightarrow & (\mathcal{K}_a)_B & \rightarrow & \mathbb{Z}([a] - [1]) & \rightarrow & 0
 \end{array}$$

Periods of  $\mathcal{K}_a$  determine  $\log(a)$  ( $\Rightarrow$  also at:

"Abel-Jacobi"  
map

$$\begin{array}{ccc}
 \mathbb{C}^* & \xrightarrow{\sim} & \mathbb{C}/2\pi i\mathbb{Z} \\
 \downarrow \psi & & \downarrow \psi \\
 a & \mapsto & \int_1^a \frac{dz}{z} \pmod{\int \frac{dz}{z}} \\
 & & \mathbb{Z}\gamma
 \end{array}$$

picture of  $\mathcal{K}_a$ :



Étale realizations :  $n \geq 1$

(i)  $k \subseteq \mathbb{C}$  :

$$\begin{array}{ccccccc}
 & & & & & & H_0(\mathbb{C}^*, \mathbb{Z}/n\mathbb{Z}) \rightarrow 0 \\
 0 \rightarrow & H_1(\mathbb{C}^*, \mathbb{Z}/n\mathbb{Z}) & \rightarrow & H_1(\mathbb{C}^*, \langle 1, a \rangle, \mathbb{Z}/n\mathbb{Z}) & \rightarrow & H_0(\langle 1, a \rangle, \mathbb{Z}/n\mathbb{Z}) & \\
 & \parallel & & \cup & & \uparrow & \\
 0 \rightarrow & \mu_n & \rightarrow & (\mathbb{Z}/n\mathbb{Z}) & \rightarrow & (\mathbb{Z}/n\mathbb{Z}) \cdot ([a] - [1]) & \rightarrow 0 \\
 & \underbrace{\hspace{2cm}} & & \underbrace{\hspace{2cm}} & & \uparrow & \\
 & \text{elts of order } n \text{ in } \bar{k}^* & & \text{elts of order } n \text{ in } \bar{k}^*/a\mathbb{Z} & & & (*)_a \\
 & & & \underbrace{\hspace{2cm}} & & & \\
 & & & \text{generated by } \mu_n, \sqrt[n]{a} & & & 
 \end{array}$$

(ii)  $k$  arbitrary, ( $n \cdot 1 \neq 0$  in  $k$ )

$\xi_n \in \mu_n$  (a fixed generator)

$\sqrt[n]{a} \in \bar{k}$  (a fixed solution of  $x^n - a = 0$ )

$$\begin{array}{ccc}
 \sigma \in G(k(\mu_n, \sqrt[n]{a})/k) & , & \sigma : \xi_n \mapsto \xi_n^{c(\sigma)} \\
 \downarrow & & \downarrow \rho_a \\
 \left( \begin{array}{cc} \xi_n^{c(\sigma)} & c(\sigma) \\ 0 & 1 \end{array} \right) \in \left( \begin{array}{cc} (\mathbb{Z}/n\mathbb{Z})^* & \mathbb{Z}/n\mathbb{Z} \\ 0 & 1 \end{array} \right) \subset GL_2(\mathbb{Z}/n\mathbb{Z}) & & \sqrt[n]{a} \mapsto \sqrt[n]{a} \cdot \xi_n^{c(\sigma)}
 \end{array}$$

is a group homomorphism.

$(*)_a$  is an exact sequence of repr. of  $G_k$

$$0 \rightarrow \mu_n \rightarrow ? \rightarrow \mathbb{Z}/n\mathbb{Z} \rightarrow 0$$

Its class lies in  $H^1(G_k, \mu_n) := \text{Ext}_{\mathbb{Z}/n\mathbb{Z}[G_k]}^1(\mathbb{Z}/n\mathbb{Z}, \mu_n)$

and is given by the class of  $\boxed{c(\sigma)}$ .

Kummer theory:

$$k^* \otimes \mathbb{Z}/n\mathbb{Z} = k^*/k^{*n} \xrightarrow{\sim} H^1(G_k, \mu_n)$$

$$\downarrow$$
$$a \otimes 1$$

$\mapsto$  class of  $\rho_a$  (= of  $c(\sigma)$ )

is an isomorphism.

Motivically:

$$0 \rightarrow \mathbb{Z}(1) \rightarrow \mathcal{K}_a \rightarrow \mathbb{Z}(0) \rightarrow 0$$

Expected:  $MM_k = \{ \text{mixed motives over } k \}$

$$\text{Ext}_{MM_k}^1(\mathbb{Z}(0), \mathbb{Z}(1)) \xleftarrow{\sim} k^*$$

$$\downarrow$$
$$\text{class of } \mathcal{K}_a \xleftarrow{\sim} a$$



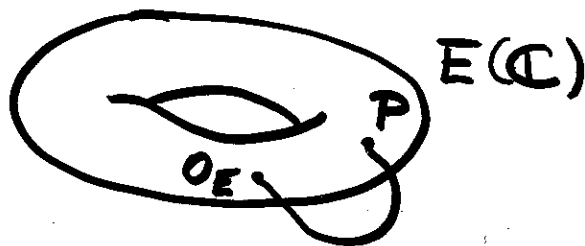
class of  $\log(a) \in \mathbb{C}/2\pi i$

class of  $a \otimes 1$   
in  $k^* \otimes \mathbb{Z}/n\mathbb{Z}$   
( $n \cdot 1 \neq 0$  in  $k$ )

Ex 2:  $E/k$  elliptic curve,  $P \in E(k)$

$M(P) \subset h_1(E, \{O_E, P\})$  consisting of paths  
 $(-)$  with  $\partial(-) = [P] - [O_E]$

$$M = h_1(E)$$



As before:

$$0 \rightarrow M \rightarrow M(P) \rightarrow \mathbb{Z}/n\mathbb{Z} \rightarrow 0$$

in various realizations.

$k \subset \mathbb{C}$ : periods of  $M(P)$  determine  $AJ(P)$   
 $(\Rightarrow P)$ :

Abel-Jacobi  
 map

$$AJ: E(\mathbb{C}) \xrightarrow{\sim} \mathbb{C}/(\mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2)$$

$$\downarrow$$

$$P \mapsto \int_{O_E}^P \omega \pmod{\int \omega}$$

closed paths

Étale realizations: ( $n \cdot 1 \neq 0$  in  $k$ )

$$0 \rightarrow E_n \rightarrow M(P)_{\mathbb{Z}/n\mathbb{Z}} \rightarrow \mathbb{Z}/n\mathbb{Z} \rightarrow 0$$

elts of order  $n$  in  $E(\bar{k})$       elts of order  $n$  in  $E(\bar{k})/\mathbb{Z} \cdot P$

gen. by  $E_n$  and  $\frac{1}{n}P$

$Q := \frac{1}{n}P \in E(\bar{k})$  - fixed solution of

$$[n]Q = P$$

mult. on  $E(\bar{k})$

analogue of  $\sqrt[n]{a}$

$$G \in G(k(E_n, \frac{1}{n}P)/k)$$

$G$  acts on  $E_n$

$$(\mathbb{Z}/n\mathbb{Z})^2$$

by  $\rho_{E_n}(G) \in GL_2(\mathbb{Z}/n\mathbb{Z})$

$$G(\frac{1}{n}P) = \frac{1}{n}P \boxplus \underbrace{z(G)}_n$$

$E_n$

$$\left( \begin{array}{ccc} \boxed{\rho_{E_n}(G)} & \begin{array}{c} z(G) \\ * \\ * \end{array} & \\ 0 & 0 & 1 \end{array} \right) \in \left\{ \begin{array}{ccc} * & * & * \\ * & * & * \\ 0 & 0 & 1 \end{array} \right\} \subset GL_3(\mathbb{Z}/n\mathbb{Z})$$

$$\underbrace{E(k) \otimes \mathbb{Z}/n\mathbb{Z}}_{E(k)/nE(k)} \hookrightarrow \underbrace{\text{Ext}^1}_{\mathbb{Z}/n\mathbb{Z}[G_k]}(\mathbb{Z}/n\mathbb{Z}, E_n)$$

$$\downarrow \quad \downarrow$$

$$P \otimes 1 \quad \longrightarrow \quad \text{class of } \underbrace{\psi}_{z(G)}$$

Unfortunately, this map is not surjective ( $\Rightarrow$ ) it is very hard

to determine  $E(k)/nE(k)$  if  $k$  is a number field, e.g.  $k = \mathbb{Q}$

Motivically:

$$0 \rightarrow h_1(E) \rightarrow M(P) \rightarrow \mathbb{Z}(0) \rightarrow 0$$

Expected:

$$E(k) \xrightarrow{\sim} \text{Ext}_{MM_k}^1(\mathbb{Z}(0), h_1(E))$$

$\downarrow$

$$P \longmapsto \text{class of } M(P)$$



Abel-Jacobi  
image of  $P$

class of  $P \otimes 1$   
in  $E(k) \otimes \mathbb{Z}/n\mathbb{Z}$   
( $n \cdot 1 \neq 0$  in  $k$ )

# ZETA FUNCTIONS

Ex:  $\zeta(s) = \sum_{n \geq 1} n^{-s} = \prod_{\substack{p \text{ prime} \\ \text{unique fact.}}} (1 + p^{-s} + p^{-2s} + \dots) = \prod_p (1 - p^{-s})^{-1}$

$p$  prime  $\leftrightarrow (p) = p\mathbb{Z}$  maximal ideal of  $\mathbb{Z}$   
 $|\mathbb{Z}/p\mathbb{Z}| = |\mathbb{F}_p| = p$

General case:  $A = \mathbb{Z}$ -algebra of finite type  
 $A = \mathbb{Z}[T_1, \dots, T_N] / (f_1, \dots, f_M)$  ( $f_j \in \mathbb{Z}[T_1, \dots, T_N]$ )  
 ideal  $\mathfrak{m} \subset A$  is maximal  $\Leftrightarrow A/\mathfrak{m}$  is a finite field  
 $N(\mathfrak{m}) := |A/\mathfrak{m}|$

Def:  $\zeta_A(s) = \prod_{\mathfrak{m} \in \text{Max}(A)} (1 - N(\mathfrak{m})^{-s})^{-1}$  (abs. conv. if  $\text{Re}(s) > \dim(A)$ )

Ex: (0)  $\zeta_{\mathbb{F}_p}(s) = (1 - p^{-s})^{-1}$

(1)  $\zeta_{\mathbb{Z}}(s) = \prod_p \zeta_{\mathbb{F}_p}(s) = \zeta(s)$

(2)  $\zeta_{\mathbb{Z}[i]}(s) = \frac{1}{4} \sum'_{(m,n) \in \mathbb{Z}^2} (m^2 + n^2)^{-s} = \zeta(s) L(s)$

$$L(s) = \sum_{\substack{n \geq 1 \\ 2 \nmid n}} (-1)^{(n-1)/2} n^{-s} = \prod_{p \neq 2} (1 - (-1)^{\frac{p-1}{2}} p^{-s})^{-1}$$

(3)  $A = \mathbb{F}_p[T]: \zeta_{\mathbb{F}_p[T]}(s) = (1 - p^{1-s})^{-1}$

Two proofs:

## Easy method:

$$\text{Max}(\mathbb{F}_p[T]) = \{ (f) \mid f \in \mathbb{F}_p[T], f \text{ monic, irred.} \}$$

$$\mathbb{F}_p[T]/(f) \cong \mathbb{F}_p^{\deg(f)} \Rightarrow \zeta_{\mathbb{F}_p[T]}(s) = Z(p^{-s}),$$

$$Z(t) = \prod_{\substack{f \in \mathbb{F}_p[T] \\ \text{monic irred.}}} (1 - t^{\deg(f)})^{-1} \stackrel{\text{U.F.}}{=} \sum_{\substack{g \in \mathbb{F}_p[T] \\ \text{monic}}} t^{\deg(g)} = \sum_{n \geq 0} p^n t^n = (1-pt)^{-1}$$

## Complicated method:

$$\mathbb{F}_{p^n} = \{ x \in \overline{\mathbb{F}_p} \mid x^{p^n} = x \} \Rightarrow \mathbb{F}_{p^d} \subset \mathbb{F}_{p^n} \Leftrightarrow d \mid n$$

in  $\mathbb{F}_p[T]$ , we have

$$T^{p^n} - T = \prod_{d \mid n} \prod_{f \in \text{Irr}_d} f \Rightarrow \sum_{d \mid n} d \cdot |\text{Irr}_d| = p^n$$

$f \in \mathbb{F}_p[T], \text{ monic, irred., } \deg(f) = d$   
 $f \in \text{Irr}_d$

$$\text{As } -\log(1-x) = \sum_{k \geq 1} \frac{x^k}{k},$$

$$\begin{aligned} \log Z(t) &= - \sum_{d \geq 1} |\text{Irr}_d| \log(1-t^d) = \sum_{d, k \geq 1} |\text{Irr}_d| \frac{t^{kd}}{k} \\ &\stackrel{n=kd}{=} \sum_{n \geq 1} \underbrace{\left( \sum_{d \mid n} d |\text{Irr}_d| \right)}_{p^n} \frac{t^n}{n} = -\log(1-pt) \end{aligned}$$

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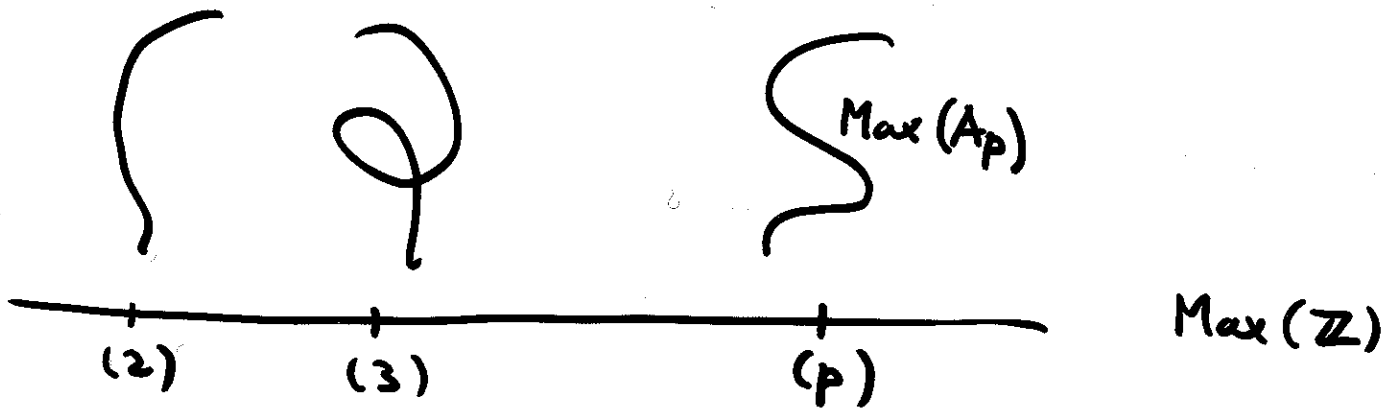
General case: geometric language

$\mathbb{Z}[T_1, \dots, T_N]$  = functions on  $\mathbb{A}_{\mathbb{Z}}^N$  (affine space over  $\mathbb{Z}$  of dim = N)

$A = \mathbb{Z}[T_1, \dots, T_N] / (f_1, \dots, f_M)$  = functions on  $X$

$X \subset \mathbb{A}_{\mathbb{Z}}^N$  given by the equations  $f_1 = \dots = f_M = 0$   
 ("  $X = \text{Spec}(A)$  ")

picture:  $p$  prime,  $A_p = A/pA$   
 $X_p = \text{Spec}(A_p) \subset \mathbb{A}_{\mathbb{F}_p}^N$  given by equations  $f_j \pmod{p} = 0$



Prop:  $|X_p(\mathbb{F}_{p^n})| = \sum_{d|n} d \left( \sum_{m \in \text{Max}(A_p)} 1 \right)$   
 $N(m) = p^d$   
 (gen. of  $p^n = \sum_{d|n} d \cdot |\text{Irr}_d|$ ).

Pf:  $\forall R$   $\mathbb{F}_p$ -algebra

$X(R) = \{ x = (x_1, \dots, x_N) \in R^N \mid \forall j \ f_j(x_1, \dots, x_N) = 0 \in R \}$   
 $\cong \text{Hom}_{\mathbb{F}_p\text{-alg}}(A_p, R)$

$x \mapsto (\text{class of } T_i \mapsto x_i)$

$m \in \text{Max}(A_p)$   
 $N(m) = p^d$

$A_p \rightarrow \underbrace{A_p/m}_{\text{isomorphic to } \mathbb{F}_p}$   
 defines  $d$  conjugate points of  $X(\mathbb{F}_p)$

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Cor:  $\zeta_{X_p}(s) := \zeta_{A_p}(s) = Z_{A_p}(p^{-s}) = Z_{X_p}(p^{-s})$ , where

$$\log Z_{X_p}(t) = \sum_{n \geq 1} |X_p(\mathbb{F}_{p^n})| \frac{t^n}{n}$$


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Ex: Affine spaces:  $X = \mathbb{A}_{\mathbb{Z}}^N$ ,  $A = \mathbb{Z}[T_1, \dots, T_N]$

$$X_p = \mathbb{A}_{\mathbb{F}_p}^N, \quad A_p = \mathbb{F}_p[T_1, \dots, T_N], \quad |X_p(\mathbb{F}_{p^n})| = p^{Nn}$$

$$\zeta_{\mathbb{A}_{\mathbb{F}_p}^N}(s) = \exp\left(\sum_{n \geq 1} p^{Nn} \frac{p^{-ns}}{n}\right) = (1 - p^{N-s})^{-1}$$

$$\zeta_{\mathbb{A}_{\mathbb{Z}}^N}(s) = \prod_p (1 - p^{N-s})^{-1} = \zeta(s-N)$$


---

Schemes of finite type over  $\mathbb{Z}$ :

$$X = \bigcup_{i=1}^r \text{Spec}(A_i), \quad \text{open covering}$$

$A_i = \mathbb{Z}$ -alg. of finite type

Ex:  $\mathbb{P}_{\mathbb{Z}}^N = \bigcup_{i=0}^N$  (standard affine subspaces  $\mathbb{A}_{\mathbb{Z}}^N$ )

Closed points of  $X$ :  $|X| = \bigcup_{i=1}^r \text{Max}(A_i)$

$$x \leftrightarrow m_i \in \text{Max}(A_i)$$

$$N(x) := N(m_i)$$

Def:  $\zeta_X(s) := \prod_{x \in |X|} (1 - N(x)^{-s})^{-1}$

Again:  $\zeta_X(s) = \prod_p \zeta_{X_p}(s)$ ,  $\zeta_{X_p}(s) = Z_{X_p}(p^{-s})$ ,

$$Z_{X_p}(t) = \exp\left(\sum_{n \geq 1} |X_p(\mathbb{F}_{p^n})| \frac{t^n}{n}\right) \quad (*)$$

Ex: Projective spaces:  $F$  field

$$\mathbb{P}^N(F) = \mathbb{A}^N(F) \amalg \mathbb{P}^{N-1}(F) = \bigsqcup_{j=0}^N \mathbb{A}^j(F)$$

(1)  $F = \mathbb{F}_{p^n}$ :  $(*) \Rightarrow \zeta_{\mathbb{P}^N_{\mathbb{F}_p}}(s) = \prod_{j=0}^N \zeta_{\mathbb{A}^j_{\mathbb{F}_p}}(s) = \prod_{j=0}^N (1 - p^{j-s})^{-1}$

$$\zeta_{\mathbb{P}^N_{\mathbb{Z}}}(s) = \prod_{j=0}^N \zeta_{\mathbb{A}^j_{\mathbb{Z}}}(s) = \prod_{j=0}^N \zeta(s-j)$$

(2)  $F = \mathbb{C}$ :  $\mathbb{P}^N(\mathbb{C}) = \bigsqcup_{j=0}^N \mathbb{C}^j$  cell decomposition

$$H_k(\mathbb{P}^N(\mathbb{C}), \mathbb{Z}) = \begin{cases} \mathbb{Z}(\text{closure of } \mathbb{C}^j), & k=2j, 0 \leq j \leq N \\ 0, & \text{otherwise} \end{cases}$$

Motivic origin of these decompositions:

$$h(\mathbb{P}^N) = \bigoplus_{j=0}^N \underbrace{h^{2j}(\mathbb{P}^N)}_{h^2(\mathbb{P}^N)^{\otimes j} = \mathbb{Z}(-j)}, \quad h^2(\mathbb{P}^N) = h^2(\mathbb{P}^1) = \mathbb{Z}(-1)$$

$\zeta(s-j)$  = zeta function of  $\mathbb{Z}(-j)$   
(base field =  $\mathbb{Q}$ )

# Zeta functions of smooth projective curves

$X/\mathbb{F}_q$  smooth irred. proj. curve,  $\mathbb{F}_q = \Gamma(X, \mathcal{O}_X)$

$g = \text{genus of } X = \dim_{\mathbb{F}_q} \Gamma(X, \Omega_{X/\mathbb{F}_q})$  field of constants

$$\zeta_X(s) = Z_X(q^{-s}), \quad Z_X(t) = \exp\left(\sum_{n \geq 1} |X(\mathbb{F}_{q^n})| \frac{t^n}{n}\right)$$

Thm (E. Artin, F. K. Schmidt, H. Hasse, A. Weil)

(1)  $Z_X(t) = \frac{P(t)}{(1-t)(1-qt)}$ ,  $P \in \mathbb{Z}[t]$ ,  $\deg(P) = 2g$

(2)  $\zeta_X(1-s) = (q^{1/2-s})^{2-2g} \zeta_X(s) \iff P(t) = \prod_{j=1}^g (1-\alpha_j t)(1-\frac{t}{\alpha_j})$

(3)  $\zeta_X(s) = 0 \implies \text{Re}(s) = \frac{1}{2} \iff \forall j \quad |\alpha_j| = \sqrt{q}$

"Pf." (1), (2)  $\exists$  additive formula for  $Z_X(t)$  (as in the "easy method" for  $X = \mathbb{A}^1$ ):

$$Z_X(t) = \sum_{\substack{D \in \text{Div}(X) \\ D \geq 0}} t^{\deg(D)} \quad D = \sum_{x \in |X|} n_x(x), \quad n_x \in \mathbb{Z} \quad (\text{finite sum})$$

$$q^{\deg(D)} = \prod_x N(x)^{n_x}$$

$$\mathcal{C}(X) = \text{Div}(X) / \langle \text{div}(f) \rangle \quad (f \in \mathbb{F}_q(X)^*)$$

$$\mathcal{L}(D) = \{f \mid (f) + D \geq 0\}, \quad \ell(D) = \dim_{\mathbb{F}_q} \mathcal{L}(D)$$

$$Z_X(t) = \sum_{C \in \mathcal{C}(X)} t^{\deg(C)} \cdot \underbrace{|\{D \in \mathcal{C}; D \geq 0\}|}_{|\mathcal{P}(\mathcal{L}(C))| = \frac{q^{\ell(C)} - 1}{q - 1}}$$

R.-R.:  $\ell(C) - \ell(K_X - C) = \deg(C) - g + 1$

$\epsilon$ :  $\exists D \in \text{Div}(X) \quad \deg(D) = 1$

$\implies (1), (2)$

Proof of (3): requires a "positivity" property

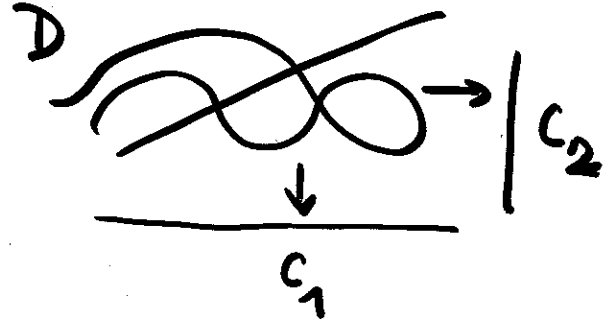
Castelnuovo-Severi inequality:  $k$  field,  $C_i$  ( $i=1,2$ )

smooth irred. proj. curve over  $k$ ,  $X = C_1 \times C_2$

$D \in \text{Div}(C_1 \times C_2)$  ( $\mathbb{Z}$ -linear combination of irred. curves on  $X$ )  
 $d_i = \deg(D \xrightarrow{\text{proj.}} C_i)$

$(CS) \quad \frac{D^2}{2} \leq d_1 d_2$

$D^2 = D \cdot D$



Ex:  $\varphi_1, \varphi_2: C_1 \rightarrow C_2$ ,  $\Gamma_{\varphi_i} = \text{graph of } \varphi_i \subset C_1 \times C_2$   
 $g_i = \text{genus}(C_i)$  ( $\Gamma_{\varphi_i}$  smooth, isomorphic to  $C_1$ )

$D = s\Gamma_{\varphi_1} + t\Gamma_{\varphi_2} \quad (s, t \in \mathbb{Z})$

Lemma:  $\Gamma_{\varphi_i}^2 = \Gamma_{\varphi_i} \cdot \Gamma_{\varphi_i} = \deg(\varphi_i)(2 - 2g_2)$

(CS) for  $D$ :  $d_1 = s+t$ ,  $d_2 = s \deg(\varphi_1) + t \deg(\varphi_2)$

$\Downarrow$  Lemma

$$g_2 \deg(\varphi_1) s^2 + (\deg(\varphi_1) + \deg(\varphi_2) - \Gamma_{\varphi_1} \cdot \Gamma_{\varphi_2}) st + g_2 \deg(\varphi_2) t^2 \geq 0$$

$\forall s, t \in \mathbb{Z}$

$\Downarrow$

$$(*) \quad |\deg(\varphi_1) + \deg(\varphi_2) - \Gamma_{\varphi_1} \cdot \Gamma_{\varphi_2}| \leq 2g_2 \sqrt{\deg(\varphi_1) \deg(\varphi_2)}$$

Special case:  $k = \mathbb{F}_2$ ,  $C_1 = C_2 = C$ ,  $g = \text{genus}(C)$

$\varphi_1 = \text{id}$ ,  $\varphi_2 = \text{Frobenius}_{2^n}: x \mapsto x^{2^n}$

$$\deg(\varphi_1) = 1, \quad \deg(\varphi_2) = 2^n, \quad \Gamma_{\varphi_1} \cdot \Gamma_{\varphi_2} = |C(\mathbb{F}_{2^n})|$$

$$(*) \Leftrightarrow |1 + 2^n - |C(\mathbb{F}_{2^n})|| \leq 2g \sqrt{2^n} \Rightarrow (3) \text{ for } C.$$

Pf of Lemma:  $X \cup E$  smooth proj. irred. surface  
————— " ————— curve

Adjunction:  $K_E = K_X + E|_E \Rightarrow \underline{2g_E - 2 = K_X \cdot E + E^2}$

$X = C_1 \times C_2$ ,  $e_i = \deg(E \xrightarrow{\text{proj.}} C_i)$

$K_X = (K_{C_1} \times C_2) + (C_1 \times K_{C_2}) \Rightarrow K_X \cdot E = (2g_1 - 2)e_1 + (2g_2 - 2)e_2$

If  $E = \Gamma_\varphi$ ,  $e_1 = 1$ ,  $e_2 = \deg(\varphi)$ ,  $E \cong C_1$ ,  $g_E = g_1$   
 $(\varphi: C_1 \rightarrow C_2)$

$0 = (2g_2 - 2) \deg(\varphi) + \Gamma_\varphi^2$

Hodge index theorem:  $k$  field (say, alg. closed)

$X$  smooth irred. proj. surface; fix  $X \hookrightarrow \mathbb{P}_k^N$   
 $H :=$  a hyperplane section of  $X$  ( $H \in \text{Div}(X)$ )

(HI)  $(\forall E \in \text{Div}(X) \otimes \mathbb{Q}) (E \cdot H = 0 \Rightarrow E^2 \leq 0)$

Prop. (HI)  $\Rightarrow$  (CS).

Pf:  $X = C_1 \times C_2$  :  $h_1 := H \cdot (C_1 \times \text{pt}) \geq 1$   $d := H \cdot D$   
 $h_2 := H \cdot (\text{pt} \times C_2) \geq 1$

$D \cdot (\text{pt} \times C_2) = d_1$ ,  $D \cdot (C_1 \times \text{pt}) = d_2$

For  $s, t \in \mathbb{Q}$ , take  $E := D + s(C_1 \times \text{pt}) + t(\text{pt} \times C_2)$   
 $E \cdot H = d + sh_1 + th_2$

$\frac{E^2}{2} = \frac{D^2}{2} + sd_2 + td_1 + st = \left(\frac{D^2}{2} - d_1d_2\right) + (sd_1)(t + d_2)$

For  $s := -d_1$ ,  $t := \frac{-d - sh_1}{h_2} = \frac{d_1h_1 - d}{h_2}$ , we have

$E \cdot H = 0$ ,  $\frac{E^2}{2} = \frac{D^2}{2} - d_1d_2 \xrightarrow{\text{(HI)}} \text{(CS)}$

General case:

$X \subset \mathbb{P}_{\mathbb{Z}}^N$  proj.,  $X_{\mathbb{Q}} \subset \mathbb{P}_{\mathbb{Q}}^N$  smooth irred.,  $d = \dim(X_{\mathbb{Q}})$

$\Rightarrow \exists$  finite set  $S_X$  of "bad primes" such that  
 $(\forall p \notin S_X) X_p \subset \mathbb{P}_{\mathbb{F}_p}^N$  is smooth irred., of  $\dim = d$ .

A. Weil's Conjectures (1949):

$(\forall p \notin S_X) \mathbb{F}_2 := \Gamma(X_p, \mathcal{O}_{X_p})$  (field of constants)

(1)  $\zeta_{X_p}(s) = \prod_{j=0}^{2d} P_j(q^{-s})^{(-1)^{j-1}}$ ,  $P_j \in \mathbb{Z}[t]$

(2)  $\deg(P_j) = b_j := \dim_{\mathbb{Q}} H^j(X(\mathbb{C}), \mathbb{Q})$

(3)  $P_j(t)$  is related to  $P_{2d-j}(1/qt)$

(4)  $P_j(t) = \prod_{k=1}^{b_j} (1 - \alpha_{j,k} t)$ ,  $|\alpha_{j,k}| = q^{j/2}$   
algebraic integers

$(P_j(q^{-s}) = 0 \Rightarrow \operatorname{Re}(s) = \frac{j}{2})$

---

Earlier work: •  $\dim(X) = 1$

• explicit formulas for diagonal equations

$$X: a_0 X_0^n + \dots + a_n X_n^n = 0.$$

Weil's suggestion: construct a (contravariant)

cohomology theory

$$Y \longmapsto H^*(Y)$$

( $Y/\mathbb{F}_2$  smooth proj.)

vector space over  
a field  $K \supset \mathbb{Q}$

axioms: Künneth formula, Poincaré duality,  
 $\exists$  cycle classes, ... ("Weil cohomology")

Lefschetz formula: for  $f: Y \rightarrow Y$

$$\Delta \cdot \Gamma_f = \sum_{j=0}^{2d} (-1)^j \text{Tr}(f^* | H^j(Y)) \quad d = \dim(Y)$$

Special case:  $f = \varphi_{2^n}: Y \rightarrow Y$   
 $y \mapsto y^{2^n}$

$$\Delta \cdot \Gamma_{\varphi_{2^n}} = |\{ \text{fixed pts of } \varphi_{2^n} \text{ in } Y(\overline{\mathbb{F}_2}) \}| = |Y(\mathbb{F}_{2^n})|$$

$$\text{As } \exp\left(\sum_{n \geq 1} \text{Tr}(A^n) \frac{t^n}{n}\right) = \frac{1}{\det(1-tA)}$$

$$\exp\left(\sum_{n \geq 1} |Y(\mathbb{F}_{2^n})| \frac{t^n}{n}\right) = \prod_{j=0}^{2d} (1 - t \varphi_2^* | H^j(Y))^{(-1)^{j-1}}$$

So: If  $\exists$  Weil cohomology  $\Rightarrow$  (1) with

$$P_j(t) = \det(1 - t \varphi_2^* | H^j(X_p)) \in \underline{\underline{K[t]}}$$

Poincaré duality  $\Rightarrow$  (3)

Do Weil cohomologies exist?

Yes! Classical Weil cohomologies: étale  
crystalline

Étale cohomology (for  $Y$  over  $\mathbb{F}_p$ ):

coefficients in  $\mathbb{Z}/n\mathbb{Z}$  ( $p \nmid n$ )

passage to a limit:  $l \neq p$  prime

coeff. in  $\mathbb{Z}_l = \varprojlim_m \mathbb{Z}/l^m\mathbb{Z}$  ( $l$ -adic integers)

" "  $\mathbb{Q}_l = \mathbb{Z}_l[\frac{1}{l}]$  (field of  $l$ -adic numbers)

$$\mathbb{Z}_l = \left\{ x = \sum_{i=0}^{\infty} x_i l^i \mid x_i \in \{0, 1, \dots, l-1\} \right\}$$

$$\mathbb{Q}_l = \left\{ x = \sum_{i \geq i_0(x) \in \mathbb{Z}} x_i l^i \mid \text{" " " } \right\}$$

$$x = \dots x_2 x_1 x_0 \cdot x_{-1} x_{-2} \dots x_{i_0} 00000 \dots$$

Ex:  $l=3$   $x = 1 + 3 + 3^2 + 3^3 + \dots \in \mathbb{Z}_3$

$$x = \dots 11111.00000 \dots$$

$$2x = \dots 22222.00000 \dots$$

$$2x+1 = \dots 00000.00000 \dots = 0$$

$$x = \frac{1}{1-3} = -\frac{1}{2}$$

Notation:  $H_l^*$  = étale coh. with coeff in  $\mathbb{Q}_l$   
( $l \neq p$ )

(crystalline coh. has coeff. in a finite extension of  $\mathbb{Q}_p$ )

$H_\ell^*$  is a Weil cohomology

(1) holds with  $P_{j,\ell}(t) = \det(1 - t\varphi_2^* | H_\ell^j(Y))$   
 $\mathbb{Z}_\ell[t]$

Questions: • is  $P_{j,\ell} \in \mathbb{Q}[t]$ ?

- is  $P_{j,\ell}$  independent of  $\ell$ ?
- is there a generalization of (HI) implying (4)?

In char=0: for varieties over  $k \subset \mathbb{C}$ ,  
étale coh. with coeff. in  $\mathbb{Z}/n\mathbb{Z} = H^*(X(\mathbb{C}), \mathbb{Z}/n\mathbb{Z})$

$(\forall \ell \text{ prime}) H_\ell^*(X) = H^*(X(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$

Question (char =  $p > 0$ ): for  $Y$  over  $\mathbb{F}_p$  as before,  
is there a Weil cohomology  $H_?^*$   
with coeff. in  $\mathbb{Q}$  s.t.  $H_\ell^*(Y) = H_?^*(Y) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ ?

Answer: NO! (Serre)

$\exists Y$  s.t. (a)  $\dim_{\mathbb{Q}_\ell} H_\ell^1(Y) = 2$

(b)  $\exists \alpha, \beta: Y \rightarrow Y$  s.t.  $A = \alpha^* \in \text{End}(H_\ell^1)$   
 $B = \beta^* \in$

satisfy  $AB = -BA$ ,  $A^2, B^2 \in \mathbb{Z}_{<0}$   
(the  $\mathbb{Q}$ -algebra generated by  $A, B$  has  
no representation over  $\mathbb{Q}$  of  $\dim = 2$ )

( $Y =$  ell. curve ~~which is~~ "supersingular" ~~( $p \equiv 3 \pmod{4}$ )~~)

Is there any motive behind this narrative?

$X \subset \mathbb{P}_{\mathbb{Z}}^N$  as before,  $P \notin S_X = S$  some Weil coh.

$$\zeta_{X_P}(s) = \prod_{j=0}^{2d} \underbrace{\det(1 - \varphi_2^* q^{-s} / H^j(X_P))}_{\zeta(h^j(X_P), s)}^{(-1)^{j-1}}$$

$$\zeta_{\{S\}}(h^j(X_{\mathbb{Q}}), s) := \prod_{P \notin S} \zeta(h^j(X_P), s) \quad X_{\mathbb{Q}} \subset \mathbb{P}_{\mathbb{Q}}^N$$

Expected: (1)  $h^j(X_{\mathbb{Q}}) = M_1 \oplus \dots \oplus M_r$   
 (submotives)

$$\zeta_{\{S\}}(h^j(X_{\mathbb{Q}}), s) = \prod_{i=1}^r \zeta_{\{S\}}(M_i, s)$$

(2)  $M = N \iff \zeta_{\{S\}}(M, s) = \zeta_{\{S\}}(N, s)$   
 pure motives over  $\mathbb{Q}$   $\forall \ell \quad M_{\ell}^{ss} = N_{\ell}^{ss}$

Ex:  $C: X_1^4 + X_2^4 = X_0^4 \subset \mathbb{P}_{\mathbb{Z}}^2 \quad S_C = \{2\}$

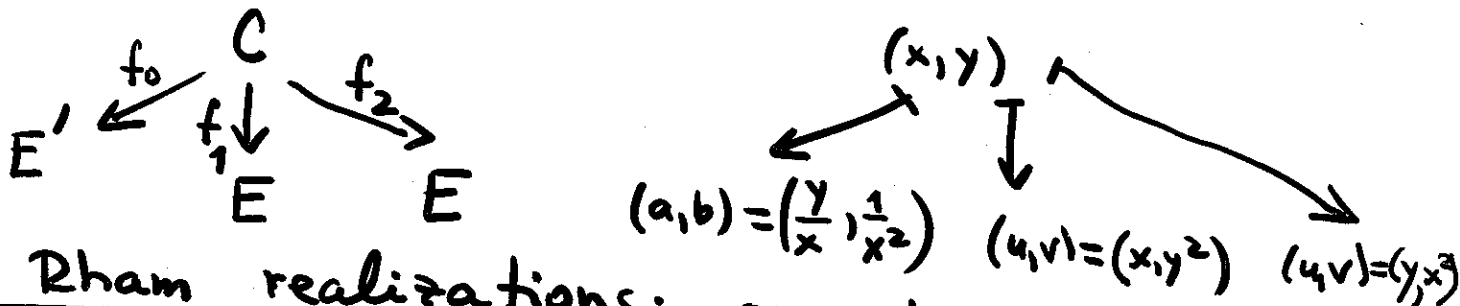
(affine equation:  $x^4 + y^4 = 1$ )  $\text{genus}(C_{\mathbb{Q}}) = 3$

$E: u^4 + v^2 = 1$  (+ 2 pts at  $\infty$ )

$E': 1 + a^4 = b^2$  ( — " — )  $\text{genus} = 1$

$$h^1(C) = h^1(E)^{\oplus 2} \oplus h^1(E')$$

Why is this true?



de Rham realizations: over  $k$ ,  $2 \cdot 1 \neq 0$  in  $k$

$$H^0(C, \Omega_{C/k}) = (k \cdot 1 \oplus k \cdot x \oplus k \cdot y) \frac{dx}{y^3}$$

$$H^0(E, \Omega_{E/k}) = k \cdot \frac{du}{v}, \quad H^0(E', \Omega_{E'/k}) = k \cdot \frac{da}{b}$$

$$f_0^*(\frac{da}{b}) = -\frac{dx}{y^3}, \quad f_1^*(\frac{du}{v}) = y \frac{dx}{y^3}, \quad f_2^*(\frac{du}{v}) = -x \frac{dx}{y^3}$$

periods: of  $E$ :  $\int_0^1 \frac{du}{\sqrt{1-u^4}} \sim B(\frac{1}{4}, \frac{1}{2})$

of  $E'$ :  $\int_0^\infty \frac{da}{\sqrt{1+a^4}} \sim B(\frac{1}{4}, \frac{1}{4})$

of  $C$ :  $\int_0^1 \frac{dx}{\sqrt{(1-x^4)^3}} \left\{ \frac{1}{\sqrt{1-x^4}}, \frac{x}{\sqrt{1-x^4}} \right\} \sim \begin{cases} B(\frac{1}{4}, \frac{1}{4}) \\ B(\frac{1}{2}, \frac{1}{4}) \\ B(\frac{1}{4}, \frac{1}{2}) \end{cases}$

zeta functions ( $\Leftrightarrow$  étale realizations):  $p \neq 2$

$$p^{n+1} - |C(\mathbb{F}_p)| \stackrel{?}{=} 2(p^{n+1} - |E(\mathbb{F}_p)|) + (p^{n+1} - |E'(\mathbb{F}_p)|)$$

amusing calculation with "Jacobi sums"  
 (= finite field analogues of  $B(a,b)$ ).

In fact:

$$\text{Jac}(C) \sim \begin{matrix} \swarrow \text{isogeneous} \\ E \times E \times E' \end{matrix}$$

## Positivity in Hodge theory

$Y \subset \mathbb{P}^N(\mathbb{C})$  complex mfd,  $d = \dim_{\mathbb{C}}(Y)$

$H \in H^2(Y, \mathbb{Q}) \cap H^{1,1}$  class of hyperplane section

Lefschetz operator:  $L: H^k(Y, \mathbb{Q}) \xrightarrow{\cup H} H^{k+2}(Y, \mathbb{Q})$

Strong Lefschetz theorem:  $(\forall k \leq d)$

$L^{d-k}: H^k(Y, \mathbb{Q}) \xrightarrow{\sim} H^{2d-k}(Y, \mathbb{Q})$  isomorphism

$\exists$   $\mathfrak{sl}_2$ -triple  $(L, \Lambda, h = [L, \Lambda])$

$H^k(Y, \mathbb{Q}) \xleftarrow{\wedge} H^{k+2}(Y, \mathbb{Q})$ ,  $h|_{H^k(Y, \mathbb{Q})} = (k-d) \cdot 1$

Primitive decomposition:

$H^k(Y, \mathbb{Q}) = \bigoplus_{j \geq 0} L^j P^{k-2j}$ ,  $P^r = H^r(Y, \mathbb{Q}) \cap \text{Ker}(\Lambda)$

Intersection pairing:

$H^k(Y, \mathbb{Q}) \times H^{2d-k}(Y, \mathbb{Q}) \rightarrow H^{2d}(Y, \mathbb{Q}) \cong \mathbb{Q}$   
 $x, y \quad \xrightarrow{\quad \quad \quad} x \cdot y$

Hodge index theorem:  $(\forall k \leq r \leq \frac{d}{2})$

$\exists$  explicit sign  $\varepsilon_{k,r} \in \{\pm 1\}$  s.t. ~~the following~~

$\forall x \in (H^{2r}(Y, \mathbb{R}) \cap H^{r,r}) \cap L^k P^{2r-2k}$ ,  $x \neq 0$

$\varepsilon_{k,r} (x \cdot L^{d-2r} x) > 0$

---

Cycle classes:  $k =$  base field,  $K \supset \mathbb{Q}$  coeff. field

$$H^* : (\text{Sm Proj}/k)^{\text{op}} \rightarrow (\text{Vect}_K) \quad \text{Weil cohomology}$$

$$X \in \text{Sm Proj}/k, \quad Z^j(X) = \left\{ \sum_{i=0}^N n_i \gamma_i \mid \begin{array}{l} \gamma_i \subset X \text{ irred.} \\ \text{def. over } k, \\ \text{codim} = j \end{array} \right\}$$

$$Z_{\text{hom}}^j(X)_{\mathbb{Q}} := \text{Im} (Z^j(X) \otimes \mathbb{Q} \rightarrow H^{2j}(X))$$

(Some of) Grothendieck's Standard Conjectures:

$X/k$  smooth proj. irred.,  $d = \dim(X)$

$$(lef) \quad Z_{\text{hom}}^r(X \times X)_{\mathbb{Q}} \xrightarrow{L^{2d-2r}} Z_{\text{hom}}^{2d-r}(X \times X)_{\mathbb{Q}}$$

is an isomorphism ( $\forall r \leq d$ )

$\Rightarrow$  primitive decomposition of  $Z_{\text{hom}}^r(X \times X)_{\mathbb{Q}}$

(Index) ( $\forall k \leq r \leq d$ )

$$(\forall x \in Z_{\text{hom}}^r(X \times X)_{\mathbb{Q}} \cap L^k P Z_{\text{hom}}^{r-k}(X \times X)_{\mathbb{Q}}, x \neq 0) \\ \epsilon_{k,r} (x \cdot L^{2d-2r} x) > 0.$$

$k \in \mathbb{C}$ : usual Hodge index thm.  $\Rightarrow$  (Index)

$k = \bar{k} \in \mathbb{C}$ : Hodge conjecture for  $X_{\mathbb{C}} \times X_{\mathbb{C}}$

$$(Z_{\text{hom}}^j(X \times X)_{\mathbb{C}} \stackrel{?}{=} H^{2j}((X \times X)(\mathbb{C}), \mathbb{Q}) \cap H^{j,j}) \Rightarrow (lef).$$

$k = \mathbb{F}_q$ : (lef) + (Index)  $\Rightarrow$  the remaining Weil's conjectures

$$(P_j(t) \in \mathbb{Q}[t], \quad |\alpha_{j,k}| = q^{j/2})$$

# Motivic Galois groups (examples)

$k$  field,  $M/k$  motive ( $M \in \text{Ch}(X)$ ) ~~\_\_\_\_\_~~

Want:  $G_{\text{mot}}(M)$  "symmetry group of  $M$ "  
 alg. group over  $\mathbb{Q}$  (coeff. field)

Ex 1.  $M = \mathbb{Z}(1)$ ,  $G_{\text{mot}}(M) = GL_1/\mathbb{Q}$ ,  $\dim(-) = 1$   
 realization with  $\mathbb{Z}/n\mathbb{Z}$ -coeff:  $\mu_n = \{\sqrt[n]{1}\}$

$$G(k(\mu_n)/k) \xrightarrow{\chi_n} GL_1(\mathbb{Z}/n\mathbb{Z})$$

bounded index if  $[k:\mathbb{Q}] < \infty$

periods:  $2\pi i$

tr.deg.  $\mathbb{Q}(2\pi i)/\mathbb{Q} = 1$

Ex 2:  $M = \mathcal{X}_a, a \in k^*$ :  $0 \rightarrow \mathbb{Z}(1) \rightarrow \mathcal{X}_a \rightarrow \mathbb{Z}(0) \rightarrow 0$

$$G(k(\mu_n, \sqrt[n]{a})/k) \hookrightarrow \left\{ \begin{pmatrix} * & ** \\ 0 & 1 \end{pmatrix} \right\} = (GL_1 \rtimes G_a)(\mathbb{Z}/n\mathbb{Z})$$

$* = \chi_n$

$**$  given by  $a \pmod{k^{*n}} \in k^*/k^{*n}$

$\mu(k) := k_{\text{tors}}^* = \{\text{roots of unity in } k\}$

Assume  $[k:\mathbb{Q}] < \infty$ :

$**$  has bounded  $\left\{ \begin{matrix} \text{order} \\ \text{index} \end{matrix} \right\}$  in  $\mathbb{Z}/n\mathbb{Z} \iff \begin{cases} a \in \mu(k) \\ a \notin \mu(k) \end{cases}$

$$G_{\text{mot}}(\mathcal{X}_a) = \begin{cases} \left\{ \begin{pmatrix} * & 0 \\ 0 & 1 \end{pmatrix} \right\} \cong GL_1, & a \in \mu(k) \quad \dim=1 \\ \left\{ \begin{pmatrix} * & ** \\ 0 & 1 \end{pmatrix} \right\} \cong GL_1 \rtimes G_a, & a \notin \mu(k) \quad \dim=2 \end{cases}$$

periods:  $2\pi i, \log(a)$

$$\text{tr.deg. } \mathbb{Q}(2\pi i, \log(a))_{\mathbb{Q}} = \begin{cases} 1 & a \in \mu(k) \\ \text{conjecturally } 2 & a \notin \mu(k) \end{cases}$$

Ex 3.  $M = h_1(E)$ ,  $E/k$  elliptic curve  
 realization with  $\mathbb{Z}/n\mathbb{Z}$ -coeff:  $E_n = \left\{ \frac{1}{n} \oplus_E \right\}$   
Endomorphism ring:  $(n \cdot 1 \neq 0 \text{ in } k)$

$$\text{End}_k(E) = \{ \alpha: E \rightarrow E, \alpha(O_E) = O_E \} \supset \mathbb{Z}$$

( $\Rightarrow \alpha$  commutes with  $\oplus$ )

Assume  
 $k \subset \mathbb{C}: E(\mathbb{C}) = \mathbb{C}/L, L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$

$$\text{End}_{\mathbb{C}}(E) = \{ \alpha \in \mathbb{C} \mid \alpha L \subseteq L \} \quad \alpha: z \mapsto \alpha z$$

(a)  $\text{End}_{\mathbb{C}}(E) = \mathbb{Z}$ :  $G(k(E_n)/k) \hookrightarrow GL_2(\mathbb{Z}/n\mathbb{Z})$

bounded index if  $[k:\mathbb{Q}]$   
 (Serre)

$$\underline{G_{\text{mot}}(E) = GL_2/\mathbb{Q}}$$

$\uparrow$   
 $\underline{\dim = 4}$

periods:  $\begin{pmatrix} \omega_1 & \eta_1 \\ \omega_2 & \eta_2 \end{pmatrix}, \quad \begin{vmatrix} \omega_1 & \eta_1 \\ \omega_2 & \eta_2 \end{vmatrix} = 2\pi i$

conjecturally: tr. deg.  $\mathbb{Q}(\omega_1, \omega_2, \eta_1, \eta_2)_{\mathbb{Q}} \stackrel{?}{=} 4$

(b)  $\text{End}_{\mathbb{C}}(E) \neq \mathbb{Z}$ : case of complex multiplication

$\mathbb{Z} + \mathbb{Z}\alpha, \quad \underline{\alpha^2 + b\alpha + c = 0}$  (CM)

$b, c \in \mathbb{Z}, \quad \mathcal{D} := b^2 - 4c < 0$

$K := \text{End}_{\mathbb{C}}(E) \otimes \mathbb{Q} = \mathbb{Q}(\sqrt{\mathcal{D}})$

(Sub) Ex:  $E: y^2 = 1 - x^4$  (+ 2 pts at  $\infty$ )

$E(\mathbb{C}) = \mathbb{C}/\Omega \cdot \mathbb{Z}[i]$ ,  $k \subset \mathbb{C}$   $O_E = (0, 1)$

(i) If  $i \in k$ :

$i: (x, y) \mapsto (ix, y)$

$\mathbb{Z} + \mathbb{Z}i = \mathbb{Z}[i] = \text{End}_k(E)$ ,  $i^2 = -1$   
 $E_n \cong \mathbb{Z}[i]/n\mathbb{Z}[i]$  (as  $\mathbb{Z}[i]$ -modules)

$G(k(E_n)/k)$

elts commute with  $\text{End}_k(E) = \mathbb{Z}[i]$

$GL_1(\mathbb{Z}[i]/n\mathbb{Z}[i]) \subset GL_2(\mathbb{Z}/n\mathbb{Z})$

$\mathbb{C} \ni a+bi \mapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \in M_{a+bi}$

bounded index

if  $[k:\mathbb{Q}] < \infty$

$G_{\text{mot}}(h_1(E)) = GL_1/\mathbb{Q}(i) \hookrightarrow GL_2/\mathbb{Q}$  (non-split Cartan subgroup)

(ii) If  $i \notin k$ :

$G(k(E_n)/k)$

elts act on  $\text{End}_k(E) = \mathbb{Z}[i]$  (by conjug.) as  $a+bi \mapsto a \pm bi$

$N(\mathbb{C}) = \mathbb{C} \cup \sigma\mathbb{C} \subset GL_2(\mathbb{Z}/n\mathbb{Z})$

$(\forall g \in \mathbb{C}) g M_{a+ib} g^{-1} = M_{a+ib}$   
 $\sigma M_{a+ib} \sigma^{-1} = M_{a-ib}$   
 (e.g.  $\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ )

$G_{\text{mot}}(h_1(E)) = GL_1/\mathbb{Q}(i) \rtimes \langle 1, \sigma \rangle \subset GL_2/\mathbb{Q}$

(normalizer of a non-split Cartan subgroup)

General CM case: similar

Summary:  $k \subset \mathbb{C}$ ,  $E/k$  ell. curve

$$K := \text{End}_k(E) \otimes \mathbb{Q}$$

no CM:  $K = \mathbb{Q}$ ,  $G_{\text{mot}}(h_1(E)) = \text{GL}_2/\mathbb{Q}$ ,  $\dim = 4$

CM:  $K = \mathbb{Q}(\sqrt{D})$

$(D < 0)$

$$G_{\text{mot}}(h_1(E)) = \begin{cases} \text{GL}_1/k & \text{End}_k(E) \otimes \mathbb{Q} = K \\ \text{GL}_1/k \rtimes \underbrace{\{1, \sigma\}}_{G(K/\mathbb{Q})} & \text{--- " ---} = \mathbb{Q} \end{cases}$$

$\dim = 2$

---

Periods in the CM case: ( $k \subset \mathbb{C}$ )

$$\mathbb{Q}(\omega_1, \omega_2, \eta_1, \eta_2) = \mathbb{Q}(2\pi i, \omega_1)$$

$$\text{tr. deg. } \mathbb{Q}(2\pi i, \omega_1)/\mathbb{Q} = 2$$

$\exists$  linear constraints on  $\left. \begin{array}{l} \text{periods} \\ G(k(E_n)/k) \end{array} \right\}$



$\exists$  non-trivial cycle  $\Gamma_\alpha \in \mathbb{Z}^1(E \times E)$   
( $\alpha \in \text{End}_k(E)$ ,  $\alpha \notin \mathbb{Z}$ )

Fact:  $E/k$  ell. curve,  $k \supset \mathbb{Q}$

$$\mathbb{Z}_{\text{hom}}^1(E \times E)_{\mathbb{Q}} = \begin{cases} \langle E \times \text{pt}, \text{pt} \times E, \Delta \rangle & \text{End}_k(E) = \mathbb{Z} \\ \langle \text{--- " ---}, \Gamma_\alpha \rangle & \alpha \in \text{End}_k(E) \\ & \alpha \notin \mathbb{Z} \end{cases}$$

General case:  $k \subset \mathbb{C}$ ,  $X, Y \in \text{Sm Proj}/k$

(1) non-zero  $Z \in Z_{\text{hom}}^r(X \times Y)_{\mathbb{Q}}$  (for  $H = H_B$ , the usual cohomology)

$\Downarrow$   
linear relation between periods of  $X$  and  $Y$   
————— " —————  
Galois gps associated to étale realizations

(2) non-zero  $Z \in Z_{\text{hom}}^r(X \times X \times \dots \times X)_{\mathbb{Q}}$   
 $\Downarrow$   
polynomial relation between periods of  $X$   
————— " —————  
for Galois gps .....

Conjecture (Grothendieck, ... ) If  $[k:\mathbb{Q}] < \infty$ , all ~~the~~ polynomial relations between periods of  $X$  (.... for Galois gps attached to étale cohomology of  $X$ ) are obtained in this way.

Reformulation:  $H_B(X) := \bigoplus_{j \geq 0} H^j(X(\mathbb{C}), \mathbb{Q})$   
 $X/k$  smooth proj.,  $k \subset \mathbb{C}$

$\Downarrow$   
 $G_{\text{mot}}(\underbrace{h(X)}_{\bigoplus_{j \geq 0} h^j(X)}) =$  subgp of elts of  $GL(H_B(X)) \times GL_1/\mathbb{Q}$   
which fix all classes in  $Z_{\text{hom}}^r(X^n)_{\mathbb{Q}} \subset H_B(X)^{\otimes n}$  ( $\forall r, n \geq 0$ )

( $t \in GL_1$  acts on  $J$  by  $t^r$ ; this technicality reflects the fact that the cycle class map really goes from  $Z^r(Y)$  to  $\underbrace{H^{2r}(Y) \otimes (H^2(\mathbb{P}^1)^\vee)^{\otimes r}}_{H^{2r}(Y)(r)}$ )

## Grothendieck's conjecture on periods:

$[k:\mathbb{Q}] < \infty$ ,  $X \in \text{Sm Proj}/k$

$\text{tr. deg. } \mathbb{Q}(\text{periods of } h(X)) / \mathbb{Q} \stackrel{?}{=} \dim G_{\text{mot}}(h(X))$   
(inequality " $\leq$ " always holds)

## Generalization to mixed motives:

$[k:\mathbb{Q}] < \infty$ ,  $M$  mixed motive over  $k$

$\text{tr. deg. } \mathbb{Q}(\text{periods of } M) / \mathbb{Q} \stackrel{?}{=} \dim G_{\text{mot}}(M)$   
(again, " $\leq$ " holds).

Ex: multiple zeta values

$$\zeta(n_1, \dots, n_r) = \sum_{1 \leq k_1 < \dots < k_r} \frac{1}{k_1^{n_1} \dots k_r^{n_r}} \quad (n_j \in \mathbb{N})$$

are periods of certain mixed motives

over  $\mathbb{Q}$  occurring inside  $\pi_1^{\text{nilp}}(\mathbb{P}^1 - \{\infty, 0, 1\})$ .

We expect (after Grothendieck):  $k$  field

(1)  $\{ \text{pure motives over } k \} = \text{semi-simple}$   
 (with coeff. in  $\mathbb{Q}$ )  $\mathbb{Q}$ -linear abelian category

(2) If  $k \subset \mathbb{C}$ :  $M$  pure motive/ $k \Rightarrow G_{\text{mot}}(M)_{/\mathbb{Q}}$  is reductive and

$\text{Rep}_{\mathbb{Q}}(G_{\text{mot}}(M)) \xleftrightarrow{\text{equiv.}} \left\{ \begin{array}{l} \text{subcategory of pure} \\ \text{motives}/k \text{ generated} \\ \text{by direct summands} \\ \text{of all } M^{\otimes m} \oplus (M^{\vee})^{\otimes n} \end{array} \right\}$   
 (alg. repr.  $G_{\text{mot}}(M) \rightarrow GL_N/\mathbb{Q}$ )  
 $\langle M \rangle^{\otimes} =$

(3) If  $[k \in \mathbb{Q}] < \infty$ :  $M$  pure motive/ $k$   
 direct summands of  $M \xrightarrow{1-1}$  "factors" of  $\zeta(M, s)$

Ex:  $k \subset \mathbb{C}$ ,  $E/k$  ell. curve,  $M = h^1(E) = h_1(E)^{\vee}$   
 $G = G_{\text{mot}}(M) \subset GL_2/\mathbb{Q}$

no CM:  $M \leftrightarrow$  standard 2-dim. repr. of  $G$   
 $G = GL_2/\mathbb{Q}$  "st"

$(\forall n \geq 1) \underbrace{S^n M}_{\text{simple}} \leftrightarrow \underbrace{S^n(\text{st})}_{\text{irreducible}}$

CM:  $G = GL_1/K$  or  $GL_1/K \rtimes G(K/\mathbb{Q})$   
 $(K = \text{End}_{\mathbb{C}}(E) \otimes \mathbb{Q})$

$M \leftrightarrow$  standard 1-dim. repr  $V$  over  $K$  of  $GL_1/K$   
 $S_{\mathbb{Q}}^2 M \leftrightarrow S_{\mathbb{Q}}^2 V = S_K^2 V \oplus \wedge_{\mathbb{Q}}^2 V$

Tensor product:  $M, N$  pure motives over  $k = \mathbb{Q}$   
 $S =$  finite set of bad primes for  $M, N$  (coeff. in  $\mathbb{Q}$ )

$$\zeta_{\{S\}}(M, s) = \prod_{p \notin S} \prod_{i=1}^m (1 - \alpha_i(p) p^{-s})^{-1}$$

$$\prod_{i=1}^m (1 - \alpha_i(p) t) = \det(1 - \underbrace{Fr_p}_l t \mid \underbrace{M_\ell}_l) \in \mathbb{Q}[t]$$

"Frobenius"  $l$ -adic repr. of  $G_{\mathbb{Q}}$  ( $l \neq p$ )

It should be possible to add correct factors also for  $p \in S \rightsquigarrow$  function  $\zeta(M, s)$  with meromorphic continuation to  $\mathbb{C}$ , finitely many poles and a functional equation relating  $\zeta(M, s)$  to  $\zeta(M^\vee, 1-s)$ .

If  $\zeta_{\{S\}}(N, s) = \prod_{p \notin S} \prod_{j=1}^N (1 - \beta_j(p) p^{-s})^{-1}$ , then

$$\zeta_{\{S\}}(M \otimes N, s) = \prod_{p \notin S} \prod_{i,j} (1 - \alpha_i(p) \beta_j(p) p^{-s})^{-1}$$

Ex:  $M = h^1(E)$ ,  $E/\mathbb{Q}$  elliptic curve

$$\zeta_{\{S\}}(M, s) = \prod_{p \notin S} [(1 - \alpha_p p^{-s})(1 - \beta_p p^{-s})]^{-1}$$

$$\begin{aligned} \zeta_{\{S\}}(S^2 M, s) &= \prod_{p \notin S} [(1 - \alpha_p^2 p^{-s})(1 - \underbrace{\alpha_p \beta_p}_p p^{-s})(1 - \beta_p^2 p^{-s})]^{-1} \\ &= \underbrace{\zeta_{\{S\}}(s-1)}_{\zeta_{\{S\}}(\mathbb{Q}(-1), s)} \prod_{p \notin S} [(1 - \alpha_p^2 p^{-s})(1 - \beta_p^2 p^{-s})]^{-1} \\ & \qquad \qquad \qquad \underbrace{\hspace{10em}}_{F(s)} \end{aligned}$$

$E$  has CM  $\Rightarrow S^2 M = \mathbb{Q}(-1) \oplus N$ ,  $F(s) = \zeta_{\{S\}}(N, s)$  and

$E$  has no CM  $\Rightarrow S^2 M$  irred.,  $F(s)$  has  $\infty$  many poles

## Tannakian formalism

$F \supset \mathbb{Q}$  field,  $G \subset GL_N/F$  affine alg. group over  $F$

Fact:  $G$  can be reconstructed from

$(\text{Rep}_F(G), \otimes, \omega_F)$

$\omega_F: \text{Rep}_F(G) \rightarrow \text{Vect}_F$   
forgetful functor

More precisely: given

$\mathcal{T} =$  abelian  $F$ -linear category with  
tensor product  $\otimes$ , duals, unit object  $\mathbb{1}$   
with  $\text{End}(\mathbb{1}) = F$

$\omega: \mathcal{T} \rightarrow \text{Vect}_F$  exact faithful  $\otimes$ -functor  
("fibre functor over  $F$ ")

Then:  $\omega$  lifts to a  $\otimes$ -equivalence  
of categories  $\mathcal{T} \xrightarrow{\cong} \text{Rep}_F(G)$ , where  
 $G = \text{Aut}^{\otimes}(\omega)$

(projective limit of affine alg. groups /  $F$ ).

If  $\mathcal{T}$  is  $\otimes$ -generated by one object  $M$ ,  
then  $G \subset GL(\omega(M))$ .

$\mathcal{T}$  semi-simple  $\Rightarrow G$  ~~reductive~~ reductive.

~~reductive~~

~~reductive~~

# Grothendieck's conditional definition of $G_{\text{mot}}^{(M)}$

$k \subset \mathbb{C}$ ,  $X \in \text{Sm Proj } k$

Goal: construct  $(\langle h(X) \rangle^{\otimes}, \otimes) \xrightarrow{H_B} \text{Vect } \mathbb{Q}$

$\mathcal{T} =$  abelian semi-simple  
Tannakian over  $\mathbb{Q}$

$$G_{\text{mot}}(h(X)) := \text{Aut}^{\otimes}(\text{this functor})$$

Objects of  $\mathcal{T}$ :  $(X^n, p, r)$  (Finite unions of)  $n \geq 0, r \in \mathbb{Z}$

$$p \in \mathbb{Z}_{\text{hom}}^{n \dim(X)} (X^n \times X^n)_{\mathbb{Q}}, p^2 = p \quad \text{Tate twist}$$

$$H_B((X^n, p, r)) = p H_B(X^n)(r) \leftarrow \text{shift in degree by } 2r$$

$$\text{Hom}_{\mathcal{T}}((X^n, p, r), (X^{n'}, p', r')) = p' \mathbb{Z}_{\text{hom}}^{n \dim(X) - r + r'} (X^{n+n'})_{\mathbb{Q}}$$

$$(X^n, p, r) \otimes (X^{n'}, p', r') = (X^n \times X^{n'}, p \times p', r+r')$$

$$(X^n, p, r)^{\vee} = (X^n, {}^t p, n \dim(X) - r)$$

$$1 = (\underline{X^0}, 1, 0)$$

$\text{Spec}(k)$

Problems: (1) Is  $\mathcal{T}$  abelian semi-simple?

(2)  $H_B: (\langle h(X) \rangle^{\otimes}, \otimes) \rightarrow (\text{Vect } \mathbb{Q}, \otimes)$

has values in super vector spaces, as

the products  $H^i \otimes H^j \rightarrow H^i \otimes H^j$

are super-commutative.  $x \otimes y \mapsto (-1)^{ij} y \otimes x$

Standard conjecture:

(Num)  $(\forall r, n \geq 1) \mathbb{Z}_{\text{hom}}^r (X^n)_{\mathbb{Q}} \times \mathbb{Z}_{\text{hom}}^{n \dim(X) - r} (X^n)_{\mathbb{Q}} \rightarrow \mathbb{Q}$

is non-degenerate.

Facts: (d) (lef) + (Index)  $\Rightarrow$  (Num)

(b) (Jannsen) (Num)  $\Rightarrow$   $\langle h(X) \rangle$  is semi-simple abelian

(c) (Num)  $\Rightarrow$  all projections  $H_B(X^n) \rightarrow H_B^j(X^n)$  are defined by elements of  $\underbrace{\mathbb{Z}_{\text{hom}}^{\text{ndim}}(X^n \times X^n)}_{P_{n,j}} \otimes \mathbb{Q}$   
( $\Rightarrow h^j(X^n) := (X^n, P_{n,j}, 0)$  is defined)

$\Rightarrow$  one can change  $\otimes$ -isomorphisms  $M \otimes N \xrightarrow{\sim} N \otimes M$  in  $\mathcal{T}$  (defined by interchanging the factors) by adding suitable signs so that  $H_B$  becomes a  $\otimes$ -functor

$$H_B^{\text{new}}: (\langle h(X) \rangle, \otimes_{\text{new}}) \rightarrow \text{Vect}_{\mathbb{Q}}$$

Def:  $G = G_{\text{mot}}(h(X)) := \underline{\text{Aut}}^{\otimes}(H_B^{\text{new}})$ .

( $\Rightarrow \text{Rep}_{\mathbb{Q}}(G) \simeq (\langle h(X) \rangle, \otimes_{\text{new}})$ )

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What if  $k > \mathbb{F}_p$ ? Even if we assume

(Num) (for a fixed  $\ell$ -adic  $H_{\ell}^*$ ), the semi-simple  $\mathbb{Q}$ -linear Tannakian category  $(\langle h(X) \rangle, \otimes_{\text{new}})$  does not admit a fibre functor into  $\text{Vect}_{\mathbb{Q}}$ . The fibre functors form a "gerbe" for a suitable Grothendieck topology on affine  $\mathbb{Q}$ -schemes.