

# Abstract

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## Number theory: Challenges of twenty first century

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Problems in number theory are sometimes easy to state and often very hard to solve. We survey some of them.

## Extended abstract

We start with prime numbers. The twin prime conjecture, Goldbach Conjecture are among the main challenges. Are there infinitely many Mersenne (resp. Fermat) prime numbers? The largest known prime numbers are Mersenne numbers. Mersenne prime numbers are also related with perfect numbers, a problem considered by Euclid and still unsolved. One the main challenges for specialists of number theory is Riemann's hypothesis, which is now more than 150 years old.

Next we discuss open problems (initiated by E. Borel in 1905 and then in 1950) on the decimal (or binary) development of algebraic numbers. Almost nothing is known on this topic.

Kontsevich and Zagier introduced the notion of *periods* and suggested a far reaching statement which would solve a large number of open problems of irrationality and transcendence.

Diophantine equations conceal plenty of mysteries. Fermat's Last Theorem has been proved by A. Wiles, but many more questions are waiting for an answer. We discuss a conjecture due to S.S. Pillai, as well as the *abc*-Conjecture of Oesterlé–Masser.

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## Hilbert's 8th Problem

August 8, 1900



David Hilbert (1862 - 1943)

Second International Congress of Mathematicians in Paris.

Twin primes,

Goldbach's Conjecture,

Riemann Hypothesis

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## The seven Millennium Problems

The Clay Mathematics Institute (CMI)  
Cambridge, Massachusetts <http://www.claymath.org>

7 million prize fund for the solution to these problems,  
with 1 million allocated to each.

Paris, May 24, 2000 :

Timothy Gowers, John Tate and Michael Atiyah.

- Birch and Swinnerton-Dyer Conjecture
- Hodge Conjecture
- Navier-Stokes Equations
- P vs NP
- Poincaré Conjecture
- Riemann Hypothesis
- Yang-Mills Theory

## Numbers

Numbers = real or complex numbers  $\mathbf{R}$ ,  $\mathbf{C}$ .

Natural integers :  $\mathbf{N} = \{0, 1, 2, \dots\}$ .

Rational integers :  $\mathbf{Z} = \{0, \pm 1, \pm 2, \dots\}$ .

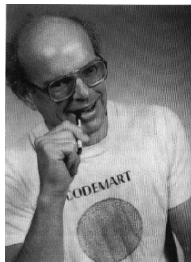
## Prime numbers

Numbers with exactly two divisors :

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, ...

The On-Line Encyclopedia of Integer Sequences

<http://oeis.org/A000040>



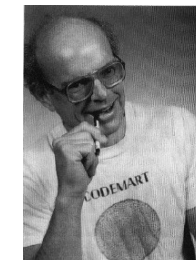
## Composite numbers

Numbers with more than two divisors :

4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20, 21, 22, 24, 25, 26, 27, ...

<http://oeis.org/A002808>

The composite numbers : numbers  $n$  of the form  $x \cdot y$  for  
 $x > 1$  and  $y > 1$ .



## Euclid of Alexandria

(about 325 BC – about 265 BC)



Given any collection  $p_1, \dots, p_n$  of primes, there is one prime which is not in this collection.

## Twin primes

**Conjecture** : there are infinitely many primes  $p$  such that  $p + 2$  is prime.

Examples : 3, 5, 5, 7, 11, 13, 17, 19...

**More generally** : is every even integer (infinitely often) the difference of two primes? of two consecutive primes?

Largest known example of twin primes (August 2009) with 100 355 decimal digits :

$$65\,516\,468\,355 \cdot 2^{333333} \pm 1.$$

<http://oeis.org/A001097>

<http://primes.utm.edu/>

## Goldbach's Conjecture



Leonhard Euler  
(1707 – 1783)

Letter of Christian Goldbach  
(1690 – 1764) to Euler,  
1742 : any integer  $\geq 5$  is sum  
of at most 3 primes.

Euler : Equivalent to  
any even integer greater than  
2 can be expressed as the sum  
of two primes.

Proof :

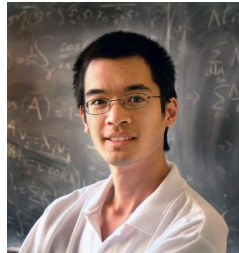
$$2n - 2 = p + p' \iff 2n = p + p' + 2 \iff 2n + 1 = p + p' + 3.$$

## Sums of two primes

$$\begin{array}{ll} 4 = 2 + 2 & 6 = 3 + 3 \\ 8 = 5 + 3 & 10 = 7 + 3 \\ 12 = 7 + 5 & 14 = 11 + 3 \\ 16 = 13 + 3 & 18 = 13 + 5 \\ 20 = 17 + 3 & 22 = 19 + 3 \\ 24 = 19 + 5 & 26 = 23 + 3 \\ & \vdots \\ & \vdots \end{array}$$

## Sums of primes

- 27 is neither prime nor a sum of two primes
- Weak (or ternary) Goldbach Conjecture : every odd integer  $\geq 7$  is the sum of three odd primes.
- Terence Tao, February 4, 2012, arXiv:1201.6656 : Every odd number greater than 1 is the sum of at most five primes.



## Circle method



Srinivasa Ramanujan  
(1887 – 1920)



G.H. Hardy  
(1877 – 1947)



J.E. Littlewood  
(1885 – 1977)

Hardy, ICM Stockholm, 1916  
Hardy and Ramanujan (1918) : partitions  
Hardy and Littlewood (1920 – 1928) :  
Some problems in Partitio Numerorum

## Circle method

Hardy and Littlewood



Ivan Matveevich Vinogradov  
(1891 – 1983)



Every sufficiently large odd integer is the sum of at most three primes.

## Largest explicitly known prime numbers

August 23, 2008 12 978 189 decimal digits

$$2^{43\,112\,609} - 1$$

June 13, 2009 12 837 064 decimal digits

$$2^{42\,643\,801} - 1$$

September 6, 2008 11 185 272 decimal digits

$$2^{37\,156\,667} - 1$$

## Large prime numbers

The nine largest explicitly known prime numbers are of the form  $2^p - 1$ .

One knows

- 43 prime numbers with more than 1 000 000 decimal digits
- 210 prime numbers with more than 500 000 decimal digits

List of the 5 000 largest explicitly known prime numbers :

<http://primes.utm.edu/largest.html>

47 prime numbers of the form of the form  $2^p - 1$  are known

<http://www.mersenne.org/>

## Mersenne

Marin Mersenne



1588 – 1648

## Mersenne prime numbers

If a number of the form  $2^k - 1$  is prime, then  $k$  itself is prime.

A prime number of the form  $2^p - 1$  is called a Mersenne prime.

47 of them are known, among them the 9 largest are also the largest explicitly known primes.

The smallest Mersenne primes are

$$3 = 2^2 - 1, \quad 7 = 2^3 - 1, \quad 31 = 2^5 - 1, \quad 127 = 2^7 - 1.$$

Are there infinitely many Mersenne primes?

## Mersenne prime numbers

In 1536, Hudalricus Regius noticed that  $2^{11} - 1 = 2047$  is not a prime number :  $2047 = 23 \cdot 89$ .

In the preface of *Cogitata Physica-Mathematica* (1644), Mersenne claimed that the numbers  $2^n - 1$  are prime for

$$n = 2, 3, 5, 7, 13, 17, 19, 31, 67, 127 \quad \text{and} \quad 257$$

and that they are composite for all other values of  $n < 257$ .

The correct list is

$$2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107 \quad \text{and} \quad 127.$$

<http://oeis.org/A000043>

## Perfect numbers

A number is **perfect** if it is equal to the sum of its divisors, excluding itself.  
For instance 6 is the sum  $1 + 2 + 3$ , and the divisors of 6 are 1, 2, 3 and 6.  
In the same way, the divisors of 28 are 1, 2, 4, 7, 14 and 28.  
The sum  $1 + 2 + 4 + 7 + 14$  is 28, hence 28 is perfect.

Notice that  $6 = 2 \cdot 3$  and 3 is a Mersenne prime  $2^2 - 1$ .  
Also  $28 = 4 \cdot 7$  and 7 is a Mersenne prime  $2^3 - 1$ .

Other perfect numbers :

$$496 = 16 \cdot 31 \quad \text{with} \quad 16 = 2^4, \quad 31 = 2^5 - 1,$$
$$8128 = 64 \cdot 127 \quad \text{and} \quad 64 = 2^6, \quad 127 = 2^7 - 1 \dots$$

## Perfect numbers

Euclid, Elements, Book IX : numbers of the form  $2^{p-1} \cdot (2^p - 1)$  with  $2^p - 1$  a (Mersenne) prime (hence  $p$  is prime) are perfect.

Euler : all even perfect numbers are of this form.

Sequence of perfect numbers : 6, 28, 496, 8128, 33550336, ...  
<http://oeis.org/A000396>

Are there infinitely even perfect numbers ?

Does there exist odd perfect numbers ?

## Fermat numbers

Fermat numbers are the numbers  $F_n = 2^{2^n} + 1$ .



Pierre de Fermat (1601 – 1665)

## Fermat primes

$F_0 = 3, F_1 = 5, F_2 = 17, F_3 = 257, F_4 = 65537$  are prime  
<http://oeis.org/A000215>

They are related with the construction of regular polygons with ruler and compass.

Fermat suggested in 1650 that all  $F_n$  are prime

Euler :  $F_5 = 2^{32} + 1$  is divisible by 641

$$4294967297 = 641 \cdot 6700417$$

$$641 = 5^4 + 2^4 = 5 \cdot 2^7 + 1$$

Are there infinitely many Fermat primes ? Only five are known.

## Leonhard Euler (1707 – 1783)



For  $s > 1$ ,

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s} = \prod_p (1 - p^{-s})^{-1}.$$

For  $s = 1$ :

$$\sum_p \frac{1}{p} = +\infty.$$

## Johann Carl Friedrich Gauss (1777 – 1855)

Let  $p_n$  be the  $n$ -th prime.

Gauss introduces

$$\pi(x) = \sum_{p \leq x} 1$$

He observes numerically

$$\pi(t + dt) - \pi(t) \sim \frac{dt}{\log t}$$

Define the density  $d\pi$  by

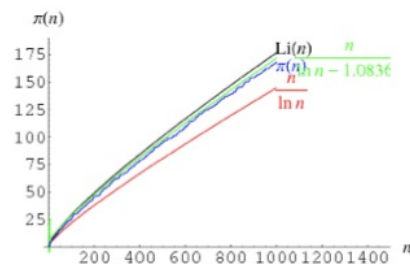
$$\pi(x) = \int_2^x d\pi(t).$$

Problem : estimate from above

$$E(x) = \left| \pi(x) - \int_2^x \frac{dt}{\log t} \right|.$$



## Plot



## Lejeune Dirichlet (1805 – 1859)

1837 :

For  $\gcd(a, q) = 1$ ,

$$\sum_{p \equiv a \pmod{q}} \frac{1}{p} = +\infty.$$



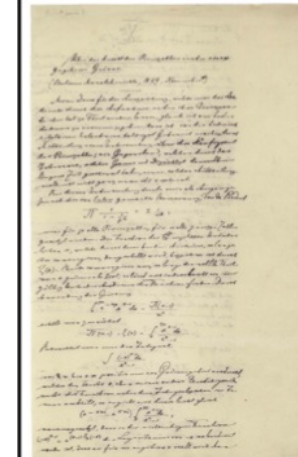
# Pafnuty Lvovich Chebyshev (1821 – 1894)



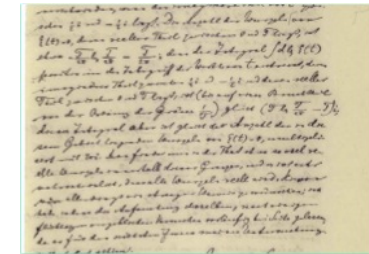
1851 :

$$0.8 \frac{x}{\log x} \leq \pi(x) \leq 1.2 \frac{x}{\log x}$$

# Riemann 1859



$\zeta(s) = 0$   
with  $0 < \Re(s) < 1$   
implies  
 $\Re(s) = 1/2$ .



## Riemann Hypothesis

*Certainly one would wish for a stricter proof here ; I have meanwhile temporarily put aside the search for this after some fleeting futile attempts, as it appears unnecessary for the next objective of my investigation.*

Über die Anzahl der Primzahlen unter einer gegebenen Grösse. (Monatsberichte der Berliner Akademie, November 1859)

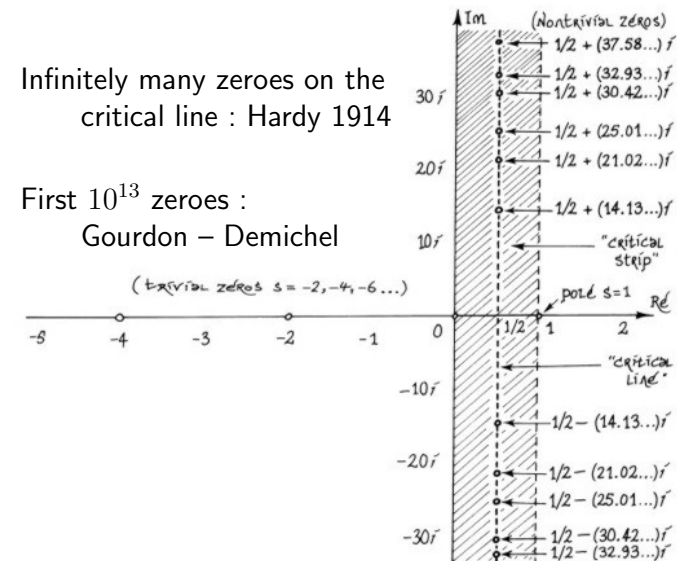
Bernhard Riemann's *Gesammelte Mathematische Werke und Wissenschaftlicher Nachlass*, herausgegeben unter Mitwirkung von Richard Dedekind, von Heinrich Weber. (Leipzig : B. G. Teubner 1892). 145–153.

<http://www.maths.tcd.ie/pub/HistMath/People/Riemann/Zeta/>

## Small Zeros Zeta

Infinitely many zeroes on the critical line : Hardy 1914

First  $10^{13}$  zeroes : Gourdon – Demichel



## Riemann Hypothesis

Riemann Hypothesis is equivalent to

$$E(x) \leq Cx^{1/2} \log x$$

for the remainder

$$E(x) = \left| \pi(x) - \int_2^x \frac{dt}{\log t} \right|.$$

Let  $\text{Even}(N)$  (resp.  $\text{Odd}(N)$ ) denote the number of positive integers  $\leq N$  with an even (resp. odd) number of prime factors, counting multiplicities. Riemann Hypothesis is also equivalent to

$$|\text{Even}(N) - \text{Odd}(N)| \leq CN^{1/2}.$$

## Prime Number Theorem : $\pi(x) \simeq x / \log x$

Jacques Hadamard  
(1865 – 1963)

Charles de la Vallée Poussin  
(1866 – 1962)



1896 :  $\zeta(1 + it) \neq 0$  for  $t \in \mathbf{R} \setminus \{0\}$ .

## Numbers : rational, irrational

**Rational numbers :**

$a/b$  with  $a$  and  $b$  rational integers,  $b > 0$ .

**Irreducible representation :**

$p/q$  with  $p$  and  $q$  in  $\mathbf{Z}$ ,  $q > 0$  and  $\text{gcd}(p, q) = 1$ .

**Irrational number :** a real (or complex) number which is not rational.

## Numbers : algebraic, transcendental

**Algebraic number :** a complex number which is root of a non-zero polynomial with rational coefficients.

**Examples :**

rational numbers :  $a/b$ , root of  $bX - a$ .

$\sqrt{2}$ , root of  $X^2 - 2$ .

$i$ , root of  $X^2 + 1$ .

The sum and the product of algebraic numbers are algebraic numbers. The set of complex algebraic numbers is a field, the algebraic closure of  $\mathbf{Q}$  in  $\mathbf{C}$ .

A **transcendental number** is a complex number which is not algebraic.

# Inverse Galois Problem

A number field is a finite extension of  $\mathbb{Q}$ .

Is any finite group  $G$  the Galois group of a number field over  $\mathbb{Q}$ ?

Equivalently :

The absolute Galois group of the field  $\mathbb{Q}$  is the group  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$  of automorphisms of the field  $\overline{\mathbb{Q}}$  of algebraic numbers. The previous question amounts to decide whether any finite group  $G$  is a quotient of  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ .



Evariste Galois (1811 – 1832)

# First decimals of $\sqrt{2}$

<http://wims.unice.fr/wims/wims.cgi>

1.41421356237309504880168872420969807856967187537694807317667973  
799073247846210703885038753432764157273501384623091229702492483  
605585073721264412149709993583141322266592750559275579995050115  
278206057147010955997160597027453459686201472851741864088919860  
955232923048430871432145083976260362799525140798968725339654633  
180882964062061525835239505474575028775996172983557522033753185  
701135437460340849884716038689997069900481503054402779031645424  
782306849293691862158057846311159666871301301561856898723723528  
850926486124949771542183342042856860601468247207714358548741556  
570696776537202264854470158588016207584749226572260020855844665  
214583988939443709265918003113882464681570826301005948587040031  
864803421948972782906410450726368813137398552561173220402450912  
277002269411275736272804957381089675040183698683684507257993647  
290607629969413804756548237289971803268024744206292691248590521  
810044598421505911202494413417285314781058036033710773091828693  
1471017111168391658172688941975871658215212822951848847 ...

# First binary digits of $\sqrt{2}$

<http://wims.unice.fr/wims/wims.cgi>

1.01101010000010011110011001100111111001110111100110010010000  
10001011001011111011000100110110011011101010100101010111110100  
11111000111010110111101100000101110101000100100111011101010000  
10011001110110100010111101011001000010110000011001100111001100  
10001010101001010111111001000001100000100001110101011100010100  
010110000111010100010110001111111001101111101110010000011110  
1101100111001000011110111010010101000010111001000011100111000  
111101101001010011110000000100100001110011011000111101111101  
00010011101101000110100100010000000101110100001110100001010101  
11100011111010011100101001100000101100111000110000000010001101  
11100001100110111101111001010101100011011110010010001000101101  
00010000100010110001010010001100000101010111100011100100010111  
10111110001001110001100111100011011010101101010001010001110001  
0111011011111010011101110011001011001010100110001101000011001  
10001111100111100100001001101111101010010111100010010000011111  
00000110110111001011000001011101110101010100100101000001000100  
110010000010000001100101001001010100000010011100101001010 ...

# Computation of decimals of $\sqrt{2}$

1 542 decimals computed by hand by Horace Uhler in 1951

14 000 decimals computed in 1967

1 000 000 decimals in 1971

$137 \cdot 10^9$  decimals computed by Yasumasa Kanada and Daisuke Takahashi in 1997 with Hitachi SR2201 in 7 hours and 31 minutes.

- Motivation : computation of  $\pi$ .

## Émile Borel (1871–1956)

- *Les probabilités dénombrables et leurs applications arithmétiques*,  
Palermo Rend. **27**, 247-271 (1909).  
Jahrbuch Database JFM 40.0283.01  
<http://www.emis.de/MATH/JFM/JFM.html>
- *Sur les chiffres décimaux de  $\sqrt{2}$  et divers problèmes de probabilités en chaînes*,  
C. R. Acad. Sci., Paris **230**, 591-593 (1950).  
Zbl 0035.08302

## Émile Borel : 1950



Let  $g \geq 2$  be an integer and  $x$  a real irrational algebraic number. *The expansion in base  $g$  of  $x$  should satisfy some of the laws which are valid for almost all real numbers (for Lebesgue's measure).*

## Conjecture of Émile Borel

**Conjecture** (É. Borel). *Let  $x$  be an irrational algebraic real number,  $g \geq 3$  a positive integer and  $a$  an integer in the range  $0 \leq a \leq g - 1$ . Then the digit  $a$  occurs at least once in the  $g$ -ary expansion of  $x$ .*

**Corollary.** *Each given sequence of digits should occur infinitely often in the  $g$ -ary expansion of any real irrational algebraic number.*

(consider powers of  $g$ ).

- An irrational number with a *regular* expansion in some base  $g$  should be transcendental.

## The state of the art

There is no explicitly known example of a triple  $(g, a, x)$ , where  $g \geq 3$  is an integer,  $a$  a digit in  $\{0, \dots, g - 1\}$  and  $x$  an algebraic irrational number, for which one can claim that the digit  $a$  occurs infinitely often in the  $g$ -ary expansion of  $x$ .

A stronger conjecture, also due to Borel, is that algebraic irrational real numbers are *normal*: each sequence of  $n$  digits in basis  $g$  should occur with the frequency  $1/g^n$ , for all  $g$  and all  $n$ .

## Complexity of the expansion in basis $b$ of a real irrational algebraic number



**Theorem** (B. Adamczewski, Y. Bugeaud 2005; conjecture of A. Cobham 1968).

If the sequence of digits of a real number  $x$  is produced by a finite automaton, then  $x$  is either rational or else transcendental.

## Srinivasa Ramanujan

Some transcendental aspects of Ramanujan's work.

*Proceedings of the Ramanujan Centennial International Conference* (Annamalainagar, 1987), RMS Publ., **1**, Ramanujan Math. Soc., Annamalainagar, 1988, 67–76.



## Open problems (irrationality)

- Is the number

$$e + \pi = 5.859\ 874\ 482\ 048\ 838\ 473\ 822\ 930\ 854\ 632 \dots$$

irrational?

- Is the number

$$e\pi = 8.539\ 734\ 222\ 673\ 567\ 065\ 463\ 550\ 869\ 546 \dots$$

irrational?

- Is the number

$$\log \pi = 1.144\ 729\ 885\ 849\ 400\ 174\ 143\ 427\ 351\ 353 \dots$$

irrational?

## Catalan's constant

Is Catalan's constant

$$\sum_{n \geq 1} \frac{(-1)^n}{(2n+1)^2} = 0.915\ 965\ 594\ 177\ 219\ 015\ 0 \dots$$

an irrational number?



## Riemann zeta function

The function

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}$$

was studied by Euler (1707– 1783)

for integer values of  $s$

and by Riemann (1859) for complex values of  $s$ .



**Euler** : for any even integer value of  $s \geq 2$ , the number  $\zeta(s)$  is a rational multiple of  $\pi^s$ .

**Examples** :  $\zeta(2) = \pi^2/6$ ,  $\zeta(4) = \pi^4/90$ ,  $\zeta(6) = \pi^6/945$ ,  
 $\zeta(8) = \pi^8/9450 \dots$

**Coefficients** : Bernoulli numbers.

## Introductio in analysin infinitorum

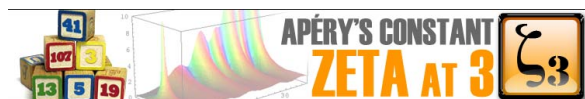


Leonhard Euler

(1707 – 1783)

Introductio in analysin infinitorum  
(1748)

## Riemann zeta function



The number

$$\zeta(3) = \sum_{n \geq 1} \frac{1}{n^3} = 1,202\,056\,903\,159\,594\,285\,399\,738\,161\,511 \dots$$

is irrational (Apéry 1978).

Recall that  $\zeta(s)/\pi^s$  is rational for any even value of  $s \geq 2$ .

**Open question** : Is the number  $\zeta(3)/\pi^3$  irrational ?

## Riemann zeta function

Is the number

$$\zeta(5) = \sum_{n \geq 1} \frac{1}{n^5} = 1.036\,927\,755\,143\,369\,926\,331\,365\,486\,457 \dots$$

irrational ?

**T. Rivoal** (2000) : infinitely many  $\zeta(2n + 1)$  are irrational.



## Euler–Mascheroni constant



Euler's Constant is

Lorenzo Mascheroni  
(1750 – 1800)

$$\gamma = \lim_{n \rightarrow \infty} \left( 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n \right)$$

$$= 0.577\ 215\ 664\ 901\ 532\ 860\ 606\ 512\ 090\ 082 \dots$$

Is it a rational number?

$$\gamma = \sum_{k=1}^{\infty} \left( \frac{1}{k} - \log \left( 1 + \frac{1}{k} \right) \right) = \int_1^{\infty} \left( \frac{1}{[x]} - \frac{1}{x} \right) dx$$

$$= - \int_0^1 \int_0^1 \frac{(1-x)dxdy}{(1-xy)\log(xy)}.$$

## Periods : Maxime Kontsevich and Don Zagier



Periods,  
*Mathematics  
unlimited—2001  
and beyond*,  
Springer 2001,  
771–808.



A *period* is a complex number whose real and imaginary parts are values of absolutely convergent integrals of rational functions with rational coefficients, over domains in  $\mathbf{R}^n$  given by polynomial inequalities with rational coefficients.

## The number $\pi$

Basic example of a *period* :

$$e^{z+2i\pi} = e^z$$

$$2i\pi = \int_{|z|=1} \frac{dz}{z}$$

$$\pi = \int \int_{x^2+y^2 \leq 1} dxdy = 2 \int_{-1}^1 \sqrt{1-x^2} dx$$

$$= \int_{-1}^1 \frac{dx}{\sqrt{1-x^2}} = \int_{-\infty}^{\infty} \frac{dx}{1-x^2}.$$

## Further examples of periods

$$\sqrt{2} = \int_{2x^2 \leq 1} dx$$

and all algebraic numbers.

$$\log 2 = \int_{1 < x < 2} \frac{dx}{x}$$

and all logarithms of algebraic numbers.

$$\pi = \int_{x^2+y^2 \leq 1} dxdy,$$

A product of periods is a period (subalgebra of  $\mathbf{C}$ ), but  $1/\pi$  is expected not to be a period.

## Relations among periods

- 1 Additivity  
(in the integrand and in the domain of integration)

$$\int_a^b (f(x) + g(x)) dx = \int_a^b f(x) dx + \int_a^b g(x) dx,$$

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx.$$

- 2 Change of variables :  
if  $y = f(x)$  is an invertible change of variables, then

$$\int_{f(a)}^{f(b)} F(y) dy = \int_a^b F(f(x)) f'(x) dx.$$

## Relations among periods (continued)



- 3 Newton–Leibniz–Stokes Formula

$$\int_a^b f'(x) dx = f(b) - f(a).$$

## Conjecture of Kontsevich and Zagier



A widely-held belief, based on a judicious combination of experience, analogy, and wishful thinking, is the following



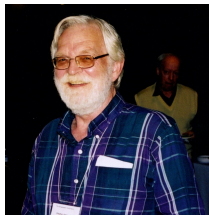
**Conjecture (Kontsevich–Zagier).** *If a period has two integral representations, then one can pass from one formula to another by using only rules 1, 2, 3 in which all functions and domains of integration are algebraic with algebraic coefficients.*

## Conjecture of Kontsevich and Zagier (continued)

*In other words, we do not expect any miraculous coincidence of two integrals of algebraic functions which will not be possible to prove using three simple rules.*

*This conjecture, which is similar in spirit to the Hodge conjecture, is one of the central conjectures about algebraic independence and transcendental numbers, and is related to many of the results and ideas of modern arithmetic algebraic geometry and the theory of motives.*

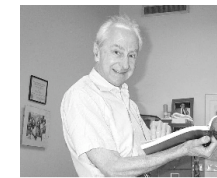
## Conjectures by S. Schanuel, A. Grothendieck and Y. André



- **Schanuel** : if  $x_1, \dots, x_n$  are  $\mathbb{Q}$ -linearly independent complex numbers, then  $n$  at least of the  $2n$  numbers  $x_1, \dots, x_n, e^{x_1}, \dots, e^{x_n}$  are algebraically independent.
- **Periods conjecture by Grothendieck** : Dimension of the Mumford–Tate group of a smooth projective variety.
- **Y. André** : generalization to motives.

## S. Ramanujan, C.L. Siegel, S. Lang, K. Ramachandra

Ramanujan : Highly composite numbers.  
 Alaoglu and Erdős (1944), Siegel,  
 Schneider, Lang, Ramachandra



## Four exponentials conjecture

Let  $t$  be a positive real number. Assume  $2^t$  and  $3^t$  are both integers. Prove that  $t$  is an integer.

Equivalently :

If  $n$  is a positive integer such that

$$n^{(\log 3)/\log 2}$$

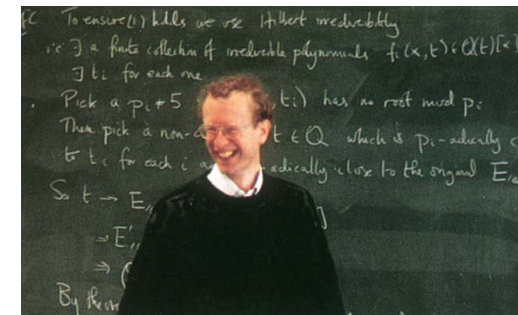
is an integer, then  $n$  is a power of 2 :

$$2^{k(\log 3)/\log 2} = 3^k.$$

## Fermat's Last Theorem $x^n + y^n = z^n$

Pierre de Fermat  
 1601 – 1665

Andrew Wiles  
 1953 –



Solution in 1994

## Diophantine problems

S.Sivasankaranarayana Pillai (1901–1950)



Collected works of S. S. Pillai,  
ed. R. Balasubramanian and R. Thangadurai, 2010.  
[http://www.geocities.com/thangadurai\\_kr/PILLAI.html](http://www.geocities.com/thangadurai_kr/PILLAI.html)

## Square, cubes...

- A **perfect power** is an integer of the form  $a^b$  where  $a \geq 1$  and  $b > 1$  are positive integers.

- **Squares** :

1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196.....

- **Cubes** :

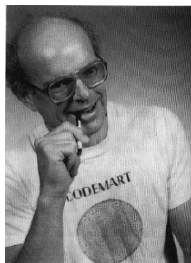
1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, 1331...

- **Fifth powers** :

1, 32, 243, 1024, 3125, 7776, 16807, 32768...

## Perfect powers

1, 4, 8, 9, 16, 25, 27, 32, 36, 49, 64, 81, 100, 121, 125,  
128, 144, 169, 196, 216, 225, 243, 256, 289, 324, 343,  
361, 400, 441, 484, 512, 529, 576, 625, 676, 729, 784...



Neil J. A. Sloane's encyclopaedia  
<http://oeis.org/A001597>

## Consecutive elements in the sequence of perfect powers

- Difference 1 : (8, 9)

- Difference 2 : (25, 27)

- Difference 3 : (1, 4), (125, 128)

- Difference 4 : (4, 8), (32, 36), (121, 125)

- Difference 5 : (4, 9), (27, 32)...



## Two conjectures



Subbayya Sivasankaranarayana Pillai  
(1901-1950)

Eugène Charles Catalan (1814 – 1894)

- **Catalan's Conjecture** : In the sequence of perfect powers, 8, 9 is the only example of consecutive integers.
- **Pillai's Conjecture** : In the sequence of perfect powers, the difference between two consecutive terms tends to infinity.

## Pillai's Conjecture :

- **Pillai's Conjecture** : In the sequence of perfect powers, the difference between two consecutive terms tends to infinity.

- **Alternatively** : Let  $k$  be a positive integer. The equation

$$x^p - y^q = k,$$

where the unknowns  $x, y, p$  and  $q$  take integer values, all  $\geq 2$ , has only finitely many solutions  $(x, y, p, q)$ .

## Pillai's conjecture

PILLAI, S. S. – *On the equation  $2^x - 3^y = 2^X + 3^Y$* , Bull. Calcutta Math. Soc. 37, (1945). 15–20.

*I take this opportunity to put in print a conjecture which I gave during the conference of the Indian Mathematical Society held at Aligarh.*

*Arrange all the powers of integers like squares, cubes etc. in increasing order as follows :*

1, 4, 8, 9, 16, 25, 27, 32, 36, 49, 64, 81, 100, 121, 125, 128, ...

*Let  $a_n$  be the  $n$ -th member of this series so that  $a_1 = 1$ ,  $a_2 = 4$ ,  $a_3 = 8$ ,  $a_4 = 9$ , etc. Then*

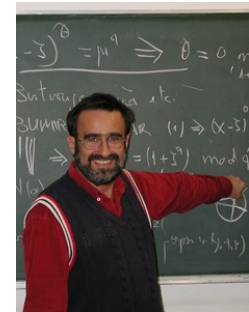
**Conjecture** :

$$\liminf(a_n - a_{n-1}) = \infty.$$

## Results

P. Mihăilescu, 2002.

Catalan was right : *the equation  $x^p - y^q = 1$  where the unknowns  $x, y, p$  and  $q$  take integer values, all  $\geq 2$ , has only one solution  $(x, y, p, q) = (3, 2, 2, 3)$ .*



Previous partial results : J.W.S. Cassels, R. Tijdeman, M. Mignotte...

## Higher values of $k$

There is no value of  $k > 1$  for which one knows that Pillai's equation  $x^p - y^q = k$  has only finitely many solutions.

Pillai's conjecture as a consequence of the  $abc$  conjecture :

$$|x^p - y^q| \geq c(\epsilon) \max\{x^p, y^q\}^{\kappa - \epsilon}$$

with

$$\kappa = 1 - \frac{1}{p} - \frac{1}{q}.$$

## The $abc$ Conjecture

- For a positive integer  $n$ , we denote by

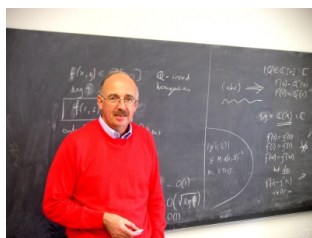
$$R(n) = \prod_{p|n} p$$

the *radical* or *square free part* of  $n$ .

- Conjecture** ( $abc$  Conjecture). For each  $\epsilon > 0$  there exists  $\kappa(\epsilon)$  such that, if  $a, b$  and  $c$  in  $\mathbf{Z}_{>0}$  are relatively prime and satisfy  $a + b = c$ , then

$$c < \kappa(\epsilon) R(abc)^{1+\epsilon}.$$

## The $abc$ Conjecture of Œsterlé and Masser



The  $abc$  Conjecture resulted from a discussion between D. W. Masser and J. Œsterlé in the mid 1980's.

## Beal Equation $x^p + y^q = z^r$

Assume

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$$

and  $x, y, z$  are relatively prime

Only 10 solutions (up to obvious symmetries) are known

$$1 + 2^3 = 3^2, \quad 2^5 + 7^2 = 3^4, \quad 7^3 + 13^2 = 2^9, \quad 2^7 + 17^3 = 71^2,$$

$$3^5 + 11^4 = 122^2, \quad 17^7 + 76271^3 = 21063928^2,$$

$$1414^3 + 2213459^2 = 65^7, \quad 9262^3 + 15312283^2 = 113^7,$$

$$43^8 + 96222^3 = 30042907^2, \quad 33^8 + 1549034^2 = 15613^3.$$

## Beal Conjecture and prize problem

“Fermat-Catalan” Conjecture H. (Darmon and A. Granville) :  
*the set of solutions to  $x^p + y^q = z^r$  with  $(1/p) + (1/q) + (1/r) < 1$  is finite.*

Consequence of the *abc* Conjecture. Hint :

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1 \quad \text{implies} \quad \frac{1}{p} + \frac{1}{q} + \frac{1}{r} \leq \frac{41}{42}.$$

R. Tijdeman, D. Zagier and A. Beal Conjecture : *there is no solution to  $x^p + y^q = z^r$  where each of  $p, q$  and  $r$  is  $\geq 3$ .*

R. D. MAULDIN, *A generalization of Fermat's last theorem : the Beal conjecture and prize problem*, Notices Amer. Math. Soc., 44 (1997), pp. 1436–1437.

## Waring's Problem

In 1770, a few months before J.L. Lagrange solved a conjecture of Bachet and Fermat by proving that every positive integer is the sum of at most four squares of integers, E. Waring wrote :



Edward Waring  
(1736 - 1798)

*“Every integer is a cube or the sum of two, three, . . . nine cubes; every integer is also the square of a square, or the sum of up to nineteen such; and so forth. Similar laws may be affirmed for the correspondingly defined numbers of quantities of any like degree.”*

## Collatz equation (Syracuse Problem)

Iterate

$$n \mapsto \begin{cases} n/2 & \text{if } n \text{ is even,} \\ 3n + 1 & \text{if } n \text{ is odd.} \end{cases}$$

Is  $(4, 2, 1)$  the only cycle ?

Example related to the *abc* conjecture :

$$109 \cdot 3^{10} + 2 = 23^5$$

Continued fraction of  $109^{1/5}$  :  $[2; 1, 1, 4, 77733, \dots]$ , approximation  $23/9$ .

N. A. Carella. *Note on the ABC Conjecture*

<http://arXiv.org/abs/math/0606221>

## Waring's function $g(k)$

- Waring's function  $g$  is defined as follows : *For any integer  $k \geq 2$ ,  $g(k)$  is the least positive integer  $s$  such that any positive integer  $N$  can be written  $x_1^k + \dots + x_s^k$ .*
- Conjecture (**The ideal Waring's Theorem**) : *For each integer  $k \geq 2$ ,*

$$g(k) = 2^k + [(3/2)^k] - 2.$$

- This is true for  $3 \leq k \leq 471\,600\,000$ , and (K. Mahler) also for all sufficiently large  $k$ .

## Waring's Problem and the *abc* Conjecture

S. David : the estimate

$$\left\| \left( \frac{3}{2} \right)^k \right\| \geq \left( \frac{3}{4} \right)^k,$$

(for sufficiently large  $k$ ) follows not only from Mahler's estimate, but also from the *abc* Conjecture!

Hence the ideal Waring Theorem  $g(k) = 2^k + [(3/2)^k] - 2$ . would follow from an explicit solution of the *abc* Conjecture.



## Waring's function $G(k)$

• Waring's function  $G$  is defined as follows : For any integer  $k \geq 2$ ,  $G(k)$  is the least positive integer  $s$  such that any sufficiently large positive integer  $N$  can be written  $x_1^k + \dots + x_s^k$ .

•  $G(k)$  is known only in two cases :  $G(2) = 4$  and  $G(4) = 16$

$$G(2) = 4$$

Joseph-Louis Lagrange  
(1736–1813)



Solution of a conjecture of Bachet and Fermat in 1770 :

Every positive integer is the sum of at most four squares of integers,

No integer congruent to  $-1$  modulo 8 can be a sum of three squares of integers.

$$G(k)$$

Kempner (1912)  $G(4) \geq 16$   
 $16^m \cdot 31$  need at least 16 biquadrates

Hardy Littlewood (1920)  $G(4) \leq 21$   
circle method, singular series

Davenport, Heilbronn, Esterman (1936)  $G(4) \leq 17$

Davenport (1939)  $G(4) = 16$

Yu. V. Linnik (1943)  $g(3) = 9$ ,  $G(3) \leq 7$

Other estimates for  $G(k)$ ,  $k \geq 5$  : Davenport, K. Sambasiva Rao, V. Narasimhamurti, K. Thanigasalam , R.C. Vaughan...

