

Centre for Advanced Studies in Mathematics (CASM)
Lahore University of Management Sciences (LUMS)

**Number theory:
Challenges of twenty first century**

Michel Waldschmidt

Institut de Mathématiques de Jussieu & Paris VI

<http://www.math.jussieu.fr/~miw/>

February 12, 2010

Abstract

Problems in number theory are sometimes easy to state and often very hard to solve. We survey some of them.

Extended abstract

We start with prime numbers. The twin prime conjecture, Goldbach Conjecture are among the main challenges. Are there infinitely many Mersenne (resp. Fermat) prime numbers? The largest known prime numbers are Mersenne numbers. Mersenne prime numbers are also related with perfect numbers, a problem considered by Euclid and still unsolved. One the main challenges for specialists of number theory is Riemann's hypothesis, which is now more than 150 years old.

Next we discuss open problems (initiated by E. Borel in 1905 and then in 1950) on the decimal (or binary) development of algebraic numbers. Almost nothing is known on this topic.

Kontsevich and Zagier introduced the notion of *periods* and suggested a far reaching statement which would solve a large number of open problems of irrationality and transcendence.

Diophantine equations conceal plenty of mysteries. Fermat's Last Theorem has been proved by A. Wiles, but many more questions are waiting for an answer. We discuss a conjecture due to S.S. Pillai, as well as the *abc*-Conjecture of Oesterlé–Masser.

Hilbert's 8th Problem

August 8, 1900



David Hilbert (1862 - 1943)

Second International Congress
of Mathematicians in Paris.

Twin primes,

Goldbach's Conjecture,

Riemann Hypothesis

The seven Millennium Problems

The Clay Mathematics Institute (CMI)

Cambridge, Massachusetts <http://www.claymath.org>

7 million prize fund for the solution to these problems,
with 1 million allocated to each.

Paris, May 24, 2000 :

Timothy Gowers, John Tate and Michael Atiyah.

- Birch and Swinnerton-Dyer Conjecture
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- Navier-Stokes Equations
- P vs NP
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Numbers

Numbers = real or complex numbers \mathbf{R} , \mathbf{C} .

Natural integers : $\mathbf{N} = \{0, 1, 2, \dots\}$.

Rational integers : $\mathbf{Z} = \{0, \pm 1, \pm 2, \dots\}$.

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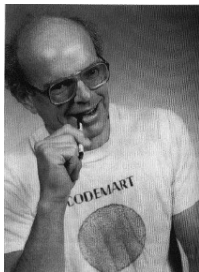
Prime numbers

Numbers with exactly two divisors :

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53,...

The On-Line Encyclopedia of Integer Sequences

<http://oeis.org/A000040>



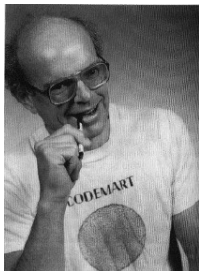
Composite numbers

Numbers with more than two divisors :

4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20, 21, 22, 24, 25, 26, 27...

<http://oeis.org/A002808>

The composite numbers : numbers n of the form $x \cdot y$ for $x > 1$ and $y > 1$.



Euclid of Alexandria

(about 325 BC – about 265 BC)



Given any collection p_1, \dots, p_n of primes, there is one prime which is not in this collection.

Twin primes

Conjecture : *there are infinitely many primes p such that $p + 2$ is prime.*

Examples : 3, 5, 5, 7, 11, 13, 17, 19...

More generally : *is every even integer (infinitely often) the difference of two primes ? of two consecutive primes ?*

Largest known example of twin primes (August 2009) with 100 355 decimal digits :

$$65\,516\,468\,355 \cdot 2^{333333} \pm 1.$$

<http://oeis.org/A001097>

<http://primes.utm.edu/>

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Goldbach's Conjecture



Leonhard Euler
(1707 – 1783)

Letter of Christian Goldbach
(1690 – 1764) to Euler,
1742 : any integer ≥ 5 is sum
of at most 3 primes.

Euler : Equivalent to
any even integer greater than
2 can be expressed as the sum
of two primes.

Proof :

$$2n - 2 = p + p' \iff 2n = p + p' + 2 \iff 2n + 1 = p + p' + 3.$$

Sums of two primes

$$4 = 2 + 2 \quad 6 = 3 + 3$$

$$8 = 5 + 3 \quad 10 = 7 + 3$$

$$12 = 7 + 5 \quad 14 = 11 + 3$$

$$16 = 13 + 3 \quad 18 = 13 + 5$$

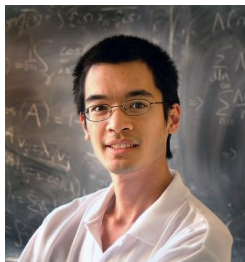
$$20 = 17 + 3 \quad 22 = 19 + 3$$

$$24 = 19 + 5 \quad 26 = 23 + 3$$

$$\vdots$$
$$\vdots$$

Sums of primes

- 27 is neither prime nor a sum of two primes
- Weak (or ternary) Goldbach Conjecture : every *odd integer ≥ 7* is the sum of three odd primes.
- Terence Tao, February 4, 2012, arXiv:1201.6656 :
Every odd number greater than 1 is the sum of at most five primes.



Circle method



Srinivasa Ramanujan
(1887 – 1920)



G.H. Hardy
(1877 – 1947)



J.E. Littlewood
(1885 – 1977)

Hardy, ICM Stockholm, 1916

Hardy and Ramanujan (1918) : partitions

Hardy and Littlewood (1920 – 1928) :

Some problems in Partitio Numerorum

Circle method

Hardy and Littlewood



Ivan Matveevich Vinogradov
(1891 – 1983)



Every sufficiently large odd integer is the sum of at most three primes.

Largest explicitly known prime numbers

August 23, 2008 12 978 189 decimal digits

$$2^{43\,112\,609} - 1$$

June 13, 2009 12 837 064 decimal digits

$$2^{42\,643\,801} - 1$$

September 6, 2008 11 185 272 decimal digits

$$2^{37\,156\,667} - 1$$

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Large prime numbers

The nine largest explicitly known prime numbers are of the form $2^p - 1$.

One knows

- 43 prime numbers with more than 1 000 000 decimal digits
- 210 prime numbers with more than 500 000 decimal digits

List of the 5 000 largest explicitly known prime numbers :

<http://primes.utm.edu/largest.html>

47 prime numbers of the form of the form $2^p - 1$ are known

<http://www.mersenne.org/>

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Mersenne

Marin Mersenne

1588 – 1648



Mersenne prime numbers

If a number of the form $2^k - 1$ is prime, then k itself is prime.

A prime number of the form $2^p - 1$ is called a Mersenne prime.

47 of them are known, among them the 9 largest are also the largest explicitly known primes.

The smallest Mersenne primes are

$$3 = 2^2 - 1, \quad 7 = 2^3 - 1, \quad 31 = 2^5 - 1, \quad 127 = 2^7 - 1.$$

Are there infinitely many Mersenne primes?

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In the preface of *Cogitata Physica-Mathematica* (1644), Mersenne claimed that the numbers $2^n - 1$ are prime for

$$n = 2, 3, 5, 7, 13, 17, 19, 31, 67, 127 \quad \text{and} \quad 257$$

and that they are composite for all other values of $n < 257$.

The correct list is

$$2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107 \quad \text{and} \quad 127.$$

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Perfect numbers

A number is **perfect** if it is equal to the sum of its divisors, excluding itself.

For instance **6** is the sum $1 + 2 + 3$, and the divisors of **6** are **1, 2, 3** and **6**.

In the same way, the divisors of **28** are **1, 2, 4, 7, 14** and **28**. The sum $1 + 2 + 4 + 7 + 14$ is **28**, hence **28** is perfect.

Notice that $6 = 2 \cdot 3$ and **3** is a Mersenne prime $2^2 - 1$.
Also $28 = 4 \cdot 7$ and **7** is a Mersenne prime $2^3 - 1$.

Other perfect numbers :

$$496 = 16 \cdot 31 \quad \text{with} \quad 16 = 2^4, \quad 31 = 2^5 - 1,$$

$$8128 = 64 \cdot 127 \quad \text{and} \quad 64 = 2^6, \quad 127 = 2^7 - 1 \dots$$

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Perfect numbers

Euclid, Elements, Book IX : numbers of the form $2^{p-1} \cdot (2^p - 1)$ with $2^p - 1$ a (Mersenne) prime (hence p is prime) are perfect.

Euler : all even perfect numbers are of this form.

Sequence of perfect numbers : 6, 28, 496, 8 128, 33 550 336,

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<http://oeis.org/A000396>

Are there infinitely even perfect number ?

Does there exist odd perfect numbers ?

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Fermat numbers

Fermat numbers are the numbers $F_n = 2^{2^n} + 1$.



Pierre de Fermat (1601 – 1665)

Fermat primes

$F_0 = 3$, $F_1 = 5$, $F_2 = 17$, $F_3 = 257$, $F_4 = 65537$ are prime

<http://oeis.org/A000215>

They are related with the construction of regular polygons with ruler and compass.

Fermat suggested in 1650 that all F_n are prime

Euler : $F_5 = 2^{32} + 1$ is divisible by 641

$$4294967297 = 641 \cdot 6700417$$

$$641 = 5^4 + 2^4 = 5 \cdot 2^7 + 1$$

Are there infinitely many Fermat primes? Only five are known.

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Leonhard Euler (1707 – 1783)



For $s > 1$,

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s} = \prod_p (1 - p^{-s})^{-1}.$$

For $s = 1$:

$$\sum_p \frac{1}{p} = +\infty.$$

Johann Carl Friedrich Gauss (1777 – 1855)

Let p_n be the n -th prime.



Gauss introduces

$$\pi(x) = \sum_{p \leq x} 1$$

He observes numerically

$$\pi(t + dt) - \pi(t) \sim \frac{dt}{\log t}$$

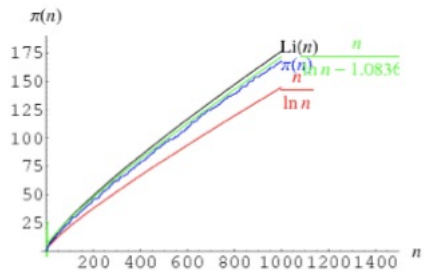
Define the density $d\pi$ by

$$\pi(x) = \int_2^x d\pi(t).$$

Problem : estimate from above

$$E(x) = \left| \pi(x) - \int_2^x \frac{dt}{\log t} \right|.$$

Plot



Lejeune Dirichlet (1805 – 1859)



1837 :

For $\gcd(a, q) = 1$,

$$\sum_{p \equiv a \pmod{q}} \frac{1}{p} = +\infty.$$

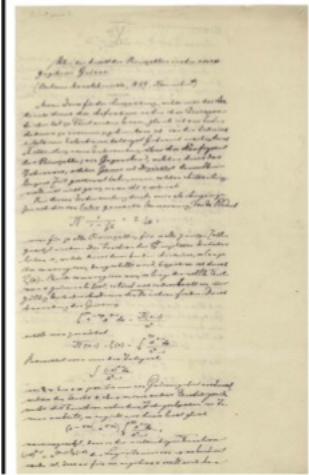
Pafnuty Lvovich Chebyshev (1821 – 1894)



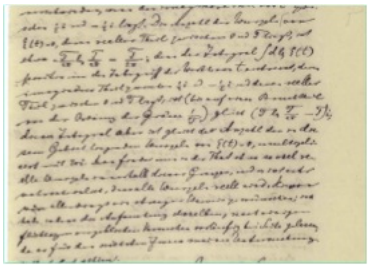
1851 :

$$0.8 \frac{x}{\log x} \leq \pi(x) \leq 1.2 \frac{x}{\log x}.$$

Riemann 1859



$\zeta(s) = 0$
 with $0 < \Re(s) < 1$
 implies
 $\Re(s) = 1/2$.



Riemann Hypothesis

Certainly one would wish for a stricter proof here; I have meanwhile temporarily put aside the search for this after some fleeting futile attempts, as it appears unnecessary for the next objective of my investigation.

Über die Anzahl der Primzahlen unter einer gegebenen Grösse.
(Monatsberichte der Berliner Akademie, November 1859)

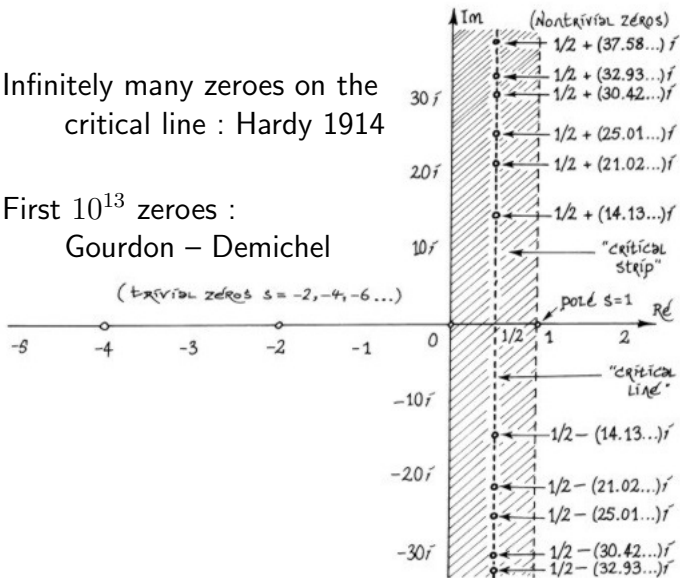
Bernhard Riemann's *Gesammelte Mathematische Werke und Wissenschaftlicher Nachlass*, herausgegeben unter Mitwirkung von Richard Dedekind, von Heinrich Weber. (Leipzig : B. G. Teubner 1892). 145–153.

<http://www.maths.tcd.ie/pub/HistMath/People/Riemann/Zeta/>

Small Zeros Zeta

Infinitely many zeroes on the critical line : Hardy 1914

First 10^{13} zeroes :
Gourdon – Demichel



Riemann Hypothesis

Riemann Hypothesis is equivalent to

$$E(x) \leq Cx^{1/2} \log x$$

for the remainder

$$E(x) = \left| \pi(x) - \int_2^x \frac{dt}{\log t} \right|.$$

Let $\text{Even}(N)$ (resp. $\text{Odd}(N)$) denote the number of positive integers $\leq N$ with an even (resp. odd) number of prime factors, counting multiplicities. Riemann Hypothesis is also equivalent to

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Prime Number Theorem : $\pi(x) \simeq x / \log x$

Jacques Hadamard
(1865 – 1963)



Charles de la Vallée Poussin
(1866 – 1962)



1896 : $\zeta(1 + it) \neq 0$ for $t \in \mathbf{R} \setminus \{0\}$.

Numbers : rational, irrational

Rational numbers :

a/b with a and b rational integers, $b > 0$.

Irreducible representation :

p/q with p and q in \mathbf{Z} , $q > 0$ and $\gcd(p, q) = 1$.

Irrational number : a real (or complex) number which is not rational.

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Numbers : algebraic, transcendental

Algebraic number : a complex number which is root of a non-zero polynomial with rational coefficients.

Examples :

rational numbers : a/b , root of $bX - a$.

$\sqrt{2}$, root of $X^2 - 2$.

i , root of $X^2 + 1$.

The sum and the product of algebraic numbers are algebraic numbers. The set of complex algebraic numbers is a field, the algebraic closure of \mathbb{Q} in \mathbb{C} .

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Inverse Galois Problem

A *number field* is a finite extension of \mathbb{Q} .

Is any finite group G the Galois group of a number field over \mathbb{Q} ?

Equivalently :

The *absolute Galois group of the field \mathbb{Q}* is the group $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ of automorphisms of the field $\overline{\mathbb{Q}}$ of algebraic numbers. The previous question amounts to decide whether any finite group G is a quotient of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.



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(1811 – 1832)

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First decimals of $\sqrt{2}$

<http://wims.unice.fr/wims/wims.cgi>

1.41421356237309504880168872420969807856967187537694807317667973
799073247846210703885038753432764157273501384623091229702492483
605585073721264412149709993583141322266592750559275579995050115
278206057147010955997160597027453459686201472851741864088919860
955232923048430871432145083976260362799525140798968725339654633
180882964062061525835239505474575028775996172983557522033753185
701135437460340849884716038689997069900481503054402779031645424
782306849293691862158057846311159666871301301561856898723723528
850926486124949771542183342042856860601468247207714358548741556
570696776537202264854470158588016207584749226572260020855844665
214583988939443709265918003113882464681570826301005948587040031
864803421948972782906410450726368813137398552561173220402450912
277002269411275736272804957381089675040183698683684507257993647
290607629969413804756548237289971803268024744206292691248590521
810044598421505911202494413417285314781058036033710773091828693
1471017111168391658172688941975871658215212822951848847 ...

First binary digits of $\sqrt{2}$

<http://wims.unice.fr/wims/wims.cgi>

1.011010100000100111100110011001111111001110111100110010010000
10001011001011111011000100110110011011101010100101010111110100
11111000111010110111101100000101110101000100100111011101010000
10011001110110100010111101011001000010110000011001100111001100
10001010101001010111111001000001100000100001110101011100010100
0101100001110101000101100011111110011011111101110010000011110
11011001110010000111101110100101010000101111001000011100111000
11110110100101001111000000001001000011100110110001111011111101
00010011101101000110100100010000000101110100001110100001010101
11100011111010011100101001100000101100111000110000000010001101
11100001100110111101111001010101100011011110010010001000101101
00010000100010110001010010001100000101010111100011100100010111
10111110001001110001100111100011011010101101010001010001110001
01110110111111010011101110011001011001010100110001101000011001
10001111100111100100001001101111101010010111100010010000011111
00000110110111001011000001011101110101010100100101000001000100
110010000010000001100101001001010100000010011100101001010 ...

Computation of decimals of $\sqrt{2}$

1 542 decimals computed by hand by Horace Uhler in 1951

14 000 decimals computed in 1967

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$137 \cdot 10^9$ decimals computed by Yasumasa Kanada and Daisuke Takahashi in 1997 with Hitachi SR2201 in 7 hours and 31 minutes.

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Émile Borel (1871–1956)

- *Les probabilités dénombrables et leurs applications arithmétiques,*

Palermo Rend. **27**, 247-271 (1909).

Jahrbuch Database

JFM 40.0283.01

<http://www.emis.de/MATH/JFM/JFM.html>

- *Sur les chiffres décimaux de $\sqrt{2}$ et divers problèmes de probabilités en chaînes,*

C. R. Acad. Sci., Paris **230**, 591-593 (1950).

Zbl 0035.08302

Émile Borel : 1950



Let $g \geq 2$ be an integer and x a real irrational algebraic number. *The expansion in base g of x should satisfy some of the laws which are valid for almost all real numbers (for Lebesgue's measure).*

Conjecture of Émile Borel

Conjecture (É. Borel). Let x be an irrational algebraic real number, $g \geq 3$ a positive integer and a an integer in the range $0 \leq a \leq g - 1$. Then the digit a occurs at least once in the g -ary expansion of x .

Corollary. Each given sequence of digits should occur infinitely often in the g -ary expansion of any real irrational algebraic number.

(consider powers of g).

- An irrational number with a *regular* expansion in some base g should be transcendental.

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The state of the art

There is no explicitly known example of a triple (g, a, x) , where $g \geq 3$ is an integer, a a digit in $\{0, \dots, g-1\}$ and x an algebraic irrational number, for which one can claim that the digit a occurs infinitely often in the g -ary expansion of x .

A stronger conjecture, also due to Borel, is that algebraic irrational real numbers are *normal*: each sequence of n digits in basis g should occur with the frequency $1/g^n$, for all g and all n .

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Complexity of the expansion in basis b of a real irrational algebraic number



Theorem (B. Adamczewski, Y. Bugeaud 2005 ; conjecture of A. Cobham 1968).

If the sequence of digits of a real number x is produced by a finite automaton, then x is either rational or else transcendental.

Srinivasa Ramanujan

Some transcendental aspects
of Ramanujan's work.

*Proceedings of the Ramanujan Centennial
International Conference*
(Annamalainagar, 1987),
RMS Publ., **1**, Ramanujan Math.
Soc., Annamalainagar, 1988, 67–76.



Open problems (irrationality)

- Is the number

$$e + \pi = 5.859\,874\,482\,048\,838\,473\,822\,930\,854\,632 \dots$$

irrational?

- Is the number

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Catalan's constant

Is Catalan's constant

$$\sum_{n \geq 1} \frac{(-1)^n}{(2n+1)^2}$$

$= 0.915\,965\,594\,177\,219\,015\,0\dots$

an irrational number?



Riemann zeta function

The function

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}$$

was studied by Euler (1707– 1783)

for integer values of s

and by Riemann (1859) for complex values of s .



Euler : for any even integer value of $s \geq 2$, the number $\zeta(s)$ is a rational multiple of π^s .

Examples : $\zeta(2) = \pi^2/6$, $\zeta(4) = \pi^4/90$, $\zeta(6) = \pi^6/945$,
 $\zeta(8) = \pi^8/9450 \dots$

Coefficients : Bernoulli numbers.

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Introductio in analysin infinitorum

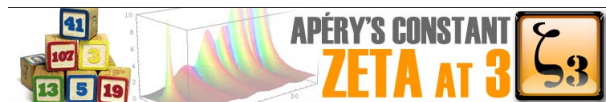


Leonhard Euler

(1707 – 1783)

Introductio in analysin infinitorum
(1748)

Riemann zeta function



The number

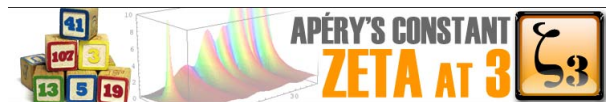
$$\zeta(3) = \sum_{n \geq 1} \frac{1}{n^3} = 1,202\,056\,903\,159\,594\,285\,399\,738\,161\,511 \dots$$

is irrational (*Apéry 1978*).

Recall that $\zeta(s)/\pi^s$ is rational for any even value of $s \geq 2$.

Open question : Is the number $\zeta(3)/\pi^3$ irrational ?

Riemann zeta function



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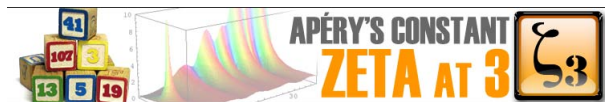
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T. Rivoal (2000) : infinitely many $\zeta(2n + 1)$ are irrational.

Riemann zeta function

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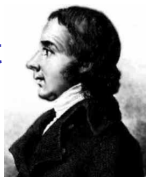
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Euler–Mascheroni constant



Lorenzo Mascheroni
(1750 – 1800)

Euler's Constant is

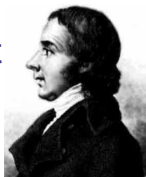
$$\begin{aligned}\gamma &= \lim_{n \rightarrow \infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n \right) \\ &= 0.577\,215\,664\,901\,532\,860\,606\,512\,090\,082\, \dots\end{aligned}$$

Is it a rational number?

$$\begin{aligned}\gamma &= \sum_{k=1}^{\infty} \left(\frac{1}{k} - \log \left(1 + \frac{1}{k} \right) \right) = \int_1^{\infty} \left(\frac{1}{[x]} - \frac{1}{x} \right) dx \\ &= - \int_0^1 \int_0^1 \frac{(1-x) dx dy}{(1-xy) \log(xy)}.\end{aligned}$$



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Lorenzo Mascheroni
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Periods : Maxime Kontsevich and Don Zagier



Periods,
*Mathematics
unlimited—2001
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Springer 2001,
771–808.



A *period* is a complex number whose real and imaginary parts are values of absolutely convergent integrals of rational functions with rational coefficients, over domains in \mathbf{R}^n given by polynomial inequalities with rational coefficients.

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The number π

Basic example of a *period* :

$$e^{z+2i\pi} = e^z$$

$$2i\pi = \int_{|z|=1} \frac{dz}{z}$$

$$\begin{aligned}\pi &= \iint_{x^2+y^2 \leq 1} dx dy = 2 \int_{-1}^1 \sqrt{1-x^2} dx \\ &= \int_{-1}^1 \frac{dx}{\sqrt{1-x^2}} = \int_{-\infty}^{\infty} \frac{dx}{1-x^2}.\end{aligned}$$

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$$\sqrt{2} = \int_{2x^2 \leq 1} dx$$

and all algebraic numbers.

$$\log 2 = \int_{1 < x < 2} \frac{dx}{x}$$

and all logarithms of algebraic numbers.

$$\pi = \int_{x^2 + y^2 \leq 1} dx dy,$$

A product of periods is a period (subalgebra of \mathbf{C}), but $1/\pi$ is expected not to be a period.

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Relations among periods

1 Additivity

(in the integrand and in the domain of integration)

$$\int_a^b (f(x) + g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx,$$

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Relations among periods (continued)



3 Newton–Leibniz–Stokes Formula

$$\int_a^b f'(x)dx = f(b) - f(a).$$

Conjecture of Kontsevich and Zagier



A widely-held belief, based on a judicious combination of experience, analogy, and wishful thinking, is the following



Conjecture (Kontsevich–Zagier). *If a period has two integral representations, then one can pass from one formula to another by using only rules $\boxed{1}$, $\boxed{2}$, $\boxed{3}$ in which all functions and domains of integration are algebraic with algebraic coefficients.*

Conjecture of Kontsevich and Zagier (continued)

In other words, we do not expect any miraculous coincidence of two integrals of algebraic functions which will not be possible to prove using three simple rules.

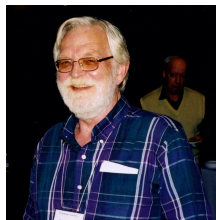
This conjecture, which is similar in spirit to the Hodge conjecture, is one of the central conjectures about algebraic independence and transcendental numbers, and is related to many of the results and ideas of modern arithmetic algebraic geometry and the theory of motives.

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Conjectures by S. Schanuel, A. Grothendieck and Y. André

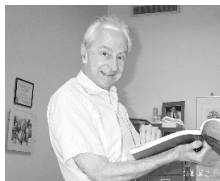


- **Schanuel** : if x_1, \dots, x_n are \mathbf{Q} -linearly independent complex numbers, then n at least of the $2n$ numbers $x_1, \dots, x_n, e^{x_1}, \dots, e^{x_n}$ are algebraically independent.
- **Periods conjecture by Grothendieck** : Dimension of the Mumford–Tate group of a smooth projective variety.
- **Y. André** : generalization to motives.

S. Ramanujan, C.L. Siegel, S. Lang, K. Ramachandra

Ramanujan : Highly composite numbers.

Alaoglu and Erdős (1944), Siegel,
Schneider, Lang, Ramachandra



Four exponentials conjecture

Let t be a positive real number. Assume 2^t and 3^t are both integers. Prove that t is an integer.

Equivalently :

If n is a positive integer such that

$$n^{(\log 3)/\log 2}$$

is an integer, then n is a power of 2 :

$$2^{k(\log 3)/\log 2} = 3^k.$$

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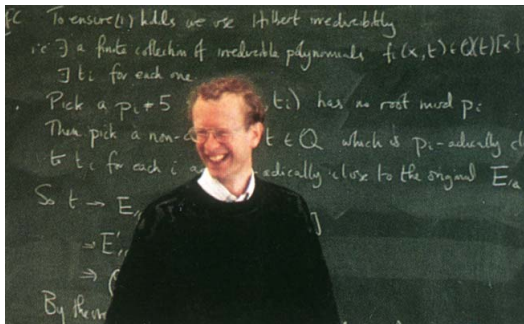
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Fermat's Last Theorem $x^n + y^n = z^n$

Pierre de Fermat
1601 – 1665



Andrew Wiles
1953 –



Solution in 1994

Diophantine problems

S.Sivasankaranarayana Pillai (1901–1950)



Collected works of S. S. Pillai,
ed. R. Balasubramanian and R. Thangadurai, 2010.
http://www.geocities.com/thangadurai_kr/PILLAI.html

Square, cubes...

- A perfect power is an integer of the form a^b where $a \geq 1$ and $b > 1$ are positive integers.

- Squares :

1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196.....

- Cubes :

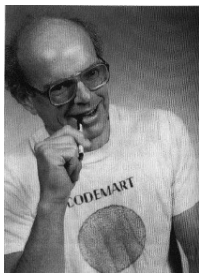
1, 8, 27, 64, 125, 216, 343, 512, 729, 1 000, 1 331...

- Fifth powers :

1, 32, 243, 1 024, 3 125, 7 776, 16 807, 32 768...

Perfect powers

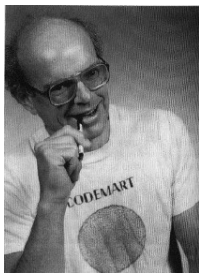
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361, 400, 441, 484, 512, 529, 576, 625, 676, 729, 784...



Neil J. A. Sloane's encyclopaedia
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Consecutive elements in the sequence of perfect powers

- Difference 1 : $(8, 9)$
- Difference 2 : $(25, 27)$
- Difference 3 : $(1, 4), (125, 128)$
- Difference 4 : $(4, 8), (32, 36), (121, 125)$
- Difference 5 : $(4, 9), (27, 32)\dots$

Two conjectures



Subbayya Sivasankaranarayana Pillai
(1901-1950)

Eugène Charles Catalan (1814 – 1894)

- **Catalan's Conjecture** : In the sequence of perfect powers, 8,9 is the only example of consecutive integers.
- **Pillai's Conjecture** : In the sequence of perfect powers, the difference between two consecutive terms tends to infinity.

Pillai's Conjecture :

- **Pillai's Conjecture** : In the sequence of perfect powers, the difference between two consecutive terms tends to infinity.
- **Alternatively** : Let k be a positive integer. The equation

$$x^p - y^q = k,$$

where the unknowns x , y , p and q take integer values, all ≥ 2 , has only finitely many solutions (x, y, p, q) .

Pillai's conjecture

PILLAI, S. S. – *On the equation $2^x - 3^y = 2^X + 3^Y$* , Bull. Calcutta Math. Soc. 37, (1945). 15–20.

I take this opportunity to put in print a conjecture which I gave during the conference of the Indian Mathematical Society held at Aligarh.

Arrange all the powers of integers like squares, cubes etc. in increasing order as follows :

1, 4, 8, 9, 16, 25, 27, 32, 36, 49, 64, 81, 100, 121, 125, 128, ...

Let a_n be the n -th member of this series so that $a_1 = 1$, $a_2 = 4$, $a_3 = 8$, $a_4 = 9$, etc. Then

Conjecture :

$$\liminf(a_n - a_{n-1}) = \infty.$$

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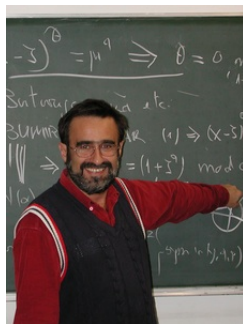
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Results

P. Mihăilescu, 2002.

Catalan was right : *the equation $x^p - y^q = 1$ where the unknowns x, y, p and q take integer values, all ≥ 2 , has only one solution $(x, y, p, q) = (3, 2, 2, 3)$.*



Previous partial results : J.W.S. Cassels, R. Tijdeman, M. Mignotte...

Higher values of k

There is no value of $k > 1$ for which one knows that Pillai's equation $x^p - y^q = k$ has only finitely many solutions.

Pillai's conjecture as a consequence of the *abc* conjecture :

$$|x^p - y^q| \geq c(\epsilon) \max\{x^p, y^q\}^{\kappa - \epsilon}$$

with

$$\kappa = 1 - \frac{1}{p} - \frac{1}{q}.$$

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The *abc* Conjecture

- For a positive integer n , we denote by

$$R(n) = \prod_{p|n} p$$

the *radical* or *square free part* of n .

- **Conjecture** (*abc* Conjecture). For each $\varepsilon > 0$ there exists $\kappa(\varepsilon)$ such that, if a , b and c in $\mathbf{Z}_{>0}$ are relatively prime and satisfy $a + b = c$, then

$$c < \kappa(\varepsilon)R(abc)^{1+\varepsilon}.$$

Beal Equation $x^p + y^q = z^r$

Assume

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$$

and x, y, z are relatively prime

Only 10 solutions (up to obvious symmetries) are known

$$1 + 2^3 = 3^2, \quad 2^5 + 7^2 = 3^4, \quad 7^3 + 13^2 = 2^9, \quad 2^7 + 17^3 = 71^2,$$

$$3^5 + 11^4 = 122^2, \quad 17^7 + 76271^3 = 21063928^2,$$

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Beal Conjecture and prize problem

“Fermat-Catalan” Conjecture H. (Darmon and A. Granville) :
the set of solutions to $x^p + y^q = z^r$ with
 $(1/p) + (1/q) + (1/r) < 1$ is finite.

Consequence of the *abc* Conjecture. Hint:

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1 \quad \text{implies} \quad \frac{1}{p} + \frac{1}{q} + \frac{1}{r} \leq \frac{41}{42}.$$

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Collatz equation (Syracuse Problem)

Iterate

$$n \mapsto \begin{cases} n/2 & \text{if } n \text{ is even,} \\ 3n + 1 & \text{if } n \text{ is odd.} \end{cases}$$

Is $(4, 2, 1)$ the only cycle?

Example related to the *abc* conjecture :

$$109 \cdot 3^{10} + 2 = 23^5$$

Continued fraction of $109^{1/5}$: $[2; 1, 1, 4, 77733, \dots]$,
approximation $23/9$.

N. A. Carella. *Note on the ABC Conjecture*

<http://arXiv.org/abs/math/0606221>

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Waring's Problem

In 1770, a few months before J.L. Lagrange solved a conjecture of Bachet and Fermat by proving that every positive integer is the sum of at most four squares of integers, E. Waring wrote :



Edward Waring
(1736 - 1798)

“Every integer is a cube or the sum of two, three, . . . nine cubes ; every integer is also the square of a square, or the sum of up to nineteen such ; and so forth. Similar laws may be affirmed for the correspondingly defined numbers of quantities of any like degree.”

Waring's function $g(k)$

- Waring's function g is defined as follows : *For any integer $k \geq 2$, $g(k)$ is the least positive integer s such that any positive integer N can be written $x_1^k + \cdots + x_s^k$.*
- Conjecture (The ideal Waring's Theorem) : *For each integer $k \geq 2$,*

$$g(k) = 2^k + \left\lceil \left(\frac{3}{2}\right)^k \right\rceil - 2.$$

- This is true for $3 \leq k \leq 471\,600\,000$, and (K. Mahler) also for all sufficiently large k .

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Waring's Problem and the *abc* Conjecture

S. David : the estimate

$$\left\| \left(\frac{3}{2} \right)^k \right\| \geq \left(\frac{3}{4} \right)^k ,$$

(for sufficiently large k) follows not only from Mahler's estimate, but also from the *abc* Conjecture !

Hence the ideal Waring Theorem $g(k) = 2^k + [(3/2)^k] - 2$. would follow from an explicit solution of the *abc* Conjecture.



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$$G(2) = 4$$

Joseph-Louis Lagrange
(1736–1813)



Solution of a conjecture of
Bachet and Fermat in 1770 :

Every positive integer is the
sum of at most four squares
of integers,

No integer congruent to -1 modulo 8 can be a sum of three
squares of integers.

$G(k)$

Kempner (1912) $G(4) \geq 16$
 $16^m \cdot 31$ need at least 16 biquadrates

Hardy Littlewood (1920) $G(4) \leq 21$
circle method, singular series

Davenport, Heilbronn, Esterman (1936) $G(4) \leq 17$

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Yu. V. Linnik (1943) $g(3) = 9$, $G(3) \leq 7$

Other estimates for $G(k)$, $k \geq 5$: Davenport, K. Sambasiva Rao, V. Narasimhamurti, K. Thanigasalam , R.C. Vaughan...

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- **Artin's Conjecture** (1927) : given an integer a which is not a square nor -1 , there are infinitely many p such that a is a primitive root modulo p .
(+ Conjectural asymptotic estimate for the density).
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