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Serge Lang's Contributions to the Theory of Transcendental Numbers[2]

Michel Waldschmidt

When Serge Lang started to work on transcendental number theory in the early 1960s, the subject was not fashionable. It became fashionable only a few years later, thanks to the work of S. Lang certainly, but also to the contributions of A. Baker. At that time the subject was considered as very technical, not part of the main stream, and only a few specialists were dealing with it. The proofs were somewhat mysterious: why was it possible to prove some results while other conjectures resisted?

With his outstanding insight and his remarkable pedagogical gifts, Lang comes into the picture and contributes to the subject in at least two very different ways: on the one hand, he simplifies the arguments (sometimes excessively) and produces the first very clear proofs which can be taught easily; on the other hand, he introduces new tools, like group varieties, which put the topic closer to the interests of many a mathematician.

His proof of the Six Exponentials Theorem is a good illustration of the simplicity he introduced in the subject. His arguments are clear; one understands for instance why the construction of an auxiliary function is such a useful tool. Probably nobody knows so far why the arguments do not lead to a proof of the four exponentials conjecture, but this is something which will be clarified only later. Several mathematicians knew the Six Exponentials Theorem; Lang was the first to publish its proof (a few years later, K. Ramachandra rediscovered it).

Another nice example is the so-called Schneider-Lang criterion. Schneider had produced a general statement on the algebraic values of meromorphic functions in 1949. This statement of Schneider is powerful; it includes a number of transcendence results, and it was the first result containing at the same time the Hermite–Lindemann Theorem on the transcendence of $\log\alpha$, the Gel'fond–Schneider solution of Hilbert's seventh problem on the transcendence of α^β, and the Six Exponentials Theorem. However, Schneider's criterion was quite complicated; the statement itself included a number of technical hypotheses. Later, in 1957 (in his book on transcendental number theory), Schneider produced a simplified version dealing with functions satisfying differential equations (at the cost of losing the Six Exponentials Theorem from the corollaries, but Schneider did not state this theorem explicitly anyway). S. Lang found nice hypotheses which enabled him to produce a simple and elegant result.

Lang also extended this Schneider-Lang criterion to several variables, again using ideas of Schneider (which he introduced in 1941 for proving the transcendence of the values $B(a,b)$ of Euler's Beta function at rational points). Lang's extension to several variables involves Cartesian products. M. Nagata suggested a stronger statement involving algebraic hypersurfaces. This conjecture was settled by E. Bombieri in 1970 using a generalization in several variables of Schwarz's Lemma, which was obtained by Bombieri and Lang using also some deep L^2 estimates from Hörmander. It is ironic that Bombieri's Theorem is not required but that the statement with Cartesian product suffices for the very surprising proof of Baker's Theorem (and its extension to elliptic curves) found by D. Bertrand and D. W. Masser in 1980.

The introduction by S. Lang of group varieties in transcendental number theory followed a conjecture of Cartier, who asked him whether it would be possible to extend the Hermite-Lindemann Theorem from the multiplicative group to a commutative algebraic group over the field of algebraic numbers. This is the result that Lang proved in 1962. At that time there were a few transcendence results (by Siegel and Schneider) on elliptic functions and even Abelian functions. But Lang's introduction of algebraic groups in this context was the start of a number of important developments in the subject.

Among the contributions of Lang to transcendental number theory (also to Diophantine approximation and Diophantine geometry), the least are not his many conjectures which shed a new light on the subject. On the contrary, he had a way of considering what the situation should be, which was impressive. Indeed, he succeeded in getting rid of the limits from the existing results and methods. He made very few errors in his predictions, especially if we compare them with the large number of conjectures he proposed. His

[2]*Editor's note: An expanded version of Waldschmidt's contribution is published in: "Les contributions de Serge Lang à la théorie des nombres transcendants",* Gaz. Math. **108** (2006), 35–46.

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description of the subject will be a guideline for a very long time.

Lang's Work on Modular Units and on Frobenius Distributions[3]

David E. Rohrlich

In 1972 Lang joined the Department of Mathematics of Yale University, where he remained a faculty member until his retirement. The move to Yale coincided with a change of direction in Lang's research, a change which reflected a broader trend in number theory as a whole: Whereas the theory of automorphic forms had previously been the exclusive domain of specialists, by the early seventies modular forms and the Langlands program were playing a central role in the thinking of number theorists of a variety of stripes. In Lang's case these influences were particularly apparent in the work with Kubert on modular units and in the work with Trotter on Frobenius distributions.

Modular Units

Two brief notes on automorphisms of the modular function field (articles [1971c] and [1973] in the *Collected Papers*) signaled Lang's developing interest in modular functions, but his primary contribution in this domain was the joint work with Kubert on modular units, expounded in a long series of papers from 1975 to 1979 and subsequently compiled in their book *Modular Units*, published in 1981. The work has two distinct components: the function theory of modular units on the one hand and the application to elliptic units on the other.

The Function-Theoretic Component

In principle the problem considered by Kubert and Lang can be formulated for any compact Riemann surface X and any finite nonempty set S of points on X. Let C_S be the subgroup of the divisor class group of X consisting of the classes of divisors of degree 0 which are supported on S. If one prefers, one can think of C_S as the subgroup of the Jacobian of X generated by the image of S under an Albanese embedding. In any case, the problem is to determine if C_S is finite, and when it is finite to compute its order.

In practice this problem is rarely of interest: if the genus of X is ≥ 1, then for most choices of S we can expect that C_S will be the free abelian group of rank $|S|-1$, and there is nothing further to say. However, in the work of Kubert and Lang X is a modular curve and S its set of cusps. In this case Manin [18] and Drinfeld [9] had already proved the finiteness of C_S, but their proof rested on a clever use of the Hecke operators and gave no information about the order of C_S. Kubert and Lang found an altogether different proof of the Manin-Drinfeld theorem in which the whole point was to exhibit a large family of functions on X with divisorial support on S. (These functions, by the way, are the "modular units". If R_S is the subring of the function field of X consisting of functions holomorphic outside S, then the modular units are indeed the elements of the unit group $R_S^\times$ of R_S.) In optimal cases, in particular when X is the modular curve usually denoted $X(N)$ and N is a power of a prime $p \geq 5$, Kubert and Lang were able to deduce an explicit formula for $|C_S|$ in terms of certain "Bernoulli-Cartan numbers" closely related to the generalized Bernoulli numbers $b_{2,\chi}$ which appear in formulas for the value of a Dirichlet L-function $L(s,\chi)$ at $s=-1$.

This work found immediate application in the proof by Mazur and Wiles [20] of the main conjecture of classical Iwasawa theory, and since then it has found many other applications as well. But quite apart from its usefulness, the work can be appreciated as a counterpoint to the "Manin-Mumford conjecture", enunciated by Lang in an earlier phase of his career (see [1965b]) in response to questions posed by the eponymous authors. The conjecture asserts that the image of a curve X of genus ≥ 2 under an Albanese embedding intersects the torsion subgroup of the Jacobian of X in only finitely many points. A strong form of the conjecture was proved by Raynaud in 1983 [23], and the subject was subsequently enriched by Coleman's theory of "torsion packets" [5]: a torsion packet on X is an equivalence class for the equivalence relation

$$P \equiv Q \Leftrightarrow \text{ The divisor } n(P-Q)$$
$$\text{is principal for some integer } n \geq 1$$

on the points of X. Of particular relevance here is the proof by Baker [1] of a conjecture of Coleman, Kaskel, and Ribet, from which it follows that for most values of N (including in particular $N = p^n$ with p outside a small finite set) the cuspidal torsion packet on $X(N)$ consists precisely of the cusps. Thus the results of Kubert and Lang provide one of the relatively rare examples of a curve for which the order of the subgroup of the Jacobian generated by the image under an Albanese embedding of a nontrivial torsion packet on the curve has been calculated explicitly.

[3] *Editor's note: In addition to contributions to both* Notices *articles about Lang, David Rohrlich has also written the following piece: "Serge Lang", Gaz. Math.* **108** *(2006), 33–34.*

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