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Canonical perturbation theory via simultaneous approximation

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"D'ailleurs, ce qui nous rend ces solutions périodiques si précieuses, c'est qu'elles sont, pour ainsi dire, la seule brèche par où nous puissions essayer de pénétrer dans une place jusqu'ici réputée inabordable."

Henri Poincaré, Les méthodes nouvelles de la mécanique céleste (§36)

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CHAPTER I

INTRODUCTION

In this article, we wish to present a new method to deal with problems related to canonical (that is, symplectic or Hamiltonian) perturbation theory. The familiar model situation consists in the perturbation of an integrable Hamiltonian system; that is, one considers the system in phase space governed by

$$(1) \qquad H(p,q) = h(p) + \varepsilon f(p,q), \quad (p,q) \in \mathbb{R}^n \times \mathbb{T}^n, \quad \mathbb{T} = \mathbb{R}/\mathbb{Z}.$$

As usual, (p, q) denote action-angle variables of the integrable Hamiltonian h, and ε is a small parameter.

Our main result here will be a substantial improvement, both quantitative and qualitative, of Nekhoroshev's results ([43], [44]) about the stability of the action variables over exponentially long times, when the unperturbed Hamiltonian h is *quasi-convex*, by which we mean (following Nekhoroshev) that the energy surface $h(p) = E$ is strictly convex (for a certain range of the energy E). Of course, any convex function is a fortiori quasi-convex.

Under such a hypothesis, the following general estimate holds:

$$(2) \qquad \|p(t) - p(0)\| \leq \mathcal{R}(\varepsilon) \text{ when } |t| \leq \mathcal{T}(\varepsilon) \text{ and } |\varepsilon| \leq \varepsilon_0,$$

where $\mathcal{R}(\varepsilon) = O(\varepsilon^b)$ and $\mathcal{T}(\varepsilon)$ has the orderof $\exp(c/\varepsilon^a)$. We call $\mathcal{T}(\varepsilon)$ the *stability time*, $\mathcal{R}(\varepsilon)$ the *radius of confinement*, and $\varepsilon_0 > 0$ the *threshold of validity*. A gross but important evaluation of the size of $\mathcal{T}(\varepsilon)$ and $\mathcal{R}(\varepsilon)$ is provided by the numbers (a, b), $0 < a, b \leqslant 1$, which we call the *stability exponents*.

(2) was proved by Nekhoroshev ([43], [44]) under the assumption that H is *analytic* (this cannot be dispensed with) and h is *steep*, which is a weak, C^∞ generic condition. Here we shall have to work under the more stringent hypothesis that h is quasi-convex and we have to insist that this does not merely allow for a technical simplification; the proof method used below simply does not carry over to the steep case. It is interesting to note in this respect that recent works also point to the specificity of quasi-convex systems (we return to this in Chapter IV, §4). We also mention that Nekhoroshev's original proof has been rewritten precisely in the convex case (see [6] and [7]).

Below (see Chapters II and III), we shall improve on the known values of the exponents in the quasi-convex case. In particular, we show that $a > 1/(2n+1) - \eta$ for any $\eta > 0$, and believe that in general $a \leqslant 1/(2n)$. The latter assertion essentially means that over a longer timescale Arnol'd's diffusion may, and in fact generically will, switch on, so that stability results are effectively broken. This of course seems quite hard to prove and we shall content ourselves with a heuristic argument which points in that direction (see Chapter V, §2). Another striking qualitative phenomenon which we explore is the fact that over finite but exponentially long times, resonance stabilizes the motion. In fact, we prove a precise local version of (2) which implies that the stability time is increased when the initial condition is resonant.

These results are easy consequences of the proof itself, which is substantially different from the usual one, and less cumbersome. To appreciate the difference, it suffices to say that the usual ingredients of canonical perturbation theory, such as small divisors, ultraviolet cut-off, resonance surfaces, or even

Fourier series in general, simply do not enter the proof at all. This stems from a change of viewpoint, and we devote the end of this introduction to a more general discussion because, as suggested in the title, we believe that this method may have a wide range of applications, some of which are explored or suggested in Chapters IV and V.

Loosely speaking, the usual point of departure of canonical perturbation theory consists in the observation that the perturbed system could be reduced to an integrable one, using the apparatus of normal form theory, in a region of phase space which is free from resonances. Since no such *open* domain of phase space exists in general (leaving aside the linear or isochronous case, that is, the perturbation of harmonic oscillators), non-integrability is the rule rather than the exception, as was brought to light essentially by Poincaré, and the aim is to understand what results can be obtained in spite of the unavoidable existence of resonances. Of such nature for instance was Kolmogorov's remarkable insight about the existence of invariant tori.

Here, in some sense, things are turned inside out, as one tries to view resonances not as a hindrance but as an opportunity. To this end, one focuses first on the fully resonant situation, which is embodied in closed orbits. Let us be more specific; at the level of "classical" perturbation theory, which culminates in results of Nekhoroshev type, one should consider the objects (resonant surfaces, closed orbits, and so on) related to the *unperturbed* system. Referring to (1) and denoting the frequency vector by $\omega(p) = \nabla h(p) \in \mathbb{R}^n$, the closed orbits of the unperturbed system simply correspond to *rational* vectors ω, that is, vectors which are multiples of integer ones: if $\omega_0 = \omega(p_0)$ is rational, the torus $p = p_0$ is filled with closed orbits of the unperturbed system, with common period T such that $T\omega_0 \in \mathbb{Z}^n$.

In order to prove estimates such as (2), one first studies stability in the neighbourhood of these periodic tori. This is the object of Chapter II, and all the analysis it requires is *one* phase or time averaging. Then, given an arbitrary point in phase space, or rather in action space, it may be approximated by points corresponding to periodic tori. The rate of approximation and the growth of the corresponding periods are related, for a generic point, by the simplest approximation result, namely Dirichlet's theorem. This procedure will enable us (in Chapter III) to prove (2) and its local version, which depends on the properties of the initial conditions.

Returning to more general considerations, one realizes that the approach relies on a kind of duality (using the word with a non-technical meaning), which may be expressed in several ways. First, at the dynamical level, there is the relationship between time and phase averaging, which lies at the root of ergodic theory. For linear flows on tori, it can of course be made much more explicit, using in particular the notion of approximate recurrence times. This corresponds, in the theory of Diophantine approximation, to the "duality" between *linear* and *simultaneous* approximations. Given $\omega \in \mathbb{R}^n$, the former deals with the size of the linear forms $\omega \cdot k$ ($k \in \mathbb{Z}^n$; ordinary dot product), the latter with the approximation of the straight line $T\omega$ ($T \in \mathbb{R}$) by the integer lattice $\mathbb{Z}^n$; that is, one is interested in the distance $\mathrm{dist}(T\omega, \mathbb{Z}^n)$ as T varies, say along the positive semiaxis. In dynamical terms, the first describes the distribution of the small divisors, the latter that of the closed orbits. In some sense, both contain the same arithmetical information about ω, as asserted by *transfer principles* which originated in the work of A. Khinchin (the simplest and most useful ones are recalled in Appendix 1). Notice however that the information is encoded in a more compact way using simultaneous approximation: it is always "one-dimensional", whatever the dimension of the ambient space. Finally, transfer principles are a — not straightforward — reflection of the projective duality between a linear subspace and its orthogonal complement, and in fact, as far as linear and simultaneous approximations are concerned, between lines and hyperplanes.

As a final word in this introduction, we mention that Appendix 2 has been inserted in order to clarify the discussion in Chapter IV, §2. Also we have tried to keep the reference list to a reasonable length and have accordingly refrained from quoting some classical — and less classical — works, the references to which can be found, for example, in the bibliographies of several of the articles we refer to.

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CHAPTER II

STABILITY IN THE NEIGHBOURHOOD OF A PERIODIC TORUS

We shall be interested in Hamiltonians of the type

$$H(p,q) = h(p) + f(p,q). \quad (p,q) \in \mathbb{R}^n \times \mathbb{T}^n, \quad \mathbb{T} = \mathbb{R}/\mathbb{Z}. \tag{1}$$

$\omega(p) = \nabla h(p)$ denotes the frequency vector of the unperturbed system and we assume in this chapter that $\omega_0 = \omega(0)$ is *rational* of (minimal) period T, that is, $T\omega_0 \in \mathbb{Z}^n$. We use the notation $\Omega = \|\omega_0\|$ (Euclidean norm). h and f are assumed to be defined and *analytic* in a neighbourhood of the origin, more precisely on a complex domain $D = D(R, \rho, \sigma)$ ($\rho > 0$, $\sigma > 0$) defined as follows: let B_R be the real ball of radius R around the origin, then

$$D(R,\rho,\sigma) = \{(p,q) \in \mathbb{C}^{2n},\ \mathrm{dist}(p, B_R) \leq \rho,\ |\mathrm{Im}\, q| \leq \sigma\}, \tag{2}$$

where $|\mathrm{Im}\, q| = \sup_i |\mathrm{Im}\, q_i|$; h and f are supposed to be continuous on the boundary of D. The real part of D is of course nothing but $B_{R+\rho} \times \mathbb{T}^n$.

When $0 \leqslant \delta \leqslant \rho$, $0 \leqslant \xi \leqslant \sigma$, we denote by $D-(\delta, \xi)$ the domain $D(R, \rho-\delta, \sigma-\xi)$.

The norm $\|\cdot\|_D$ is simply the sup norm (L^∞) over D and we write $\|f\|_D = \varepsilon E$. Notice that we have not written the small parameter $\varepsilon \geqslant 0$ explicitly in (1) and we introduce the letter E essentially because in this way ε becomes a non-dimensional quantity and all the formulae we get will be dimensionally correct. The reader who does not find it useful may set $E = 1$ in the sequel; he may also set $\Omega = 1$, using a rescaling of the time variable, but again we find the formulae more suggestive this way. Finally, to define the size of the perturbation, one could also compare the norm of ∇f with Ω; we do not even assume, as one could, that f has zero mean with respect to q, for any p.

We consider the case when h is *convex*; the slight modifications needed when h is only assumed to be *quasi*-convex will be indicated at the end of this section. We denote the Hessian matrix by $A(p) = \nabla^2 h(p)$ and suppose that it is positive definite (if not, change t into $-t$ and H into $-H$), more precisely that m (respectively M) is a lower (respectively upper) bound of the spectrum of A over D. Explicitly:

$$\|A(p)v\| \leq M\|v\|, \quad (A(p)v, v) \geq m\|v\|^2, \quad \text{for any} \quad p \in D \cap \mathbb{R}^n \quad \text{and} \quad v \in \mathbb{R}^n,$$

where $0 < m \leqslant M$ and the dot denotes the usual scalar product.

We shall first prove an iterative lemma which consists in a simple one-phase averaging procedure and constitutes the only analytical result that will be needed in all this paper. From it there will easily follow three allied statements which describe the stability near periodic tori. To give a more precise idea of the type of results we have in mind, let us state Theorem 1B (see below) in a slightly informal way.

Let $H(p, q) = h(p)+f(p, q)$ be a perturbation of a convex integrable Hamiltonian such that $p = 0$ is a periodic torus of period T. Let $(p(t), q(t))$ denote the trajectory starting at $(p(0), q(0))$. Then, if $\|p(0)\| \leqslant r_0\varepsilon^{1/3}$, the estimate $\|p(t)\| \leqslant R_0\varepsilon^{1/3}$ holds when $\varepsilon \leqslant \varepsilon_0$ and $|t| \leqslant T(\varepsilon) = T_0 \exp((\tau/T)\varepsilon^{-1/3})$.

All the constants will be explicitly computed as simple functions of the parameters Ω, m, M, ρ, σ, E and T, the physical meaning of which is clear. The number n of degrees of freedom will not appear.

In order to state the iterative lemma, we need yet another simple notion. With a function $g(q)$ on the torus one associates its *time average* $\langle g\rangle$ along the orbits of the linear flow defined by ω_0:

$$\langle g\rangle(q) = \frac{1}{T}\int_0^T g(q+\omega_0 t)dt.$$

We shall say that g is *resonant* (with respect to ω_0) if $g = \langle g\rangle$, which simply means that g is constant along the orbits of the flow directed along ω_0.

Iterative lemma. *Let $H(p, q)$ be an analytic Hamiltonian on $D = D(R, \rho, \sigma)$ such that*

$$H(p,q) = h(p) + Z(p,q) + N(p,q), \tag{3}$$

where Z is resonant with respect to ω_0 (p comes in as a parameter) whereas $\langle N\rangle = 0$. Suppose that $\|Z+N\|_D \leqslant \varepsilon E$ and $\|N\|_D \leqslant \eta E$. Let ξ and δ satisfy

$$0 < \delta < \rho, \quad 0 < \xi < \sigma \quad \text{and} \quad \xi\delta \geq 2TE\eta; \tag{4}$$

then there exists a canonical transformation $C : D' \to D$ with $D' = D-(\delta, \xi)$ such that C is one-to-one and its image $C(D')$ satisfies

$$D - (\delta/2, \xi/2) \supset C(D') \supset D - (3\delta/2, 3\xi/2).$$

C is analytic and preserves reality, that is, $C(D' \cap \mathbb{R}^{2n}) \subset D \cap \mathbb{R}^{2n}$. Moreover if $C(p', q') = (p, q)$, one has the estimates $\|p'-p\| \leqslant \delta/2$, $\|q'-q\| \leqslant \xi/2$, and denoting $H' = H \circ C$, the function H' can be written in the form (3) *(using primed letters) with*

$$\varepsilon' \leq \varepsilon + \frac{1}{2}\eta', \tag{5$_1$}$$

$$\eta' \leq 5T\eta\left[M(R+\rho)\xi^{-1} + 4E\varepsilon(\xi\delta)^{-1}\right]. \tag{5$_2$}$$

To anticipate a bit, the idea is that by using the lemma one will progressively make the *non*-resonant part N as small as possible, keeping the size of the resonant term Z roughly constant. One then uses Hamilton's equation $\dot{p} = -\dfrac{\partial H}{\partial q}$ and the crucial fact that $\omega_0 \cdot \dfrac{\partial Z}{\partial q} = 0$ (here one may think of the "standard" case when $\omega_0 = (\nu, 0, ..., 0)$, with period $T = 1/\nu$). This implies that $\omega_0 \cdot \dot{p} = -\omega_0 \cdot \dfrac{\partial N}{\partial q}$, which will be very small. Thus, one has almost eliminated the possibility of a drift along the direction of ω_0. This will in turn guarantee stability, using a simple geometric argument (see (21) and the reasoning below).

Turning to the proof of the iterative lemma, C is built with the help of a Lie series, that is, as the time 1 map of an auxiliary Hamiltonian $\chi(p, q)$. We shall need a lemma in order to estimate gradients and Poisson brackets, which represents a slightly elaborate use of the Cauchy formula. Below, $\partial f/\partial p$ and $\partial f/\partial q$ (n-vectors) denote of course the gradients of f with respect to the variables p and q; $\{\cdot,\cdot\}$ is the Poisson bracket:

$$\{f,g\} = \frac{\partial f}{\partial p}\frac{\partial g}{\partial q} - \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} \quad \text{(dot products)}.$$

Finally the norm of a vector-valued function on D is defined as the supremum over D of the Euclidean norm of its value.

***Lemma* 1.** *Let f and g be analytic on D (continuous at the boundary); then*

$$(6)\qquad \left\|\frac{\partial f}{\partial q}\right\|_{D-(0,\xi)} \le \frac{1}{\xi}\|f\|_D, \quad \left\|\frac{\partial f}{\partial p}\right\|_{D-(\delta,0)} \le \frac{1}{\delta}\|f\|_D.$$

Suppose that $0 \leqslant \xi' < \xi$ *and* $0 \leqslant \delta' < \delta$; *then*

$$(7)\qquad \|\{f,g\}\|_{D-(\delta,\xi)} \le \Big(\inf[\xi(\delta-\delta'),\delta(\xi-\xi')]\Big)^{-1}\|f\|_D\|g\|_{D-(\delta',\xi')}.$$

In particular,

$$(8)\qquad \begin{aligned} &\|\{f,g\}\|_{D-(\delta,\xi)} \le \frac{1}{\xi\delta}\|f\|_D\|g\|_D \quad \textit{and} \\ &\|\{f,g\}\|_{D-(\delta,\xi)} \le \frac{2}{\xi\delta}\|f\|_D\|g\|_{D-\left(\frac{\delta}{2},\frac{\xi}{2}\right)}. \end{aligned}$$

In order to prove the first inequality in (6), one observes that at a given point (p, q)

$$\left\|\frac{\partial f}{\partial q}(p,q)\right\| = \sup_{\|e\|=1}\left\|\frac{d}{dt}\right|_{t=0} f(p,q+te)\Big\|.$$

One then applies Cauchy's formula to the function $t \to f(p, q+te)$ of the complex variable t, defined for $|t| \leqslant \xi$ and continuous at the boundary, when $(p, q) \in D-(0, \xi)$; the second inequality in (6) is proved analogously.

To prove (7), one writes, in a similar way:

$$\{f,g\}(p,q) = \frac{d}{dt}\Big|_{t=0} g\Big(p - t\frac{\partial f}{\partial q}, q + t\frac{\partial f}{\partial p}\Big)$$

and again applies Cauchy's formula to the function of t appearing on the right-hand side. Here one uses the circle $|t| = \inf[\xi(\delta-\delta'), \delta(\xi-\xi')]\cdot(\|f\|_D)^{-1}$. To justify this, one first notices that $|g(p-t\partial f/\partial q, q+t\partial f/\partial p)| \leqslant \|g\|_{D-(\delta',\xi')}$ when $(p, q) \in D-(\delta, \xi)$ if t satisfies $|t\partial f/\partial q| \leqslant \delta-\delta'$ and $|t\partial f/\partial p| \leqslant \xi-\xi'$; one then applies (6) to show that the function at hand is indeed analytic inside the circle mentioned above (and continuous at the boundary). □

Let us now come back to the construction of the canonical transformation C, as the time 1 map of the Hamiltonian χ. One demands that it satisfy $\|\chi\|_D \leqslant \xi\delta/4$, so that (6) provides the following evaluations for the Hamiltonian vector field:

$$(9)\qquad \left\|\frac{\partial \chi}{\partial q}\right\|_{D-\left(0,\frac{\xi}{2}\right)} \le \frac{2}{\xi}\|\chi\|_D \le \frac{\delta}{2}, \quad \left\|\frac{\partial \chi}{\partial p}\right\|_{D-\left(\frac{\delta}{2},0\right)} \le \frac{2}{\delta}\|\chi\|_D \le \frac{\xi}{2}.$$

This allows one to define C and ensures that the inclusions and inequalities following (4) are satisfied. Let L_χ denote Liouville's operator ($L_\chi(f) = \{\chi, f\}$). The transformation $C = \exp L_\chi$ acts on functions defined on D' and one can compute

$$(10)\qquad H' = \exp(L_\chi)H = h + Z + N + \{\chi,h\} + \{\chi, Z+N\} + H' - H - \{\chi, H\},$$

to be estimated on D' (below, however, we write (p, q) instead of (p', q') for simplicity). One has $\{\chi, h\} = -\omega\cdot\partial\chi/\partial q$ ($\omega = \omega(p)$) and we shall define χ such that it satisfies $\omega_0\cdot\partial\chi/\partial q = N$, which is made possible by the condition $\langle N\rangle = 0$ (see Lemma 2 below). One may then write $H' = h+Z+N''$ with

$$N'' = (\omega_0 - \omega)\frac{\partial\chi}{\partial q} + \{\chi, Z+N\} + H' - H - \{\chi, H\}. \tag{11}$$

Then, one defines $Z' = Z+\langle N''\rangle, N = N'' - \langle N''\rangle$ and finds that

$$\|Z' + N'\|_{D'} \le \|Z\|_{D'} + \|N\|_{D'} \le \varepsilon E + \|N''\|_{D'},$$
$$\|N'\|_{D'} = \left\|N'' - \langle N''\rangle\right\|_{D'} \le 2\|N''\|_{D'},$$

so that $\varepsilon' \leqslant \varepsilon+(1/2)\eta'$, which is $(5)_1$, with $\eta' \leqslant 2\|N''\|_{D'}/E$. We have used the fact that the operation $\langle\cdot\rangle$ of averaging is a projection of unit norm, which means that for any function g, $\|\langle g\rangle\|_D \leqslant \|g\|_D$. This is obvious from the definition of $\langle\cdot\rangle$.

It remains to evaluate $\|N''\|_{D'}$. To this end one writes

$$\|N''\|_{D'} \le \|\omega_0 - \omega\|_{D'}\left\|\frac{\partial\chi}{\partial q}\right\| + \left\|\{\chi, Z+N\}\right\|_{D'} + \frac{1}{2}\left\|\{\chi,\{\chi,H\}\}\right\|_{D-(\frac{\delta}{2},\frac{\xi}{2})}. \tag{12}$$

The last term comes from Taylor's formula and the fact that $C(D')\subset D-(\delta/2, \xi/2)$.

The first two terms on the right-hand side are easily estimated:

$$\|\omega_0 - \omega\|_{D'} \le M(R+\rho) \quad\text{and}\quad \left\|\frac{\partial\chi}{\partial q}\right\|_{D'} \le \xi^{-1}\|\chi\|_D$$

(using (6));

$$\left\|\{\chi, Z+N\}\right\|_{D'} \le (\xi\delta)^{-1}\|\chi\|_D\|Z+N\|_D \le \varepsilon E(\xi\delta)^{-1}\|\chi\|_D$$

(using (8) and the definition of ε).

To estimate the third term, one takes advantage of the definition of χ and writes

$$\{\chi, H\} = -N + (\omega_0 - \omega)\frac{\partial\chi}{\partial q} + \{\chi, Z+N\},$$

so that

$$\left\|\{\chi,\{\chi,H\}\}\right\| \le \|\{\chi,N\}\| + \left\|\left\{\chi,(\omega_0-\omega)\frac{\partial\chi}{\partial q}\right\}\right\| + \left\|\{\chi,\{\chi,Z+N\}\}\right\|,$$

and there are again three terms to be estimated over $D-(\delta/2, \xi/2)$. The first is dealt with as above:

$$\left\|\{\chi,N\}\right\|_{D-(\frac{\delta}{2},\frac{\xi}{2})} \le 4(\xi\delta)^{-1}\|\chi\|_D\|N\|_D \le 4\eta E(\xi\delta)^{-1}\|\chi\|_D.$$

The other two necessitate a repeated application of the inequalities (8):

$$\begin{aligned}\left\|\left\{\chi,(\omega_0-\omega)\frac{\partial\chi}{\partial q}\right\}\right\|_{D-(\frac{\delta}{2},\frac{\xi}{2})} &\le 8(\xi\delta)^{-1}\|\chi\|_D\left\|(\omega_0-\omega)\frac{\partial\chi}{\partial q}\right\|_{D-(\frac{\delta}{4},\frac{\xi}{4})}\\ &\le 8(\xi\delta)^{-1}\|\chi\|_D M(R+\rho)4\xi^{-1}\|\chi\|_D\\ &= 32M(R+\rho)\xi^{-2}\delta^{-1}\|\chi\|_D^2;\end{aligned}$$

and

$$\|\{\chi,\{\chi,Z+N\}\}\|_{D-(\frac{\delta}{2},\frac{\xi}{2})} \le 8(\xi\delta)^{-1}\|\chi\|_D\|\{\chi,Z+N\}\|_{D-(\frac{\delta}{4},\frac{\xi}{4})}$$
$$\le 8(\xi\delta)^{-1}\|\chi\|_D^2 16(\xi\delta)^{-1}\|Z+N\|_D$$
$$= 128\varepsilon E\xi^{-2}\delta^{-2}\|\chi\|_D^2.$$

Gathering terms together, this elementary (but admittedly forbidding) computation furnishes

$$\|N''\|_{D'} = \eta'\frac{E}{2} \le M(R+\rho)\xi^{-1}\|\chi\|_D + \varepsilon E(\xi\delta)^{-1}\|\chi\|_D + 2\eta E(\xi\delta)^{-1}\|\chi\|_D + 16M(R+\rho)\xi^{-2}\delta^{-1}\|\chi\|_D^2 + 64\varepsilon E\xi^{-2}\delta^{-2}\|\chi\|_D^2. \tag{13}$$

To go further, we must compute χ, to which end we use the following lemma, where the variable p is omitted because it plays the role of a dummy parameter.

***Lemma* 2.** *Let $g(q)$ be a function with zero mean value (with respect to ω_0): $\langle g\rangle = 0$; then the equation*

$$\omega_0\frac{\partial\chi}{\partial q} = g \tag{14}$$

possesses the explicit solution

$$\chi(q) = \frac{1}{T}\int_0^T g(q+\omega_0 t)t\,dt. \tag{15}$$

In particular, it satisfies

$$\|\chi\| \le \frac{T}{2}\|g\| \tag{16}$$

for any translation-invariant norm $\|\cdot\|$ defined on the space of measurable functions on the torus.

The two-line proof reduces to an integration by parts:

$$\omega_0\frac{\partial\chi}{\partial q} = \frac{1}{T}\int_0^T \omega_0\frac{\partial g}{\partial q}(q+\omega_0 t)t\,dt = \frac{1}{T}\int_0^T \frac{d}{dt}g(q+\omega_0 t)dt$$
$$= \frac{1}{T}g(q+\omega_0 t)t\Big|_0^T - \frac{1}{T}\int_0^T g(q+\omega_0 t)dt = g(q). \quad \square$$

Coming back to (13), χ satisfies (14) with $g = N$, and choosing the solution (15), we get $\|\chi\|_D \leqslant (T/2)\eta E$, so (13) becomes

$$\eta' \le \eta T\left[M(R+\rho)\xi^{-1} + \varepsilon E(\xi\delta)^{-1} + 2\eta E(\xi\delta)^{-1} + 8MET(R+\rho)\xi^{-2}\delta^{-1}\eta + 32E^2T\xi^{-2}\delta^{-2}\varepsilon\eta\right]. \tag{17}$$

All these estimates are valid under the hypothesis that $\|\chi\|_D \leqslant \xi\delta/4$, that is, $2TE\eta \leqslant \xi\delta$. It allows for a simplification of (17), which also underlines its dimensional correctness. In fact, after inserting this inequality in (17), one gets the second inequality in (5):

$$(5)_2 \qquad \eta' \leq 5T\eta\left[M(R+\rho)\xi^{-1} + 4E\varepsilon(\xi\delta)^{-1}\right].$$

This finishes the proof of the iterative lemma. □

A few remarks may be in order.

1. The contribution of the second order term (that is, $\{\chi, \{\chi, H\}\}$) has been reduced to a numerical factor.

2. η'/η is bounded by a quantity proportional to T: the larger the frequency over which one averages, the better the resulting estimate.

3. The appearance of the terms on the right-hand side of $(5)_2$ is easy to understand. The second stems from the "quadratic error" and has the size of the Poisson bracket $\{\chi, f\}$ of the auxiliary Hamiltonian (or generating function) with the perturbation. The first represents a frequency shift and comes in because we always solve (14), instead of adapting the frequency, that is solve the same equation with ω_0 replaced by $\omega = \omega(p)$. The above algorithm is thus in principle worse than the usual Picard method, not to mention Newton's.

We shall now apply the iterative lemma a certain number of times, say $s = s(\varepsilon)$, so as to eliminate the resonant part of the Hamiltonian to a high order. We start from $H = H^{(0)} = h+f$, defined on $D^{(0)}$ (with $D^{(0)} \subset D(R, \rho, \sigma)$; see below) and in the decomposition (3) set

$$Z = \langle f\rangle,\ N = f - \langle f\rangle,\ \varepsilon = \varepsilon_0,\ \eta = \eta_0 = \left\|f - \langle f\rangle\right\|_D/E \leq 2\varepsilon.$$

In the end, we get a Hamiltonian $H' = H^{(s)}$, defined on $D' \subset D = D^{(0)}$ and characterized by $\varepsilon' = \varepsilon_s$ and $\eta' = \eta_s$, having gone through a sequence of intermediate quantities $H^{(j)}$, $D^{(j)}$, ε_j, η_j, $j = 0, ..., s$.

Because of the frequency shift, one cannot work on a domain of order 1, so we use the smaller domain

$$D = D^{(0)} = D\big(R(\varepsilon), \rho(\varepsilon), \sigma\big) \subset D(R, \rho, \sigma).$$

Any trajectory with initial condition (in action space) lying in the (real) ball of radius $r(\varepsilon)$ around the origin will stay, until time $T(\varepsilon)$, in the ball of radius $R(\varepsilon) \leqslant R$. One has $D^{(j)} = D^{(j-1)} - (\delta_j, \xi_j)$ and all the pairs (δ_j, ξ_j) are chosen to be equal for $j = 1, ..., s$: $\xi_j = \xi$, $\delta_j = \delta$; this choice implies that

$$D' = D^{(0)} - (s\delta, s\xi) = D\big(R(\varepsilon), \rho(\varepsilon) - s\delta, \sigma - s\xi\big).$$

We build a sequence of canonical transformations $C^{(j)} : D^{(j)} \rightarrow D^{(j-1)}$ using the iterative lemma and denote by C their composition, from D' into D.

The iterative lemma ensures that

$$D^{(0)} - \left(\frac{s\delta}{2}, \frac{s\xi}{2}\right) \supset C(D') \supset D^{(0)} - \left(\frac{3s\delta}{2}, \frac{3s\xi}{2}\right) = D\left(R(\epsilon), \rho(\epsilon) - \frac{3s\delta}{2}, \sigma - \frac{3s\xi}{2}\right).$$

There are two requirements on the domains:

1) The system is defined on $D = D^{(0)}$, that is, $D \subset D(R, \rho, \sigma)$, which implies that $R(\varepsilon) \leqslant R$ and $\rho(\varepsilon) \leqslant \rho$.

2) The image of C contains the real ball of radius $R(\varepsilon)$ centred at the origin in action space. For this to be true, it is enough that $3s\delta/2 \leqslant \rho(\varepsilon)$ and $3s\xi/2 \leqslant \sigma(\varepsilon)$, which leads to the choice

$$\delta_j = \delta = \frac{\rho(\epsilon)}{2s}, \quad \xi_j = \xi = \frac{\sigma}{2s}.$$

Moreover, in the sequel, we choose $R(\varepsilon) = \rho(\varepsilon)$, which is no essential restriction; $r(\varepsilon) < R(\varepsilon)$ will be of the same order (with respect to ε). Finally, to simplify the notation, we shall often write r, R and ρ, without making the dependence on ε explicit. This slight ambiguity should cause no confusion as the original quantities R and ρ will not play any role and we shall rewrite the conditions $R(\varepsilon) \leqslant R$ and $\rho(\varepsilon) \leqslant \rho$ at the very end.

Rewriting formula $(5)_2$ with these values of the parameters, we get

$$\eta_j \leq \eta_{j-1} 20\sigma^{-1} T\left(M\rho s + \frac{4E\epsilon_j s^2}{\rho}\right), \quad j = 1, \ldots, s. \tag{18}$$

This is in some sense the fundamental inequality and it reflects rather accurately the data of the problem in its dependence with respect to the various parameters. We shall exploit it in three ways, but first go on with what is common to the three variants.

Each time we perform s transformations and ask that $s \geqslant 2$ (in fact otherwise the construction is of no interest). We also require that the sequence η_j decrease at least geometrically with ratio $1/e$ ($e = 2.718...$), which is of course somewhat arbitrary. From $\eta_j \leqslant \eta_0 e^{-j}$, $\eta_0 \leqslant 2\varepsilon$ and $\varepsilon_j \leqslant \varepsilon_{j-1} + (1\backslash 2)\eta_j$, one finds that $\varepsilon_j \leqslant 2\varepsilon_0 = 2\varepsilon$. Inserting this into (18) we find that $\eta_j \leqslant (1/e)\eta_{j-1}$ is guaranteed provided that

$$X \stackrel{\text{def}}{=} 20\sigma^{-1} T\left(M\rho s + \frac{8E\epsilon s^2}{\rho}\right) \leq \frac{1}{e}. \tag{19}$$

The final remainder η' is thus estimated as

$$\eta' = \eta_s \leq \eta_0 e^{-s} \leq 2\epsilon e^{-s}. \tag{20}$$

We now have at our disposal a "resonant normal form", in which the non-resonant harmonics of the original Hamiltonian have been eliminated to a high order s (still to be determined), with the help of a canonical transformation C such that $C(p', q') = (p, q)$.

To make use of it, we now add the ingredients of energy conservation and convexity of the unperturbed Hamiltonian. The functions $s = s(\varepsilon)$, $r(\varepsilon)$ and $R(\varepsilon)$ are still free parameters, and the reasoning below concerns the *real* parts of the various domains. We denote by $c(\varepsilon)$ the size of the canonical transformation in action space: $\|p'-p\| \leqslant c(\varepsilon)$; from the construction $c(\varepsilon) \leqslant (1/2)\rho(\varepsilon)$, but we shall need a somewhat more precise estimate.

We thus start from the initial condition $(p(0), q(0))$ with $\|p(0)\| \leqslant r(\varepsilon)$; one has

$$h(p'(t)) = h(p'(0)) + \omega(p'(0))\cdot(p'(t) - p'(0)) + \frac{1}{2}\Big(A(p^*)(p'(t) - p'(0))\cdot(p'(t) - p'(0))\Big),$$

where p^* is situated between $p'(t)$ and $p'(0)$ (the domain is convex). The convexity of h then implies that

$$(21)\quad \frac{1}{2}m\|p'(t) - p'(0)\| \le \Big|h(p'(t)) - h(p'(0))\Big| + \Big|\,\omega(p'(0))\cdot(p'(t) - p'(0))\,\Big|.$$

The first term on the right-hand side is estimated using conservation of energy for the full system: $H'(p'(t), q'(t)) = H'(p'(0), q'(0))$ implies that

$$(22)\quad \Big|h(p'(t)) - h(p'(0))\Big| \le \Big|Z'(p'(t))\Big| + \Big|Z'(p'(0))\Big| + \Big|N'(p'(t))\Big| + \Big|N'(p'(0))\Big| \le 2\varepsilon E + 2\varepsilon E + \eta' E + \eta' E \le 5\varepsilon E.$$

We have used $\|Z'\| \leqslant 2\varepsilon E$ and $\eta' \leqslant (1/2)\varepsilon$ (which comes from $s \geqslant 2$).

To estimate the second term on the right-hand side of (21), one considers the projections of the vectors $\omega(p'(0))$ and $p'(t)-p'(0)$ on ω_0 and on the orthogonal complement. We denote the corresponding projection operators as Π and $\Pi^\perp$ respectively. First:

$$\Big\|\Pi^\perp\omega(p'(0))\Big\| = \Big\|\Pi^\perp(\omega(p'(0)) - \omega(0))\Big\| \le \Big\|\omega(p'(0)) - \omega(0)\Big\| \le M\|p'(0)\| \le 2rM.$$

$c(\varepsilon) \leqslant r(\varepsilon)$ has been used, and that will be yet another requirement to keep in mind.

Projecting now on ω_0 we use the crucial fact that $\Pi\left(\dfrac{\partial}{\partial q}Z'\right) = 0$, because Z' is resonant. This is the only place where use is made of the normal form; to rephrase this, one can say that we have eliminated *one* degree of freedom by time averaging over the motion of period T. Because of the Hamiltonian character of the equation, this corresponds to motion orthogonal to the (unperturbed) energy surface. Convexity then provides quadratic potential wells, which prevent motion tangent to the surface. So from

$$\Pi\dot{p}' = -\Pi\Big(\frac{\partial Z'}{\partial q} + \frac{\partial N'}{\partial q}\Big) = -\Pi\frac{\partial N'}{\partial q}$$

we get

$$\Big\|\Pi(p'(t) - p'(0))\Big\| \le |t|\Big\|\frac{\partial}{\partial q}N'\Big\| \le \frac{2}{\sigma}|t|\|N'\| \le \frac{2}{\sigma}T(\varepsilon)\eta' E.$$

We have applied the Cauchy inequality to *real* points of the domain D' and used the fact that the analyticity width σ' of H' is equal to $\sigma/2$ because of the choice of ξ. The time of validity $\mathcal{T}(\varepsilon)$ is still a free parameter at this point. We thus obtain

$$\left|\omega(p'(0))\cdot(p'(t)-p'(0))\right| \le 2rM\|p'(t)-p'(0)\| + \frac{2}{\sigma}\mathcal{T}(\varepsilon)\eta' E\left\|\omega(p'(0))\right\|.$$

Since $p'(0)\in B_{2r}$, one has $\|\omega(p'(0))\| \leqslant \Omega+2rM \leqslant 2\Omega$ if $2rM \leqslant \Omega$, which is a condition on $r = r(\varepsilon)$. Writing $a = \|p'(t)-p'(0)\|$, we collect the above estimates as

$$\frac{1}{2}ma^2 \le 5\varepsilon E + \frac{4}{\sigma}\mathcal{T}(\varepsilon)\Omega\eta' E + 2rMa. \tag{23}$$

We choose $\mathcal{T}(\varepsilon)$ such that $(4/\sigma)\mathcal{T}(\varepsilon)\Omega\eta' E \leqslant \varepsilon E$, that is, $\mathcal{T}(\varepsilon) \leqslant \varepsilon(\sigma/4\Omega)\eta'^{-1}$; since $\eta' \leqslant 2\varepsilon e^{-s}$, this is larger than

$$\mathcal{T}(\varepsilon) = \frac{\sigma}{8\Omega}e^{s(\varepsilon)}, \tag{24}$$

which is the value we finally adopt. Putting this into (23), we find that

$$a \le 2r\frac{M}{m} + \frac{1}{m}(12m\varepsilon E + 4r^2M^2)^{\frac{1}{2}}. \tag{25}$$

One may notice that the quantity on the right-hand side is at least of the order of $\sqrt{\varepsilon}$, which could have been predicted. In fact, convexity provides *quadratic* potential wells, so that the energy increases as a^2 (the square of the distance to the bottom); adding a perturbation of order ε, both terms have to be at least of the same size to ensure confinement. This implies that the confinement radius is at least of the order of $\sqrt{\varepsilon}$, so that the second stability exponent satisfies $b \leqslant 1/2$.

We now require that the second term in the bracket be the larger: $12m\varepsilon E \leqslant 4r^2M^2$. Under this condition (25) implies that $a \leqslant 5rM/m$, so

$$\|p(t)-p(0)\| \le 2c(\varepsilon) + \|p'(t)-p'(0)\| \le 7r\frac{M}{m},$$

which entails $\|p(t)\| \leqslant 8rM/m$.

Before gathering everything together, we make the condition $r(\varepsilon) \geqslant c(\varepsilon)$ explicit, by estimating the latter quantity as follows:

$$c(\varepsilon) \le \sum_{j=1}^{s}\frac{2}{\varepsilon_j}\|\chi_j\|_{D_j} \le \sum_{j=0}^{s-1}\frac{4s}{\sigma}\frac{T}{2}\eta_j E = \frac{2ET}{\sigma}s\sum_{j=0}^{s-1}\eta_j.$$

From $\eta_j \leqslant 2\varepsilon e^{-j}$ we deduce the bound: $c(\varepsilon) \leqslant (8ET/\sigma)s\varepsilon$.

Everything we have done above is valid under the assumptions (4) of the iterative lemma. Now the first two are satisfied by construction and it is easy to see that the inequality $\xi\delta \geqslant 2TE\eta$ holds if $(5)_2$ is satisfied. This is shown by looking back at $(5)_2$ and estimating the right-hand side from below, keeping only the second term in the bracket. To compare the resulting inequalities, one uses the fact that $\eta \leqslant 2\varepsilon$ (we apply the iterative lemma with $\eta = \eta_j$, $\eta' = \eta_{j+1}$ and $\varepsilon_j \leqslant 2\varepsilon$).

We can now produce a statement from which Theorems 1A, 1B, 1C below will easily follow, by specifying the parameters in different ways.

Model statement. *In the situation above, suppose* (19) *is satisfied, so that one can perform* $s(\varepsilon)$ *steps of the algorithm. We choose* $R(\varepsilon) = \rho(\varepsilon)$ *and* $r(\varepsilon) = \frac{m}{8M}R(\varepsilon)$. *Then if the initial condition satisfies* $\|p(0)\| \leqslant r(\varepsilon)$, *one has* $\|p(t)\| \leqslant R(\varepsilon)$ *for* $|t| \leqslant T(\varepsilon) = \frac{\sigma}{8\Omega}e^{s(\varepsilon)}$, *provided that the following conditions obtain:*

i) $R(\varepsilon) = \rho(\varepsilon) \leqslant \inf(R, \rho)$; *R and* ρ *are the original quantities (see* (2)*). One simply requires that the Hamiltonian be defined over the domain one is working on.*

ii) $2Mr \leqslant \Omega$; *the frequency should not vary too much over the ball where the initial conditions are chosen.*

iii) $12m\varepsilon E \leqslant 4M^2r^2$, *that is,* $r^2 \geqslant 3\frac{m}{M^2}\varepsilon E$; *the energy of the perturbation is balanced by that arising from the quadratic wells for the "kinetic" part.*

iv) $r(\varepsilon) \geqslant c(\varepsilon)$, or $r \geqslant \frac{8ET}{\sigma}s\varepsilon$: *the size of the ball prescribed for the initial conditions is larger than that of the canonical transformation (in action space).*

v) $s \geqslant 2$: *one can perform at least two steps in the algorithm.*

Before stating Theorem 1A, we define three quantities which will be useful in the sequel; they are dimensionally correct and their occurrence, except for a numerical factor, is easy to understand; so we let

$$\lambda = 10^{-3}\frac{\sigma m}{M^2}, \quad \tau = 3 \cdot 10^{-3}\frac{\sigma}{\sqrt{EM}}, \quad \mathcal{T}_0 = 4 \cdot 10^{-2}\frac{\sigma}{\Omega}. \tag{26}$$

We retain, of course, the setting defined at the beginning of this section.

***Theorem* 1A.** *Let* α *be such that* $0 < \alpha \leqslant 1/3$; *assume that* $\|p(0)\| \leqslant r(\varepsilon) = \lambda\varepsilon^\alpha/T$ *with T satisfying* $1 \leqslant T \leqslant \tau\varepsilon^{-\frac{1}{2}(1-3\alpha)}$. *Then* $\|p(t)\| \leqslant R(\varepsilon) = 8M/m \cdot r(\varepsilon) \leqslant \leqslant 10^{-2}\sigma/M \cdot \varepsilon^\alpha/T$ *when* $|t| \leqslant T(\varepsilon) = \mathcal{T}_0 \exp(\varepsilon^{-\alpha})$, *provided that* ε *satisfies the following inequalities*:

$$\begin{aligned} &\varepsilon^\alpha \leq 100\frac{M}{\sigma}\inf(R,\rho), \quad \varepsilon^\alpha \leq 10^3\frac{\Omega M}{2\sigma m}, \\ &\varepsilon^\alpha \leq 4 \cdot 10^{-2}\frac{m}{M}, \qquad \varepsilon^{\frac{1}{2}(1-3\alpha)} \leq \tau. \end{aligned} \tag{27}$$

This may not look like the most natural statement in the framework of this chapter, but its merit lies in that the time of validity is independent of the chapter of the orbit when this is short enough. This will be crucial in the next section. To prove the result, set

$$\rho = \rho_0\frac{T_0}{T}\varepsilon^\alpha, \quad s = [s_0\varepsilon^\alpha], \quad T \leq T_0\varepsilon^{-\beta}, \quad \beta = \frac{1}{2}(1-3\alpha). \tag{28}$$

ρ_0, T_0 and s_0 are to be determined; $[x]$ denotes the integer part of a real number x. The factor X in (19) becomes

$$X = \frac{20}{\sigma}\left[M\rho_0 s_0 T_0 + 8ET^2\frac{s_0^2}{\rho_0 T_0}\varepsilon^{1-3\alpha}\right].$$

The first term in the bracket is larger than the second provided that

$$M\rho_0 s_0 T_0 \geq 8E\frac{T_0 s_0^2}{\rho_0}.$$

Choose $s_0 = 1$ and $\rho_0 = 2(2E/M)^{1/2}$, which ensures equality. (19) is reduced to

$$\frac{40}{\sigma}M\rho_0 s_0 T_0 \leq \frac{1}{e}.$$

Assuming equality again, we find that

$$T_0 = \frac{1}{80e\sqrt{2}}\frac{\sigma}{\sqrt{EM}} > \tau,$$

so we may adopt the value $T_0 = \tau$ and compute $\rho(\varepsilon) = R(\varepsilon)$ and $r(\varepsilon)$. Notice that

$$R(\varepsilon) \leq \rho_0 T_0 \varepsilon^{\alpha} \text{ and } R(\varepsilon) \geq \rho_0 \varepsilon^{\alpha+\beta} = \rho_0 \varepsilon^{\frac{1}{2}(1-\alpha)} > \rho_0 \varepsilon^{\frac{1}{2}}.$$

Theorem 1A is then a consequence of the "model statement" above. Concerning the time of validity, one writes $s \geqslant s_0\varepsilon^{-\alpha}-1$, which leads to the value of T_0 defined in (26), by slightly reducing the quantity $\frac{\sigma}{8e\Omega}$, obtained from this inequality and the model statement.

The computation of the thresholds is straightforward, using the five conditions of the model statement. In the first two, one should use *upper* bounds for $r(\varepsilon)$ and $R(\varepsilon)$, that is, set $T = 1$, whereas in the third one has to use the *lower* bound for $r(\varepsilon)$. This leads to the first three inequalities in (27). Conditions iv) and v) are both weaker than iii). The last threshold ensures that the upper bound for T is indeed larger than 1. □

The next result will sound more natural in this setting; we write again

$$\rho_0 = R_0 = 2\sqrt{\frac{2E}{M}}, \quad r_0 = \frac{m}{8M}R_0 = \frac{m}{4M}\sqrt{\frac{2E}{M}}. \tag{29}$$

***Theorem* 1B.** *If* $\|p(0)\| \leqslant r_0\varepsilon^{1/3}$, *then* $\|p(t)\| \leqslant R_0\varepsilon^{1/3}$ *if* $|t| \leqslant T_0 \exp\left(\frac{\tau}{T\varepsilon^{1/3}}\right)$ *(however, if* $T \leqslant \tau$, *one should replace the factor* τ/T *by* 1) *provided that* ε *satisfies*

$$\begin{aligned} &\varepsilon^{\frac{1}{3}} \leq \frac{1}{2}\sqrt{\frac{M}{2E}}\inf(R,\rho), && \varepsilon^{\frac{1}{3}} \leq \frac{\Omega}{\sqrt{EM}}, \\ &\varepsilon^{\frac{1}{3}} \leq 4\cdot 10^{-2}\frac{m}{M}, && \varepsilon^{\frac{1}{3}} \leq \frac{\tau}{2T} = 1.5\cdot 10^{-3}\frac{\sigma}{T\sqrt{EM}}. \end{aligned} \tag{30}$$

The proof is similar, and in fact simpler. One defines

$$\rho = \rho_0 \varepsilon^{\frac{1}{3}}, \quad s = [s_0 \varepsilon^{-\frac{1}{3}}];$$

inserting these values in (19) gives

$$X = \frac{20}{\sigma} T \Big[M \rho_0 s_0 + \frac{8E}{\rho_0} s_0^2 \Big] \le \frac{1}{e}.$$

We assume that $s_0 \leqslant 1$ (hence the restriction in brackets in the statement), which allows one to replace s_0^2 by s_0, slightly strengthening the condition. Assuming equality, one finds that

$$s_0 = \frac{\sigma}{20eT} \Big(M \rho_0 + \frac{8E}{\rho_0} \Big)^{-1}.$$

Upon maximizing this expression with respect to ρ_0, one finds the value in (29) and

$$s_0 = \frac{1}{80e\sqrt{2}} \frac{\sigma}{T\sqrt{EM}} > \frac{\tau}{T}.$$

It remains to determine the thresholds of validity, which does not pose any particular problem. Again iv) is weaker than iii), and ii) has been slightly strengthened for aesthetic purposes. □

We are concerned with three parameters of physical interest, T, s and r, which are connected, respectively, with the *period* of the linear flow on the given torus, the *time of validity* of the stability estimate, and what we shall call the radius of the *influence zone* of the torus. In our last statement, we shall put the latter quantity in a privileged position and treat r as a free variable, trying to make it as large as possible. We may assume that $r \geqslant r_0 \varepsilon^{1/3}$, since otherwise Theorem 1B applies.

***Theorem* 1C.** *Let $(p(0), q(0))$ be an initial condition such that $\|p(0)\| \leqslant r$; then $\|p(t)\| \leqslant 8(M/m)r$ if $|t| \leqslant T_0 \exp[\lambda/(rT)]$ (in the case $\lambda/(rT) \geqslant \varepsilon^{-1/3}$ one should use the latter quantity), provided that the following conditions hold:*

a) $r \geqslant r_0 \varepsilon^{1/3}$, *where* $r_0 = \dfrac{m}{4M}(2E/M)^{1/2}$.

b) r *satisfies the following four inequalities*:

$$(31) \quad r \le \frac{m}{8M} \inf(R, \rho), \; r \le \frac{\Omega}{2M}, \; r^2 \ge \frac{3m}{M^2} \varepsilon E, \; r \le \frac{\lambda}{2T} = 10^{-3} \frac{\sigma m}{2TM^2}.$$

Combining the last two inequalities provides a threshold for ε, and Theorems 1C and 1B connect nicely in the vicinity of $r = r_0 \varepsilon^{1/3}$. The proof is again straightforward, using the model statement. One has $\rho = \dfrac{8M}{m} r \geqslant \rho_0 \varepsilon^{1/3}$, so in (19) one finds that

$$X = \frac{20}{\sigma} T \Big[M \rho s + 8E \frac{\varepsilon s^2}{\rho} \Big] \le \frac{20}{\sigma} T \big[M \rho s + 2\sqrt{2EM} (\varepsilon^{\frac{1}{3}} s)^2 \big].$$

One assumes that $s \leqslant \varepsilon^{-1/3}$ (hence the restriction in brackets in the statement) which is in fact true if for instance $\tau \leqslant T$, because $r \geqslant r_0\varepsilon^{1/3}$. Under this assumption, the first term in the bracket is the larger and one proceeds as in Theorems 1A and 1B. □

The three above results constitute the "building bricks" from which the general stability theorems over finite times will be obtained in the next section, using only simple approximation properties, without any additional work. Since these arithmetical considerations cannot be improved upon, all the non-optimal features of the results should be blamed on the above.

It remains in this chapter to indicate briefly the necessary modifications when h is only assumed to be *quasi*-convex. This is the natural geometrical assumption: the unperturbed energy surface is convex in angle-action variables (when considered in action space). It allows one in particular to include the case of periodic perturbations of convex Hamiltonians (see below) and the related situation of symplectic maps with sign-definite twist matrices (see Chapter IV, §2).

We still denote the Hessian matrix as $A(p) = \nabla^2 h(p)$ and M its largest eigenvalue on the given domain: $\|A(p)v\| \leqslant M\|v\|$ for any $v \in \mathbb{R}^n$, $p \in D$. Now $m > 0$ is defined by the inequality

$$A(p)v\cdot v \geq m\|v\|^2 \text{ if } \omega(p)\cdot v = 0,\ v \in \mathbb{R}^n,\ p \in D\cap\mathbb{R}^n,$$

that is, v is tangent to the unperturbed energy surface ($\omega(p) = \nabla h(p)$).

For illustrative purposes, let us compute this quantity for a periodically perturbed convex Hamiltonian, that is, let $H(p, q, t) = h(p)+f(p, q, t)$, where h is convex and has associated quantities m and M when considered as a convex functional; f is assumed to be 1-periodic with respect to t. One may regard this as an autonomous problem in $n+1$ dimensions, with Hamiltonian

$$H_1(p_1, q_1) = h_1(p_1) + f_1(p_1, q_1),$$

where $p_1 = (p, e)$, $q_1 = (q, t)$, $h_1 = h(p)+e$ and $f_1(p_1, q_1) = f(p, q_1)$. The frequency is $\omega_1 = (\omega, 1)$. The Hessian matrix is singular, since we have added a constant frequency, but the system is isoenergetically non-degenerate (see the beginning of Chapter III). The function h_1 is quasi-convex, as we shortly see; let m_1 and M_1 denote the associated quantities. Obviously one has $M_1 = M$, and m_1 is computed as follows: if $v_1 = (v, w) \in \mathbb{R}^{n+1}$, $v \in \mathbb{R}^n$, $w \in \mathbb{R}$, the condition $v_1 \cdot \omega_1 = 0$ reads $\omega \cdot v + w = 0$. By definition,

$$A_1 v_1 \cdot v_1 = Av\cdot v \geq m\|v\|^2$$

and under the assumption $v_1 \cdot \omega_1 = 0$

$$\|v_1\|^2 = \|v\|^2 + |w|^2 \leq (1 + \|\omega\|^2)\|v\|^2,$$

hence

$$A_1 v_1 \cdot v_1 \geq m(1 + \|\omega\|^2)^{-1}\|v_1\|^2.$$

Working in a domain such that, say, $\|\omega(p)\| \leqslant 2\Omega = 2\|\omega(0)\|$, we may take

$$m_1 = m(1+4\Omega^2)^{-1}.$$

Returning to the proof of the results, the difference between convexity and quasiconvexity occurs in the geometrical reasoning only; the iterative lemma remains untouched (notice that m does not appear in it). (21) can still be written

$$\frac{1}{2}\Big(A(p^*)\big(p'(t)-p'(0)\big)\cdot\big(p'(t)-p'(0)\big)\Big) \leq \Big|h\big(p'(t)\big)-h\big(p'(0)\big)\Big| + \Big|\,\omega\big(p'(0)\big)\cdot\big(p'(t)-p'(0)\big)\Big|,$$

where p^* lies somewhere between $p'(t)$ and $p'(0)$.

The right-hand side is estimated as above by the right-hand side of (23). Let $\omega^* = \omega(p^*)$, $A^* = A(p^*)$, $u = p'(t)-p'(0)$, $\|u\| = a$. Choosing $\mathcal{T}(\varepsilon)$ again as in (24), one thus finds that

$$\frac{1}{2}(A^*u,u) \leq 6\varepsilon E + 2rMa, \tag{32}$$

and it only remains to estimate the left-hand side from below. To this end, let Π^* be the orthogonal-projection operator on ω^*, and $\Pi^{*\perp}$ the projection operator on the orthogonal complement; we write the explicit decomposition

$$A^*u\cdot u = A^*\Pi^*u\cdot\Pi^*u + A^*\Pi^{*\perp}u\cdot\Pi^{*\perp}u + 2\,A^*\Pi^*u\cdot\Pi^{*\perp}u\,.$$

Then

$$\frac{1}{2}A^*\Pi^*u\cdot\Pi^{*\perp}u \geq \frac{1}{2}m\|\Pi^{*\perp}u\|^2 = \frac{1}{2}m\big(a^2-\|\Pi^*u\|^2\big).$$

Hence

$$\begin{aligned}\frac{1}{2}A^*u\cdot u &\geq \frac{1}{2}ma^2 - \frac{1}{2}m\|\Pi^*u\|^2 - \frac{1}{2}M\|\Pi^*u\|^2 - Ma\|\Pi^*u\| \\ &\geq \frac{1}{2}ma^2 - 2Ma\|\Pi^*u\|,\end{aligned}$$

which yields

$$\frac{1}{2}ma^2 \leq 6\varepsilon E + 2rMa + 2Ma\|\Pi^*u\|. \tag{33}$$

With Π still denoting the projection on $\omega_0 = \omega(0)$ and $\Omega = \|\omega_0\|$, one has

$$\|\Pi^*u\| = \|\Pi u\| + \big\|(\Pi-\Pi^*)u\big\|.$$

We then use

$$\|\Pi u\| \leq \frac{2}{\sigma}\mathcal{T}(\varepsilon)\eta' E \leq \frac{\varepsilon E}{2\Omega},$$

which still holds. From $\|p^*\| \leqslant 2r+a$, one concludes that

$$\|(\Pi - \Pi^*)u\| \leq 2a(2r+a)\frac{M}{\Omega},$$

and in the end

$$\|\Pi^* u\| \leq \varepsilon\frac{E}{2\Omega} + 2a(2r+a)\frac{M}{\Omega}. \tag{34}$$

We now insist on obtaining the *same* radius of confinement $R(\varepsilon)$, in order not to have to alter the domains in the iterative lemma. A simple way to achieve this is to require that $\|\Pi^* u\| \leqslant r$. In view of (33), one may then leave R unchanged, replacing r by $r/2$. We therefore define

$$R(\varepsilon) = \rho(\varepsilon) \text{ and } r(\varepsilon) = \frac{m}{16M}R(\varepsilon) \quad \left(\text{instead of } \frac{m}{8M}R(\varepsilon)\right).$$

We need $\|\Pi^* u\| \leqslant r$, knowing that $a \leqslant R$. This amounts to a simple bootstrap argument: (34) yields a condition on R (or r), which is satisfied in particular if

$$r \geq \varepsilon\frac{E}{\Omega} \text{ and } r \leq 10^{-3}\frac{m^2\Omega}{M^3}.$$

Model statement (version for quasi-convex Hamiltonians). *It differs from the version for convex Hamiltonians only in the following points:*

1) *One still defines* $R(\varepsilon) = \rho(\varepsilon)$, *but now* $r(\varepsilon) = \dfrac{m}{16M}R(\varepsilon)$.

2) *Condition* ii) *is replaced by*

$$\text{ii bis)} \quad r \leq 10^{-3}\frac{m^2\Omega}{M^3};$$

which is stronger.

3) *One adds the condition* $r \geqslant \varepsilon\dfrac{E}{\Omega}$, *which for small* ε *is weaker than* iii).

We leave it to the interested reader to modify Theorems 1A, 1B, 1C accordingly. The modifications are of minor significance.

Chapter III

STABILITY FOR ARBITRARY INITIAL CONDITIONS

In this chapter, we use Theorems 1A, 1B, 1C, especially Theorem 1A, to obtain information about stability of points in phase space. The general idea is to apply one of these results whenever a given point lies in the influence zone of some periodic torus. This amounts to studying the distribution of rational points in frequency space, which corresponds to that of unperturbed periodic tori, provided that the frequency map $p \to \omega(p)$ enjoys some non-degeneracy condition. We mention that this is really all that is needed for the approximation process; in particular, analyticity and quasi-convexity are used only inasmuch as the theorems of Chapter II are applied.

To make all this precise, let us consider again the Hamiltonian (1) of Chapter II; the only difference here is that we look at the neighbourhood of some arbitrary point (or rather torus) $p = p^*$, so that in (2) (Chapter II) one should use a ball with centre at p^*.

If the unperturbed Hamiltonian h is convex, the Hessian matrix $A(p) = \nabla^2 h(p)$ is non-singular ($\det A(p) \neq 0$), and the frequency map is a local diffeomorphism. Simultaneous approximation will however deal rather with the ratios of the frequencies to one of them, which corresponds to isoenergetic non-degeneracy. For the sake of completeness, we briefly recall the definition and show that it is satisfied by quasi-convex (in particular convex) Hamiltonians.

Let Σ be the unperturbed energy surface $h(p) = h(p^*)$; one wants the $n-1$ ratios of the frequencies to a given one to yield a local chart of Σ. This is the same as requiring that the map

$$p \in \Sigma \to \omega \in \mathbb{P}\mathbb{R}^{n-1}$$

be a local diffeomorphism near p^*, where the frequency is considered in projective space. To check this condition, one should make sure that the Hessian matrix of the "homogeneous" map

$$(p, \lambda) \in \mathbb{R}^n \times \mathbb{R} \to \lambda h(p)$$

is non-singular at $(p^*, 1)$. The matrix reads

$$\mathcal{A}(p^*) = \begin{pmatrix} A & \omega \\ \omega & 0 \end{pmatrix},$$

where $\omega = \omega(p^*)$ is written as a column on the right and as a row at the bottom. The isoenergetic non-degeneracy condition thus reads $\det \mathcal{A} \neq 0$ (this is sometimes called "the Arnol'd determinant"). Suppose now that h is quasi-convex, and let $u \in \mathbb{R}^{n+1}$ with $\mathcal{A}u = 0$. We write $u = (v, w)$, $v \in \mathbb{R}^n$, $w \in \mathbb{R}$. The condition $\mathcal{A}u = 0$ splits into

$$Av + w\omega = 0 \quad \text{and} \quad (\omega, v) = 0.$$

Taking the dot product of the first equality with v, we find that

$$(Av, v) = 0 \quad \text{and} \quad (\omega, v) = 0,$$

which implies that $v = 0$ by the very definition of quasi-convexity, and then $w = 0$, so $u = 0$.

Quasi-convex Hamiltonians are thus isoenergetically non-degenerate. We recall that in the opposite direction, and for low dimensions, one has the following simple results: when $n = 2$, isoenergetic non-degeneracy is equivalent to quasi-convexity; when $n = 3$ quasi-convexity is equivalent to the condition $\det \mathcal{A} < 0$, so that, so to speak, "half" the non-degenerate Hamiltonians are quasi-convex. We also recall, as a word of caution, that outside the realm of convexity and quasi-convexity, non-degeneracy and isoenergetic non-degeneracy are independent conditions; neither of them follows from the other.

Below we shall again, for simplicity, treat the case of convex Hamiltonians; the modifications needed in the quasi-convex case are of minor interest. Geometrically speaking, in both cases the locus, in action space, where $\omega \in \mathbb{P}\mathbb{R}^{n-1}$ is constant, is a smooth curve which intersects the unperturbed energy surface transversally. So on this curve the n-vector ω varies along a straight line. Now in the convex case it does indeed vary and the frequency itself constitutes a local parameter; in the *quasi*-convex case, however, the frequency may well be constant along the curve (think of the periodic perturbation of a convex Hamiltonian; compare the end of the previous chapter).

So let h be convex, p^* a hitherto arbitrary point, and $\omega^* = \omega(p^*)$. Of course we use the notation of Chapter II. By the usual implicit function theorem, one has the following. Let $B(p^*)$ be a ball centred at p^* with radius S, such that for $p \in B(p^*)$

$$\|A(p) - A(p^*)\| \leq \frac{m}{2}.$$

Here $\|\cdot\|$ denotes the usual operator norm associated with the Euclidean norm. One can take $S = \frac{m}{2|h|_3}$, where $|h|_3$ is an upper bound of the third derivative of h. Then the frequency map is one-to-one on $B(p^*)$, and $\omega(B(p^*)) \supset B(\omega^*)$, a ball with centre at ω^* and radius $(m/2)S$.

The above determines in a quantitative way the local inversion properties of the frequency map. Because of this, the procedure to determine the domain is as follows. One starts from a fixed ball $B_0(p^*)$ over which $H = h+f$ is defined with analyticity widths ρ and σ. One then determines m, M and $|h|_3$ on $B_0(p^*)$, and then restricts oneself to a ball of radius S, which may be assumed to be included in $B_0(p^*)$, decreasing m if necessary.

Let us now get closer to the heart of the matter, which will necessitate one more piece of notation. For real x one has

$$x = [x] + \{x\},$$

with $[x] \in \mathbb{Z}$ the integer part, and $\{x\} \in (0, 1)$. We use the notation

$$\|x\|_{\mathbb{Z}} \stackrel{\text{def}}{=} \inf(\{x\}, 1 - \{x\}) = \operatorname{dist}(x, \mathbb{Z}).$$

Although $\|\cdot\|_{\mathbb{Z}}$ is not a norm, this is a commonly used notation, even without the index $\mathbb{Z}$, which we have added to avoid confusion. If now $x \in \mathbb{R}^n$ with components $x^{(j)}$, one sets

$$\|x\|_{\mathbb{Z}} \stackrel{\text{def}}{=} \sup_{j=1,\dots,n} \|x^{(j)}\|_{\mathbb{Z}} = \inf_{\zeta \in \mathbb{Z}^n} \|x - \zeta\|_\infty,$$

where $\|\cdot\|_\infty$ is the norm of the largest component, which naturally occurs in approximation theory. In particular,

$$\|x\|_{\mathbb{Z}} \leq \operatorname{dist}(x, \mathbb{Z}^n) \leq \sqrt{n}\|x\|_{\mathbb{Z}}.$$

With this notation, one has the following result.

Theorem (Dirichlet, see, for example, [11] or [52]). *Let $\alpha \in \mathbb{R}^n$ and Q a real number, $Q > 1$. There exists an integer q, $1 \leqslant q < Q$, such that*

$$\|q\alpha\|_{\mathbb{Z}} \leq Q^{-\frac{1}{n}}.$$

We shall apply this basic result of approximation theory with $\alpha = \omega^* = \omega(p^*)$ and $q = T$ playing the role of a period. Before we do this, however, it is important to notice that one can gain one dimension, which bears directly on the value of the stability exponents. This reflects the fact that simultaneous approximation corresponds to *inhomogeneous* linear approximation (see Appendix 1) or, very concretely, that q in the above theorem is an integer, so that approximating α is equivalent to approximating $(1, \alpha) \in \mathbb{R}^{n+1}$. One way to put this to use is precisely to consider that $\alpha \in \mathbb{P}\mathbb{R}^{n-1}$, that is, one should in fact approximate the ratios of the components to a fixed one. In the framework we are interested in, we may in fact simply rescale one of the components of ω^* to unity. To this effect, let $w > 0$ denote the modulus of a non-zero component of ω^*, for example, but not necessarily the largest one: $w = \|\omega^*\|_\infty$. One introduces the scaling

$$(1)\quad t' = wt,\ H' = \frac{H}{w},\ \omega^{*\prime} = \frac{\omega^*}{w},\ m' = \frac{m}{w},\ M' = \frac{M}{w},\ E' = \frac{E}{w},\ \varepsilon' = \varepsilon.$$

Relabelling if necessary, we are reduced to the case when the first component of the frequency is equal to unity. Below, for the sake of clarity, we write everything using the original quantities and at the very end remember that one should *first* perform the transformations (1) and change the results accordingly.

Let us now apply Dirichlet's theorem with $\alpha = \omega^*$; for any $Q > 1$ there exists an integer T, $1 \leqslant T < Q$, and $\zeta \in \mathbb{Z}^n$, such that

$$\|T\omega^* - \zeta\| \leq \sqrt{n}Q^{-\frac{1}{n}} \text{ (Euclidean norm).}$$

Thus, $\omega = T^{-1}\zeta$ is a rational vector of period T, satisfying

$$(2)\qquad \|\omega - \omega^*\| \leq \frac{\sqrt{n}}{TQ^{1/n}}.$$

We assume that ω is close enough to ω^* so that the frequency map can be inverted. From what was recalled above, to secure this it is enough to require that

$$\frac{\sqrt{n}}{Q^{1/n}} \leq |h|_3^{-1}\frac{m^2}{4},$$

where T has been removed from the left-hand side because it is $\geqslant 1$. Under this condition, there exists a point p such that $\omega = \omega(p)$ and

$$\|p - p^*\| \leq \frac{\sqrt{n}}{m} \frac{1}{TQ^{1/n}}, \tag{3}$$

where the factor $1/m$ estimates from above the norm of the inverse of the frequency map.

We wish to apply Theorem 1A *around the point p*, which is rational with period T, and we want p^* to lie in the influence zone of p, which will be the case if

$$r(\varepsilon) = \frac{\lambda \varepsilon^{\alpha}}{T} \geq \|p - p^*\|.$$

It follows from (3) that this is in turn guaranteed if we choose

$$Q^{\frac{1}{n}} = \frac{\sqrt{n}}{\lambda m} \varepsilon^{-\alpha}, \tag{4}$$

which *defines* the value of Q. This is where it is crucial to apply Theorem 1A, with its time of stability independent of the period and a radius of the influence zone inversely proportional to it. The latter feature yields (4), in which T does not appear. We shall point out below what can be inferred if one tries to use Theorems 1B and 1C.

To apply Theorem 1A, there remains yet another important condition to be satisfied: the period should not be too long. More precisely, since $T < Q$, it is enough to require that

$$Q \leq \tau \varepsilon^{-\frac{1}{2}(1-3\alpha)},$$

that is, referring to (4),

$$\left(\frac{\sqrt{n}}{\lambda m}\right)^n \varepsilon^{-n\alpha} \leq \tau \varepsilon^{-\frac{1}{2}(1-3\alpha)},$$

or

$$\varepsilon^{\frac{1-3\alpha}{2} - n\alpha} \leq \tau \left(\frac{\lambda m}{\sqrt{n}}\right)^n. \tag{5}$$

This simple reasoning is very important, because in fact it unveils the meaning of the first stability exponent, the more important one. The inequality (5) defines a threshold for ε provided that $\frac{1}{2}(1-3\alpha) - n\alpha > 0$, or

$$\alpha < \frac{1}{2n+3}.$$

The value on the right-hand side is thus not accessible, but any smaller value is; at this point one should remember that we shall then substitute $n-1$ for n, and the stability exponent $a(n)$ will be given by $a(n) = \alpha(n-1)$. For the time being, we go on with the initial quantities and write a statement with

$\alpha = \dfrac{1}{2n+4}$, which satisfies $\frac{1}{2}(1-3\alpha)-n\alpha = \frac{1}{2}\alpha$. This allows us to rewrite (5) as

$$\varepsilon^{\alpha} \leq \tau^{2}\left(\frac{\lambda m}{\sqrt{n}}\right)^{2n}. \tag{6}$$

The radius of stability can be easily computed since Theorem 1A specifies the distance from p and $\|p-p^*\| \leqslant r(\varepsilon)$. In fact, returning to the text immediately above the "model statement" of Chapter II, one notices that we have proved slightly more than what was actually included in the statement itself; in fact, it was shown that

$$\|p(t)-p(0)\| \leq 7\tau\frac{M}{m} < R(\varepsilon).$$

We use this estimate to get

$$\|p(t)-p(0)\| < R(\varepsilon) < 10^{-2}\frac{\sigma}{M}\frac{\varepsilon^{\alpha}}{T} \leq 10^{-2}\frac{\sigma}{M}\varepsilon^{\alpha}.$$

To compute the threshold of validity, one should essentially copy inequalities (27) of Chapter II, add the invertibility condition and, most important, inequality (5), in the form (6) because of our choice of α. Let us quickly go into some details.

We leave the first of (27) unchanged: H is defined and analytic over the domain $D = D(R, \rho, \sigma)$ around p^*, and then one should possibly restrict the domain because of the invertibility condition, as explained above. Applying Theorem 1A around p (and not p^*) does not change anything.

In the second of (27), however, one should beware of the fact that Ω refers to p ($\Omega = \|\omega(p)\|$), and not p^* ($\Omega^* = \|\omega^*\|$). In order to express everything with parameters centred at p^*, we may add for example the following condition:

$$|\Omega - \Omega^*| \leq \frac{1}{2}\Omega^*,$$

which holds in particular if

$$M\|p-p^*\| \leq M r \leq \frac{1}{2}\Omega^*.$$

But this is precisely equivalent to the second inequality in (27) (see the "model statement", condition ii) in Chapter II), with ω replaced by Ω^*. In short, to take care of these details, it is enough to substitute Ω^* for Ω in the second of (27). Then in the definition of T_0 (compare (26) in Chapter II) one replaces Ω by $\frac{3}{2}\Omega^*$.

The third inequality in (27) remains unaltered. As regards the fourth, one notices that the exponent $\frac{1}{2}(1-3\alpha)$ is larger than 1/5 (or even 1/4 if $n \geqslant 3$); here we have already taken into account the substitution $n \to n-1$, to be effected after the transformation (1). One then adds the invertibility condition together with (6), with $n-1$ instead of n. Summarising, we have proved the following statement.

Theorem 2. *For any initial point $(p(0), q(0))$ $(p(0) = p^*)$, the trajectory $(p(t), q(t))$ starting at $(p(0), q(0))$ satisfies*

$$\|p(t) - p(0)\| \leq 10^{-2} \frac{\sigma}{M} \varepsilon^{\alpha} \quad \text{if} \quad |t| \leq \mathcal{T}(\varepsilon) = \mathcal{T}_0^* \exp(\varepsilon^{-a}),$$

where $a = a(n) = \dfrac{1}{2(n+1)}$, $\Omega^* = \| \omega(p(0)) \|$, $\mathcal{T}_0^* = 3 \cdot 10^{-2} \dfrac{\sigma}{\Omega^*}$.

This holds provided that ε satisfies the following inequalities:

$$(7) \quad \begin{array}{lll} \varepsilon^a \leq 100 \dfrac{M}{\sigma} \inf(R, \rho), & \varepsilon^a \leq 200 \dfrac{M\Omega^*}{\sigma m}, & \varepsilon^a \leq 4 \cdot 10^{-2} \dfrac{m}{M}, \\ \varepsilon \leq \tau^5, & \varepsilon^a \leq 200 \dfrac{M^2}{\sigma |h|_3}, & \varepsilon^a \leq \tau^2 \left(\dfrac{\lambda m}{\sqrt{n-1}} \right)^{2(n-1)}, \end{array}$$

where λ and τ are defined in formula (26) *of Chapter* II, *and* $|h|_3$ *is the maximum of the third derivative of h over the domain D.*

In this statement, all the parameters connected with the Hamiltonian, along with time t, are those which are obtained after the rescalings (1) *have been performed, that is, one should use the primed quantities in* (1).

Although the threshold conditions seem to proliferate somewhat dangerously, only the last one is really significant. In particular, all but this last one read exactly the same whatever the value of a, inside the interval $\left(0, \dfrac{1}{2n+1}\right)$. On the other hand, the last condition is essentially a rewriting of (5), which is a direct consequence of the Dirichlet estimate. In short, the value of the first exponent, which governs the time of stability, is a very direct descendant of the exponent $1/n$ which appears in Dirichlet's theorem. In particular, when the number of degrees of freedom increases, results deteriorate, not because of the invasion of the phase space by the resonance surfaces, but because of the relative scarcity of rational points, that is, periodic tori. We shall see below how this new point of view may be exploited further.

Returning to the last condition, we notice that this is also the only place in which n appears explicitly. The factor $\sqrt{n}$ (or $\sqrt{n-1}$) is simply the length of the diagonal of the unit cube, and occurs because one uses Euclidean norms, whereas the sup norm is more natural when dealing with approximation theory; this is of little importance. Apart from this, the last threshold is very sensitive to the value of α (or a), and it vanishes when $a = 1/(2n+1)$. If one chooses for instance $\alpha = 1/(2n+5)$, which satisfies $\frac{1}{2}(1-3\alpha) - n\alpha = \alpha$, one comes up with the condition

$$\varepsilon^a \leq \tau \left(\frac{\lambda m}{\sqrt{n-1}} \right)^{n-1}, \quad a = \frac{1}{2n+3},$$

which is much weaker than the last inequality in (7).

It is important to notice that the lack of optimality of Theorem 2 is exactly the same as that of Theorem 1A. Indeed, the only new ingredient we have added is Dirichlet's theorem, which is optimal (at least as far as the quantities we are interested in are concerned).

It is in our opinion quite remarkable that essentially the best possible perturbation result over finite times may be obtained using the most basic approximation result, but various important refinements and improvements of Theorem 2 are easily derived for certain classes of initial conditions. We devote the end of this chapter to some of them, using freely the notions and notations of Appendix 1.

Placing arithmetical conditions on the frequency is essentially equivalent to ensuring some comparison between q (or T) and Q in Dirichlet's theorem. In particular, the following statement is true.

***Corollary* 1.** *Assume that after the rescalings* (1) *one has* $\omega^* = \omega(p^*) = (1, \omega')$, *with* $\omega' \in \Omega_{n-1}(\delta, \gamma)$, $(\gamma, \delta > 0$; *we avoid the letter* τ, *which has been used already in this context). Then in Theorem* 2 *one may replace the radius of confinement by*

$$\|p(t) - p^*\| \leq 10^{-2}\frac{\sigma}{M}\frac{\varepsilon^a}{T} \leq 10^{-2}\frac{\sigma}{M}\left(\frac{\lambda m}{\sqrt{n-1}}\right)^{\frac{n-1}{1+\delta}}\frac{\varepsilon^b}{\gamma},$$

where $b = a\dfrac{n+\delta}{1+\delta}, a = \dfrac{1}{2(n+1)}$.

The time of stability and the threshold conditions remain the same.

In particular, almost all points in phase space admit, for any $\eta > 0$, *the stability exponents*

$$(a, b) = \left(\frac{1}{2n+1} - \eta, \frac{n}{2n+1} - \eta\right).$$

The proof is straightforward; we have in fact already written the first inequality on the norm $\|p(t)-p^*\|$. Now, when $\omega^* \in \Omega_n(\delta, \gamma)$, T may be estimated from below, since

$$Q^{-\frac{1}{n}} \geq \|T\omega^*\|_{\mathbb{Z}} \geq \left(\frac{\gamma}{T}\right)^{\frac{1}{n}(1+\delta)}.$$

Q is given by (4) and one only needs to substitute it; of course, (1) is used first in order to rescale one of the components. The last assertion comes from the proposition in Appendix 1, together with the fact that in Theorem 2 one may replace $1/(2n+2)$ by $\dfrac{1}{2n+1} - \eta$ for any $\eta > 0$ (and vanishing threshold when η goes to zero). □

Note that the pair of exponents comes very close to the would-be optimal pair $\left(\dfrac{1}{2n}, \dfrac{1}{2}\right)$. The assertion may be slightly misleading, because although almost every point belongs to $\Omega_n(\delta)$ for any $\delta > 0$, it is also the case that for almost every point the corresponding constant γ approaches 0 together with δ.

One may devise a result of a slightly different nature; fix $\delta > 0$ and consider $\gamma = \gamma(\varepsilon)$, going to infinity as ε goes to 0, for example, $\gamma = \gamma_0\varepsilon^{-\xi}$, $\gamma_0 > 0$, $0 < \xi < b = b(n, \delta)$. One then gets the same result, over the set (after rescaling) $\Omega_{n-1}(\delta, \gamma(\varepsilon))$, whose relative measure goes to 1 as ε goes to 0, with a second exponent $b(n, \delta)-\xi$, and a *fixed* value γ_0.

Suppose now that we wanted to apply Theorem 1B or 1C in order to find results for general initial conditions. We would come across a kind of intermittency phenomenon, which is perhaps worth noticing. Let $(T_i)_{i\geqslant 0}$ be the sequence of the periods of ω^*, and $(\omega_i)_{i\geqslant 0}$ the related best approximations. The rational vectors ω_i converge to ω^*, so one has $T_i\omega_i \in \mathbb{Z}^n$, and the estimate

$$\|\omega_i - \omega^*\| \leq \frac{\sqrt{n}}{T_i T_{i+1}^{1/n}}.$$

For i large enough, let us define the corresponding points p_i, converging to p^* ($\omega(p_i) = \omega_i$), and let

$$r_i = \|p_i - p^*\| \leq \frac{\sqrt{n}}{m}\frac{1}{T_i T_{i+1}^{1/n}}. \tag{8}$$

Finally, we define the sequence of values ε_i satisfying $r_i = r_0\varepsilon_i^{1/3}$, where r_0 still denotes the constant in formula (29) of Chapter II. With these definitions and the same setting as above, the following statement holds.

Let $\varepsilon > 0$, $\varepsilon_{i-1} \geqslant \varepsilon > \varepsilon_i$, i large enough; then

$$\|p(t) - p^*\| \leq \frac{8M}{m} r_{i-1}$$

if $|t| \leqslant \mathcal{T}_i^* = \mathcal{T}_0^* \exp(\mu T_i^{1/(n-1)})$, with $\mathcal{T}_0^* = 3 \cdot 10^{-2}\frac{\sigma}{\Omega^*}, \mu = 10^{-2}\frac{\sigma}{\sqrt{n-1}}\left(\frac{m}{M}\right)^2$.

As usual, the parameters relate to the situation after transformation (1).

This holds for any initial condition ($p^* = p(0)$, $q(0)$), but we do not state this rather unnatural assertion as a "theorem", nor bother to mention the thresholds, which could be easily computed. The proof is again very short; just apply Theorem 1C around the point p_{i-1}, which is valid because $r_{i-1} \geqslant r_0\varepsilon^{1/3}$, by the definition of ε and of the sequence (ε_i). Next, to estimate the exponent $\lambda/(r_{i-1}T_{i-1})$ from below, use (8):

$$\frac{\lambda}{r_{i-1}T_{i-1}} \geq \frac{\lambda m}{\sqrt{n}} T_i^{\frac{1}{n}}.$$

Finally, notice that the factor $\lambda m/\sqrt{n}$ is slightly larger than μ, after the substitution of $n-1$ for n. □

Both the radius of confinement and the time of stability remain constant when ε belongs to an interval $(\varepsilon_i, \varepsilon_{i-1})$. In fact, the most favourable situation occurs when ε is equal to one of the ε_i's. This is also apparent if one applies Theorem 1B: if $\varepsilon = \varepsilon_{i-1}$, it may be applied around the point p_{i-1}; but as soon as ε crosses this value, p^* leaves the influence zone of p_{i-1} and one must use the point p_i instead. This causes the time of validity to drop discontinuously from a quantity proportional to $\exp(\tau/(T_{i-1}\varepsilon^{1/3}))$ to one proportional to $\exp(\tau/(T_i\varepsilon^{1/3}))$.

We notice that the quantities T_i, ω_i, p_i, r_i have an obvious intrinsic meaning, but the sequence (ε_i) is rather artificial, again partially because of the exponent 1/3 instead of 1/2 in Theorem 1B. What is really important is that the distribution of the rational vectors around a given one may be quite erratic (for example, nothing can be said in general about the sequences T_i/T_{i+1} or r_i/r_{i+1}) and there arises a sequence of values for which the closed orbit approximation is relatively best possible.

With this in mind, it is not surprising that one can prove statements of the same type as Corollary 1, using Theorem 1B and the Diophantine sets $\Omega(\tau, \gamma)$; in fact, by the very definition of these sets (see Appendix 1), it is then possible to estimate, for example, the ratios T_i/T_{i+1} from below, which enables one to derive results that are valid for all sufficiently small perturbations over a set of large measure (or even almost everywhere). Recalling from Appendix 1 the inclusions

$$\Omega_n(\tau,\gamma) \subset \Omega(\tau, \gamma^{-(1+\tau)}),$$

one may then compare the statement obtained from Theorem 1B with Corollary 1, over sets of type $\Omega_n(\tau, \gamma)$, and make sure that they are essentially equivalent. We shall not go into the easy details.

As a final remark on this topic, we note that it may sound somewhat paradoxical to introduce Diophantine conditions while studying the behaviour of a system over *finite* times, because these arithmetical conditions are essentially of asymptotic nature. In fact, there are two additional flexibilities which we have not used.

1. We are interested in phenomena which occur over exponentially long times; on the other hand, the sequence of the periods (T_i) of *any* vector increases at least geometrically (see Appendix 1). Therefore we could restrict attention to indices i such that $i = O(\varepsilon^{-c})$ for some $c > 0$; this is the "simultaneous" analogue of the ultraviolet cut-off, and it is naturally interpreted in terms of approximate recurrence times (again see Appendix 1).

2. There is an additional freedom related to the initial condition. Suppose that we divide the radius $r(\varepsilon)$ of the influence zone into—say—two equal parts; it is then sufficient to find a point whose frequency has nice arithmetical properties and which lies within $r(\varepsilon)/2$ of the given initial condition p^*.

It may be that these two remarks can be combined to show that the estimate of the radius of confinement which appears in Corollary 1 holds in fact for *any* point in phase space, so the second stability exponent would indeed always be close to 1/2. In any case, the second remark will be put to use below, to derive Corollary 3.

Let us now pass to the interpretation of the resonances, and to what happens when the initial condition is resonant or even only *nearly* resonant. Loosely speaking, the important property that emerges is that the more resonant the initial condition, the more stable the corresponding trajectory will be. In fact it should be remembered that the more resonant points correspond precisely to rational vectors, that is, unperturbed closed orbits, and that we have already noticed that Theorems 1A, 1B, 1C are essentially independent of the number of dimensions. We strongly emphasize that this stabilization *through* resonance is very specific of quasi-convex systems and cannot possibly hold for generic steep Hamiltonians.

Let us first recall some usual notions. Let $\mathcal{M}$ be a submodule (or sublattice of $\mathbb{Z}^n$ of rank (dimension) r, generated over $\mathbb{Z}$ by the linearly independent vectors $k_1, ..., k_r$ of $\mathbb{Z}^n$. A vector $\omega \in \mathbb{R}^n$ is said to be resonant with *multiplicity* r and *associated module* $\mathcal{M}$ (we write $\mathcal{M}$-resonant) if $\omega \cdot k = 0$ for any $k \in \mathcal{M}$, which is of course equivalent to $\omega \cdot k = 0$, $i = 1, ..., r$. With $\mathcal{M}$ we also associate the corresponding *resonant surface* $\Sigma_{\mathcal{M}}$, consisting of the points p in action space whose corresponding frequency $\omega(p)$ is $\mathcal{M}$-resonant:

$$\Sigma_{\mathcal{M}} = \left\{p \in \mathbb{R}^n,\ \left(\omega(p), k_i\right) = 0,\ i = 1, \dots, r\right\}.$$

Since the frequency map is a local diffeomorphism, $\Sigma_{\mathcal{M}}$ is a smooth manifold of dimension $d = n-r$. In this classical framework, we prove the following result.

***Corollary* 2.** *Let $\mathcal{M}$ a submodule of $\mathbb{Z}^n$ of rank r, $0 \leqslant r \leqslant n-1$, $p^* \in \mathbb{R}^n$, $\omega(p^*) = \omega^*$. Assume that ω^* is $\mathcal{M}$-resonant, that is, $p^* \in \Sigma_{\mathcal{M}}$. Then there exist positive constants $c(\mathcal{M})$ and $c'(\mathcal{M})$ (to be constructively defined in the proof) such that Theorem* 2 *and Corollary* 1 *remain valid at the points $(p(0) = p^*, q(0))$ ($q(0)$ arbitrary) with the following changes:*

a) *one replaces everywhere n by $d = n-r$ and $\sqrt{n-1}$ by $c(\mathcal{M})\sqrt{d-1}$.*

b) *in the transformation* (1) *one replaces $w = \|\omega^*\|_\infty$ by $c'(\mathcal{M})w$.*

c) *finally, in Corollary* 1, *the expression "almost all points" should be interpreted as "almost everywhere on the resonant surface $\Sigma_{\mathcal{M}}$", equipped with the natural superficial measure.*

The basic idea is that resonant surfaces should be viewed as loci which contain abnormally many rational vectors. Thus, Corollary 2 will follow almost immediately from the next lemma.

***Lemma* 3.** *Let $\mathcal{M}$ a submodule of $\mathbb{Z}^n$ of rank r, $0 \leqslant r \leqslant n$, and let $\alpha \in \mathbb{R}^n$, be an $\mathcal{M}$-resonant vector. Writing $d = n-r$, there exists $c(\mathcal{M})$ such that if $Q > 1$ is real one can find an integer q satisfying $1 \leqslant q < Q$ and such that*

$$\|q\alpha\|_{\mathbb{Z}} \leq c(\mathcal{M})Q^{-\frac{1}{d}}.$$

$c(\mathcal{M})$ is defined as the smallest constant with the above property and the integer $c(\mathcal{M})/c(\{0\})$ will be called the order of the resonance (or submodule).

The assertion is obvious when $\mathcal{M}$ defines the "standard" resonance, that is, when α has the form $\alpha = (0, \alpha')$, where $0 \in \mathbb{R}^r$ is the zero vector and $\alpha' \in \mathbb{R}^d$. The proof of the lemma is then nothing but an exercise in linear algebra, by means of which we can reduce everything to this case, but we shall go into some details, for the sake of completeness. Before this, we note that Dirichlet's theorem asserts that $c(\{0\}) \leqslant 1$, but that equality does *not* hold (see Appendix 1), which is the reason why we defined the order as above; since however $c(\{0\})$ is close to 1 and the difference is completely irrelevant for our purpose, we shall occasionally indulge in calling $c(\mathcal{M})$ itself the order of the resonance.

Let $K = (k_i^{(j)})$ be an $r \times n$ matrix whose rows are vectors k_i which generate $\mathcal{M}$ over $\mathbb{Z}$: $k_i = k_i^{(j)}$, $j = 1, \ldots, n$. Since K has integer entries, a classical result from linear algebra asserts that it may be written $K = B\Delta A$, with $B \in Gl_r(\mathbb{Z})$ and $A \in Gl_n(\mathbb{Z})$ *invertible* square matrices; Δ has the form $\Delta = [D \,|\, 0_d]$. Here 0_d is the zero matrix of order $d = n-r$ and D is diagonal: $D = \operatorname{diag}(d_1, \ldots, d_r)$; moreover, d_j is a multiple of d_i for $i \leqslant j$. The positive integers d_i are often called the invariants of $\mathcal{M}$. We say that the module is *primitive* when they are all equal to unity, which is the same as requiring that $d_r = 1$ or else that the determinants of all the $r \times r$ submatrices of K be mutually prime. One has then $D = \mathbb{1}_r$, and we write $\Delta = \Pi$, because this is a projection operator. Any module is contained in a unique primitive one (obtained by replacing the original Δ by Π) which defines the same resonance, so that one may restrict attention to primitive modules. This stems from the obvious equivalences

$$\alpha \ \mathcal{M}\text{-resonant} \iff K\alpha = 0 \iff \Delta A\alpha = 0 \iff \Pi A\alpha = 0.$$

So let $\mathcal{M}$ be primitive and denote by (e_i), $i = 1, \ldots, n$, the standard basis of $\mathbb{Z}^n$. The $\mathcal{M}$-resonant vectors are generated over $\mathbb{R}$ by the vectors $A^{-1}e_i$, $i = r+1, \ldots, n$. Let $\|u\|_\infty$ denote the sup norm as usual and if $M = (m_{ij})$ is a matrix, let $\|M\|_\infty$ denote the corresponding operator norm, that is:

$$\|M\|_\infty = \sup_{u, \|u\|=1} \|Mu\|_\infty = \sup_i \sum_j |m_{ij}|.$$

Now suppose one wants to approximate an $\mathcal{M}$-resonant vector α; $A\alpha$ lies in the subspace $\mathbb{R}^d$ spanned by (e_i), $i = r+1, \ldots, n$. Apply Dirichlet's theorem in this space to find $q \in \mathbb{N}$ such that $\|qA\alpha\|_{\mathbb{Z}} \leqslant Q^{-1/d}$; then

$$\|q\alpha\|_{\mathbb{Z}} \leq \|A^{-1}\|_\infty Q^{-\frac{1}{d}},$$

because A has *integer* entries. This proves Lemma 3 and the estimate $c(\mathcal{M}) \leqslant \|A^{-1}\|_\infty$. □

Corollary 2 is a direct consequence of this lemma, because if ω^* is $\mathcal{M}$-resonant, then using Lemma 3 the estimate (2) may be changed to

$$\|\omega - \omega^*\| \leq \frac{\sqrt{d}c(\mathcal{M})}{TQ^{1/d}},$$

and the reader will check that everything follows from this, except for the preliminary rescaling. To be more specific, one uses the linear symplectic transformation $(p, q) \to (p', q') = ({}^tA^{-1}p, Aq)$ to reduce the situation to the standard resonance case. One then uses transformation (1) to gain one more dimension, and obtain a frequency vector of type $(0, \ldots, 0, 1, \omega')$ with $\omega' \in \mathbb{R}^{d-1}$. In this way, one has rescaled a component of $A\omega^*$, and this is how the factor $c'(\mathcal{M})$ arises. In fact, this shows that $c'(\mathcal{M}) \leqslant \|A\|_\infty$.

Finally, the last assertion of the corollary about the interpretation of the expression "almost all points" which arises in Corollary 1 should be clear from the above. □

We shall add some simple remarks about the geometric meaning of the constants $c(\mathcal{M})$ and $c'(\mathcal{M})$ and obtain slightly better estimates for them.

By construction, the last d columns of A^{-1} provide integer vectors which are orthogonal to $\mathcal{M}$, and in fact they generate over $\mathbb{Z}$ the primitive module $\mathcal{M}^\perp$ orthogonal to $\mathcal{M}$. In other words, the $n \times d$ matrix E composed of the last d columns of A^{-1} defines a linear embedding of $\mathbb{Z}^d$ into $\mathbb{Z}^n$ whose image coincides with $\mathcal{M}^\perp$. One may obviously refine the estimate of $c(\mathcal{M})$ to

$$c(\mathcal{M}) \leq \|E\|_\infty = \sup_{i=1,\ldots,n} \sum_{j=r+1}^{n} |(A^{-1})_{ij}|.$$

Since A has determinant ± 1, its inverse is simply the cofactor matrix. One can still minimize this with respect to the possible matrices E, that is, with respect to the possible embeddings of $\mathbb{Z}^d$ into $\mathbb{Z}^n$ with image $\mathcal{M}^\perp$. In other words, E can be replaced by ET, where $T \in Gl_d(\mathbb{Z})$, which corresponds to changing A into

$$\begin{pmatrix} \mathbb{1}_r & 0 \\ 0 & T^{-1} \end{pmatrix} A,$$

using block notation.

In a parallel way, the d last *rows* of A provide vectors which generate a module, or lattice, $\mathcal{M}'$ such that $\mathcal{M} \oplus \mathcal{M}' = \mathbb{Z}^n$ and one has the estimate

$$c'(\mathcal{M}) \leq \sup_{i=r+1,\ldots,n} \sum_{j=1}^{n} |A_{ij}|,$$

with a further minimization over the possible choices.

Of course the matrix B, whose value does not enter, simply corresponds to a possible change of basis of $\mathcal{M}$ itself: changing the basis changes K into PK with $P \in Gl_r(\mathbb{Z})$, and choosing $P = B^{-1}$ reduces the general situation to the case $B = \mathbb{1}_r$.

We also note, because it may sometimes be useful, that it is easy to write down explicitly a basis of *integer vectors* for the real subspace orthogonal to $\mathcal{M}$ ("resonant plane"). Assume in fact that $k_1, \ldots, k_r, e_{r+1}, \ldots, e_n$ span the whole space $\mathbb{R}^n$, which is always the case, up to a possible relabelling; then set

$$l_i = k_1 \wedge \cdots \wedge k_r \wedge e_{r+1} \wedge \cdots \wedge \widehat{e_{r+1}} \wedge \cdots \wedge e_n, \quad i = 1, \ldots, d,$$

where $\wedge$ denotes the ordinary exterior product and the vector under the "hat" is omitted. The l_i's generate over $\mathbb{Z}$ a module $\mathcal{L}, \mathcal{L} \subset \mathcal{M}^\perp$, and the span over $\mathbb{R}$ ($\mathcal{L} \otimes \mathbb{R}$) is the plane orthogonal to $\mathcal{M}$; in general $\mathcal{L}$ is not primitive, so $\mathcal{L} \neq \mathcal{M}^\perp$.

As a final remark, let us consider the case when $r = n-1$, that is, the "maximally resonant" case, or that of rational vectors. Then Corollary 2 should and indeed does reduce to Theorem 1A, except for a few minor losses which occur while going all the way round. When $r = n-1$, and writing ω instead of α, one has $A\omega = (0, \ldots, 0, \nu)$ with $\nu > 0$ (up to a possible change of sign in A), so

$$\nu = \sum_j A_{nj}\omega_j,$$

where A_{nj} is the last row of A. The period $T = 1/\nu$ and $T\omega \in \mathbb{Z}^n$ is the last column of A^{-1}. In other words, suppose that $\omega \cdot k_i = 0$, $i = 1, \ldots, n-1$, and the $n-1$ square matrices of size $n-1$ obtained by deleting a column from the matrix K have mutually prime determinants. If $k_n \in \mathbb{Z}^n$ satisfies $\det(k_1, \ldots, k_n) = \pm 1$, the period is given by $T = |\omega \cdot k_n|^{-1}$.

We shall now refine Corollary 2, showing that the initial point p^* need not be situated exactly on the resonant surface. This uses the remark we made above, that instead of approximating the initial point itself, one may use another, sufficiently close point. So, consider again the module $\mathcal{M}$, the associated resonant surface $\Sigma_{\mathcal{M}}$, and a point p^* lying at a distance from $\Sigma_{\mathcal{M}}$ less than $r(\varepsilon)/2$. Here (see Theorem 1A), one has $r(\varepsilon) = \lambda\frac{\varepsilon^\alpha}{T}$, and we want to estimate this from *below* as T runs through the values prescribed in Theorem 1A. This was done already in Chapter II, towards the end of the proof of Theorem 1A, to the effect that

$$r(\varepsilon) \geq r_0\varepsilon^{\frac{1}{2}(1-\alpha)} > r_0\varepsilon^{\frac{1}{2}},$$

with r_0 defined in (29) of Chapter II. So let p^* satisfy

$$\mathrm{dist}(p^*, \Sigma_{\mathcal{M}}) \leq \frac{r_0}{2}\varepsilon^{\frac{1}{2}};$$

we apply Theorem 2, in the version of Corollary 2, to a point of $\Sigma_{\mathcal{M}}$ as close to p^* as possible (it is not necessarily unique but it does not matter). The only difference is that the influence zones should be shrunk, so that the result effectively applies to p^*. So we also approximate the point on the surface by means of rational points within $r(\varepsilon)/2$, instead of $r(\varepsilon)$. Looking back to equation (3), one sees that formally this is equivalent to changing $\sqrt{n}$ into $2\sqrt{n}$, or rather $c(\mathcal{M})\sqrt{d-1}$ into $2c(\mathcal{M})\sqrt{d-1}$. This proves the following result.

***Corollary* 3.** *Let $\mathcal{M}$ be a submodule of $\mathbb{Z}^n$ of rank r, and $\Sigma_{\mathcal{M}}$ the associated resonant surface of dimension $d = n-r$; let p^* be a point in action space satisfying*

$$\operatorname{dist}(p^*,\Sigma_{\mathcal{M}}) \leq \frac{r_0}{2}\varepsilon^{\frac{1}{2}} = \frac{m}{8M}\sqrt{\frac{2E}{M}}\varepsilon^{\frac{1}{2}}.$$

Then for any point $(p(0) = p^*, q(0))$, Theorem 2 applies with the replacement of n by d and $\sqrt{n-1}$ by $2c(\mathcal{M})\sqrt{d-1}$; in the preliminary transformation (1), $w = \|\omega^*\|_\infty$ *should be changed to $c'(\mathcal{M})w$.*

Of course, one could also devise a — somewhat far-fetched — statement in the spirit of Corollary 1. We believe that Corollaries 2 and 3 should have important and far-reaching consequences. Roughly speaking, one may remember that initial conditions which belong to a tubular neighbourhood of thickness $O(\sqrt{\varepsilon})$ of a resonant surface of dimension d will be stable (in action space) for a time of the order of $\exp(c\varepsilon^{-1(2d)})$; but, of course, the order of the resonance comes into play and, given ε, this will break if this order is too high. We slightly elaborate on this heuristic picture in Chapter V, §2.

To put it differently, define subsets of phase space by

$$\mathcal{F}(d_0, c_0, \varepsilon) = \{(p^*, q^*) \in \mathbb{R}^n \times \mathbb{T}^n, \text{ such that there exists } \mathcal{M}, \text{ a submodule of } \mathbb{Z}^n, \operatorname{corank} \mathcal{M} \leqslant d_0, c(\mathcal{M}) \leqslant c_0, \text{ and } \operatorname{dist}(p^*, \Sigma_{\mathcal{M}}) \leqslant \frac{r_0}{2}\cdot\varepsilon^{1/2}\},$$

with $c_0 > 0$ and $d_0 \in \mathbb{N}$ $(1 \leqslant d_0 \leqslant n)$. Then on such a subset the stability of the action variables is essentially that of a system with d_0 degrees of freedom. It is of course tempting to let n tend to infinity (thermodynamical limit) or simply be infinite from the start (see Chapter IV, §3).

Corollaries 2 and 3 also demonstrate that there should be, *for quasi-convex systems*, a competition between stability over finite times and perpetual KAM stability, which applies, roughly speaking, to "very non-resonant" frequencies. This may be relevant in particular in celestial mechanics, as detailed below (Chapter IV, §1).

CHAPTER IV

TRANSPOSITIONS, APPLICATIONS, PROSPECTS

§1. Additional variables and an application to celestial mechanics

There is one important extension of the above results which does not require any extra work, namely one may add canonical variables in the perturbation. That is, let

$$H(p,q,I,\phi)=h(p)+\varepsilon f(p,q,I,\phi),\quad (p,q)\in\mathbb{R}^n\times\mathbb{T}^n,\quad (I,\phi)\in\mathbb{R}^m\times\mathbb{T}^m;$$

if h is quasi-convex, all the results above carry over, obtaining of course stability of the p variables only. To check this, just go through the proofs again and make sure that nothing is altered by the addition of "dummy" variables (this remark also applies in the general steep case; see [43], §1.5). It should be emphasized that such systems are degenerate from the standpoint of KAM theory, which extends to them only under some rather restrictive additional assumptions. Essentially, one should have

$$f(p,q,I,\phi)=f_1(p,I)+\varepsilon f_2(p,q,I,\phi),$$

along with the corresponding non-degeneracy condition.

As a first class of applications, one may treat in this way the "adiabatic-integrable" situations, that is, Hamiltonians of the form

$$H(p,q,\varepsilon t)=h(p)+\varepsilon f(p,q,\varepsilon t),$$

where f is periodic in $\tau=\varepsilon t$. Introducing the variable e, canonically conjugate to τ, brings this to the form

$$H(p,q,e,\tau)=h(p)+\varepsilon\left[e+f(p,q,\tau)\right],$$

which is of the type considered above.

Here we want to mention an important application, which may have far-reaching consequences in celestial mechanics: the problem of planetary systems. Since it is discussed at length by Arnol'd ([2]) in connection with the conservation of tori and by Nekhorochev ([43], §1.18 and §12) from the same viewpoint as ours, namely stability over exponential times, we shall be quite sketchy about the setting of the problem. Our results will however be not only quantitatively better than those of [43], but also qualitatively different, because Corollaries 2 and 3 seem indeed to open new perspectives, when applied in this context.

So, one wants to examine the particular case of the many-body problem in which one of them (the sun) is much heavier than the others (the planets). If the interactions among the planets is neglected, these travel along mutually independent Keplerian orbits, which are determined by their elliptic elements: the major semi-axis, the eccentricity, and the inclination, along with the corresponding angles. For reasons to be sketched below, one has to restrict attention to the case of small eccentricities and small mutual inclinations, that is, to a neighbour of the plane circular problem. The best suited variables are then the so-called Poincaré heliocentric variables. We refer to [45] (§§8–12) or to [2] (Chapter III, §2) for their definition. They read $(\Lambda, H, Z, \lambda, h, \zeta)\in(\mathbb{R}^n)^3\times(\mathbb{T}^n)^3$; the action variables (Λ, H, Z) are simple functions of the semi-axes, eccentricities, and mutual inclinations; when eccentricities and inclinations are small, it is best to pass to symplectic polar coordinates in the pairs (H, h) and (Z, ζ), obtaining the variables $(\Lambda, \lambda, \xi, \eta, p, q)$ where, componentwise,

$$H = \frac{1}{2}(\xi^2 + \eta^2), \quad Z = \frac{1}{2}(p^2 + q^2).$$

The mass of the sun may be normalized to unity and those of the planets written as $m_i = \varepsilon \mu_i$, where ε is the ratio of the mass of the heaviest planet to that of the sun (for the solar system, $\varepsilon \approx 10^{-3}$). With these notations, the Hamiltonian reads

$$H = h(\Lambda) + \varepsilon f(\Lambda, \lambda, \xi, \eta, p, q, \varepsilon), \quad h(\Lambda) = -\frac{1}{2}\sum_{i=1}^{n} \frac{\mu_i^3}{\Lambda_i^2}.$$

Moreover, $\Lambda_i = \mu_i \sqrt{a_i}$, a_i being the major semi-axis of the ellipse osculating to the trajectory of the i-th planet at a given time; hence, controlling Λ is equivalent to controlling the semi-axes.

Before applying any theorem, one should make sure that the perturbation is indeed small, and this is true only so long as the $n+1$ bodies do not come too close to each other. Since we shall have control on the a_i's only, the only region in phase space when this implies estimates on the eccentricities and inclinations is near the plane circular problem. This is because plane circular motion with the planets travelling in the same direction achieves a maximum of the angular momentum of the system, and the latter is a conserved quantity. More precisely, let G_i be the angular momentum vector of the i-th planet ($\|G_i\| = m_i(a_i(1-e_i^2))^{1/2}$, e_i the eccentricity), $G = \Sigma_i G_i$, the total angular momentum, and $N = \varepsilon^{-1}G$, independent of ε. Pick $2n$ positive numbers α_i, β_i satisfying

$$0 \le \alpha_1 \le \beta_1 \le \alpha_2 \le \cdots \le \alpha_n \le \beta_n.$$

A *domain of planetary motion* $B(\alpha, \beta, \gamma)$ is a region of phase space such that

$$\alpha_i \le a_i \le \beta_i, \quad i = 1, \ldots, n \text{ and } \|N\| \ge \gamma;$$

γ is a number such that $0 < \gamma_0(\alpha, \beta) < \gamma < \gamma_m(\alpha, \beta)$. Here $\gamma_m(\alpha, \beta)$ is the maximal possible value of $\|N\|$ under the conditions imposed on the a_i's; it corresponds to plane circular motions with radii β_i, and $\gamma_0(\alpha, \beta)$ is the largest value of $\|N\|$ corresponding to possible collisions among the planets and/or with the sun. We refer to [43] (§12) for a detailed discussion of the fact that on a domain $B(\alpha, \beta, \gamma)$ one may indeed apply the results about stability over exponentially long times; this discussion carries over without any change.

Since $h(\Lambda)$ is a convex function, Theorem 2 applies, yielding stability of the major semi-axes over exponentially long times. The implications of Corollaries 2 and 3, however, are much more intriguing. Indeed, these assert that resonant, or even nearly resonant, trajectories are *privileged*, from the point of view of finite time stability. Resonance here simply means resonance between the inverses of the periods of the motions along the instantaneous ellipses, that is, the inverses of the "years"; the relation between these periods of revolution and the values of the semi-axes, that is, Kepler's third law, follows from the expression for $h(\Lambda)$.

But there has been a long-standing discussion about the fact that celestial bodies seem to pick resonant trajectories more often than could be expected from a mere statistical effect. These speculations about "harmonic motions" could be traced back to Pythagoras, Plato or Kepler, but in modern terms this was forcefully advocated by Molchanov (see [40], [41], [5], [27]) who noticed the existence of many "simple" resonance relations between the planets of the solar system and inside the satellite subsystems around Jupiter, Saturn and Uranus. He was immediately strongly criticized on the ground that these relations were not really "astonishing" and would often occur among numbers or vectors picked "at random"; he then replied to these criticisms, trying in particular to give a precise definition of the adjective "simple" used above. Since no repeatable experiment can be performed in this case, the evidence is bound to remain fragile. In any case, since then a lot of work has been devoted to the subject, including resonances which involve artificial satellites. Many of these resonances are ascribed to non-Hamiltonian causes, for example tidal effects, but there seems to remain some "mystery" buried in a mass of controversial observations. In Molchanov's terms: "Why are planets and satellites locked into simple resonances, whereas the rings of Saturn or the asteroid belt have gaps in these places?" Even if particular assertions may be challenged, this seems to ask us an authentic riddle.

The results above offer the first purely Hamiltonian partial explanation for this; if the bodies must linger much longer about resonant trajectories than elsewhere, after some time these will become indeed the most populated places. This is not so simple, however, and in accordance with the spirit of the above quotation, we have really set up a "competition" between finite time stability and perpetual stability of the KAM type, since the latter favours very *non*-resonant trajectories. According to the concrete situation at hand, it is quite possible that one or the other kind of stability actually prevails. In this context, we insist that the stability estimates imply that the bodies remain locked in resonance zones for exponentially long times, but of course they do not preclude small amplitude $(O(\sqrt{\varepsilon}))$ "chaotic" motions inside such a zone, on much shorter timescales.

One should also note that the present results have a wider range of validity than KAM results (specifically the theorem proved in [2]). First, from a practical point of view, although it is not realistic, the threshold of validity which we obtain is not nearly as small as the one of KAM theory; it could even perhaps be pushed to some realistic value, using computer assisted estimates. Second, finding invariant tori of *maximal* dimension (half the dimension of the phase space) requires that the unperturbed system be integrable with respect to *all* the variables. Here, this translates into the fact that one must perturb from the *exact* plane circular problem (as in [2]) and so, in the perturbed problem, the eccentricities and the inclinations should be of the order of (a power of) the perturbation parameter, that is, extraordinarily small. It should be noted however that a version of KAM theory has been developed in which one looks for *low-dimensional* tori, that is, tori which are *not* of maximal dimension (see in particular [10], [47], [53], [57] and references therein); this in turn requires only *partial* integrability of the system, as is the case here. To our knowledge, this theory has never been applied specifically to the planetary problem, although the difficulties are probably of a technical nature only (see however [51]). More significant may be the fact that the set of tori one thus finds is of zero Lebesgue measure. We shall briefly comment on this when discussing Arnol'd's diffusion in Chapter V, §2.

Returning to the results on stability over finite times, these only require that the system be close enough to the plane circular problem so as to avoid collisions. This is the condition $||N|| > \gamma_0$ in the definition of a domain of planetary motions, where γ_0 is independent of ε. It defines an "order 1" neighbourhood of the plane circular problem, the most favourable case arising when all the planets have the same mass; indeed no condition on the momentum can possibly prevent collision as the mass of at least one planet vanishes, as in the restricted three-body problem.

§2. Transposition to other contexts and degenerate cases

The results of Chapters II and III can be transposed, at least to some extent, to the other circumstances under which classical perturbation theory applies. We mention:

i) perturbation of an integrable Hamiltonian vector field;

ii) neighbourhood of an elliptic fixed point of a Hamiltonian vector field;

iii) neighbourhood of a Lagrangian torus over which a Hamiltonian vector field induces a flow conjugate to a linear one.

Each situation has its discrete analogue where Hamiltonian vector fields are replaced by symplectic maps. Of course, i) is the problem we have been dealing with, but we listed it for the sake of completeness. We refer to [4] (and [21] as far as iii) is concerned) for the elementary details. Continuous and discrete problems essentially correspond under the two inverse operations of *section* and *suspension*. Let us briefly illustrate this on i). It is well known how to construct a local Poincaré section for an autonomous Hamiltonian vector field. On the other hand, start from the discrete problem, which is described as follows: let B_δ be the open ball of radius $\delta > 0$ centred at the origin in $\mathbb{R}^n$, and $\mathbb{A}_\delta = \mathbb{T}^n \times B_\delta$ an annulus. Let f_0 be defined as

$$(1) \qquad (\theta, r) \to f_0(\theta, r) = \big(\theta + \omega(r) \bmod \mathbb{Z}^n, r\big), \quad (\theta, r) \in \mathbb{A}_\delta.$$

We assume that $\omega = \nabla h$ is the gradient of a function h; then f_0 is an integrable globally canonical map with generating function h. We consider the map f generated by a perturbation of h, $\Sigma(\theta, r') = h(r')+\sigma(\theta, r')$, where σ is small (of order ε), and f is implicitly defined by

$$(\theta', r') = f(\theta, r) = \left(\theta + \omega(r') + \frac{\partial\sigma}{\partial r'} \bmod \mathbb{Z}^n, r - \frac{\partial \sigma}{\partial \theta}\right).$$

Assume that h and σ are analytic and that h is a *convex* function; then one has the following stability result for the variable $r \in B_\delta$:

If $\varepsilon = \|\sigma\| \leqslant \varepsilon_0$, *then* $\|r_s - r\| \leqslant c\,\varepsilon^b$ *when* $|s| \leqslant c \exp(\varepsilon^{-a})$, $s \in \mathbb{Z}$; we use the notation $(\theta_s, r_s) = f^s(\theta, r)$, $\theta_0 = \theta$, $r_0 = r$.

The definition of the norms is as in Chapters II and III, and the exponents (a, b) are as in Theorem 2, *but* in dimension $n+1$. All the later refinements could be added.

To view such a result as a corollary of those of Chapter III, one must build a suspension of the map f, that is, realize it as the time 1 map of a flow associated with a Hamiltonian $H(\theta, r, t)$ which is periodic of period 1 in the time variable t. Here the real difficulty lies in the regularity assumption; in fact, the construction is quite easy in a C^∞ setting, much less so if one requires analyticity, as is necessary here. There is no obstruction however, and Kuksin proves (in [34]) the existence of H, which is an $O(\varepsilon)$-perturbation of h, of which f_0 is the time 1 map. So one is led to the quasi-convex case, having to deal with a periodic perturbation of a convex Hamiltonian (this is why h must be convex, not *quasi*-convex). Of course, it would still be useful to write a direct proof of the result above. Note that one has to cope with the fact that energy conservation is not available any more.

We shall now dwell a bit more on situation ii), which has been the subject of much study, for the past century at least. We shall not mention any more the discrete cases corresponding to ii) and iii). Situation ii) is degenerate from the point of view of perturbation theory, and before we turn to it, it is useful to look at another, slightly simpler but quite similar problem: the perturbation of harmonic oscillators; of course, this is also interesting for its own sake. So let

$$H(p,q) = \omega_0 \cdot p + \varepsilon h_1(p) + \varepsilon^2 f(p,q), \tag{3}$$

where $\omega_0 \in \mathbb{R}^n$ is a non-zero vector, h_1 and f are analytic functions, and h_1 is quasi-convex. We perform the scalings $t \to \varepsilon t$, $H \to \varepsilon^{-1} H$, and obtain, keeping the same notations for simplicity,

$$H(p,q) = \frac{\omega_0}{\varepsilon} \cdot p + h_1(p) + \varepsilon f(p,q). \tag{4}$$

We write $\omega_1 = \nabla h_1$; degeneracy manifests itself through the fact that the frequency $\omega = \varepsilon^{-1}\omega_0 + \omega_1$ is of the order of ε^{-1}. Fix $\varepsilon > 0$ small enough, suppose that $\omega(0)$ is rational of period T, and go through Chapter II again.

The quantities m and M which measure the non-linearity and convexity now refer to h_1 and are independent of ω_0; this implies that the iterative lemma carries over without any change. However, in the geometric reasoning leading to equation (23) of Chapter II, one should take into account the fact that $\|\omega(0)\|$ is of order ε^{-1}; so just replace Ω by $2\varepsilon^{-1}\Omega$, where Ω here stands for $\|\omega_0\|$ (indeed $\|\omega(0)\| \leqslant 2\varepsilon^{-1}\|\omega_0\|$ for ε small enough). Then (24) still defines $\mathcal{T}(\varepsilon)$, with the replacement $\Omega \to 2\varepsilon^{-1}\Omega$, and the rest is unaltered. So the "model statement" also carries over with only this modification.

Finally, Theorem 1A is valid for the Hamiltonian (4), except for the substitution $\Omega \to 2\varepsilon^{-1}\|\omega_0\|$. We said we have *fixed* $\varepsilon > 0$ so that $\omega(0)$ is rational of period T. Now the only requirement is that ε satisfy inequalities (27) of Chapter II. The second of these inequalities is very much weakened by the substitution on Ω, but one has to add the requirement $\|\omega(0)\| \leqslant 2\varepsilon^{-1}\|\omega_0\|$, that is, $\|\omega_1\| \leqslant \varepsilon^{-1}\|\omega_0\|$, which is a weak bound on ε. Note that Ω does not enter in the definition (26) of the quantities λ and τ.

Now let $p^* \in \mathbb{R}^n$ be a point in action space, $\omega_1^* = \omega_1(p^*)$, $\omega^* = \varepsilon^{-1}\omega_0 + \omega_1^*$, and we wish to approximate ω^*. Here comes the key observation: although we are working at *high* frequencies (of order ε^{-1}), there are always *low* frequency (of order 1) orbits close to a given one, and this phenomenon is uniform in ε as this quantity goes to zero. Indeed it only expresses the fact that for any value of $\varepsilon > 0$, $\varepsilon^{-1}\omega_0$ can be shifted back into the unit cube, using an integer vector. This simple but physically significant property will allow one to cope with the degeneracy. Let us now implement the above: with our notations, formula (2) of Chapter III is unchanged; *define* ω_1 by the equality $\omega = \varepsilon^{-1}\omega_0 + \omega_1$, so

$$\|\omega_1 - \omega_1^*\| \leq \frac{\sqrt{n}}{TQ^{1/n}}.$$

Since the map $p \to \omega_1(p)$ is locally invertible, one finds p close to p^* such that $\omega_1 = \omega_1(p)$, and the rest of the reasoning needs no modification at all. Of course the matrix A now denotes the Hessian matrix of h_1, and analogously for the other quantities. Let us state the result.

***Theorem* 3.** *Consider the Hamiltonian* (3) *above. Then the result stated as Theorem* 2 *of Chapter* III *holds, with the following qualifications:*

i) Ω^* *is replaced by* $2\varepsilon^{-1}\|\omega_0\|$, *with the additional threshold condition on* ε:

$$\|\nabla h_1\| \leq 2\varepsilon^{-1}\|\omega_0\|;$$

ii) *the constant* $\mathcal{T}_0^*$ *now has the value*

$$\mathcal{T}_0^* = 1.5 \cdot 10^{-2} \frac{\sigma}{\|\omega_0\|};$$

iii) *scalings* (1) *of Chapter* III *are not performed, so* n *should be changed to* $n+1$ *in the statement, and quantities* m, M, *and so on, refer to the original Hamiltonian* h_1.

Although this result has been obtained in an almost effortless way, we have stated it as a "theorem", because we believe it is quite significant: indeed, this is the first non-linear stability result over exponential times to be obtained in a degenerate case. Before we comment on this, let us briefly return to the statement above: ii) comes from the fact that we have been working with the Hamiltonian (4); returning to (3) involves a scaling of the time variable which gains back the factor ε that had been lost before. Scalings (1) of Chapter III cannot be performed because they involve the frequency, which is here of order ε^{-1}; hence iii). In particular, we obtain for the time of stability an exponent $a = \frac{1}{2n+3} - \eta$ for any $\eta > 0$.

The Hamiltonian (3) may arise naturally, for example in the following context: consider again a perturbation of a system of harmonic oscillators:

$$H(p,q) = \omega_0 \cdot p + \varepsilon g(p,q). \tag{5}$$

Assume that g contains only a finite number of harmonics, that is, it is a trigonometric polynomial in q. Then, away from a *finite* number of resonance surfaces, one can perform one step of the reduction to normal form, which leads (after a change of variables) to

$$H(p,q) = \omega_0 \cdot p + \varepsilon \langle g \rangle (p) + \varepsilon^2 f(p,q).$$

So, if the space average $\langle g \rangle$ is quasi-convex, we are reduced to the Hamiltonian (3).

To appreciate the significance of Theorem 3, one should beware of an important possible misunderstanding. We have proved a result which is completely independent of ω_0, in particular its arithmetical properties. In fact, if one sets $\omega_0 = 0$ in (3), it reduces to the non-degenerate case, and apart from some details which we leave to the reader to settle, we do recover the corresponding result. Now, if ω_0 is strongly non-resonant, say satisfies the usual Diophantine condition

$$\exists \gamma > 0,\ \tau > n-1, \text{ such that } \ |\omega_0 \cdot k| \geq \gamma |k|^{-\tau},\ \forall k \in \mathbb{Z}^n \backslash \{0\}, \tag{6}$$

it is easy to derive a stability result over exponential times. Indeed, starting from the less explicit form (5), one simply builds up the Birkhoff series, and makes use of (6) to control the process, using either an iterative method or a majorant series. This completely algebraic construction allows us to prove a stability result over exponential times, but one which is very sensitive to the arithmetics of ω_0.

Such elementary estimates are derived for example in [7], and we propose to call them *Gevrey type estimates*, the reason for this terminology being clarified in Appendix 2. In results of this kind, one considers the *linear* Hamiltonian $h(p) = \omega_0 \cdot p$ as the unperturbed system. Theorem 3 lies definitely deeper: one considers the *non-linear* Hamiltonian $h(p) = \omega_0 \cdot p + \varepsilon h_1(p)$ as the unperturbed part and takes advantage of the non-linearity (anharmonicity) and convexity to derive an estimate which is independent of the unperturbed frequency ω_0; such estimates we propose to call *Nekhoroshev type estimates.* A similar result should be valid (with other exponents) if h_1 is only assumed to be *steep*, but this seems very cumbersome to obtain if one applies Nekhoroshev's original method.

We finally note that Zaslavskii and coworkers (see [58]) have recently studied, mostly from a physical and numerical standpoint, systems which are perturbations of Hamiltonians of type

$$h(p) = \omega_0 \cdot p_0 + h_1(p_1), \quad p = (p_0, p_1) \in \mathbb{R}^{l+m} = \mathbb{R}^n,$$

where h_1 is non-degenerate (say convex); in such a situation, instabilities usually occur on much shorter time-scales, and Nekhoroshev type results are excluded in general.

We now return to the problem of studying a Hamiltonian vector field in the neighbourhood of an elliptic fixed point, and we shall use Theorem 3 in order to derive a result for this situation. We denote by $\pm i\alpha_j (i = \sqrt{-1}), j = 1, \ldots, n$, the eigenvalues of the linearized system at the fixed point, which we take as the origin of the coordinate system (x_j, y_j), $j = 1, \ldots, n$, of $\mathbb{R}^{2n}$; we write $z = (x, y) \in \mathbb{R}^{2n}$. We assume that the linear part can be diagonalized and that there is no resonance of order $\leqslant s$ (a positive integer), which means, writing $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$, that

$$\forall k \in \mathbb{Z}^n \setminus \{0\} \quad \alpha \cdot k \neq 0 \quad \text{if} \quad |k| = |k_1| + \cdots + |k_n| \leq s.$$

Let $r_j = (1/2)(x_j^2 + y_j^2)$, $r = (r_1, \ldots, r_n) \in \mathbb{R}^n_+$. Following Birkhoff, one can perform a canonical transformation so as to put the Hamiltonian in the form

$$H(z) = H(x, y) = H^{(s)}(r) + O(\|x, y\|^{s+1}). \tag{7}$$

$H^{(s)}(r)$ is a polynomial of degree at most $[s/2]$ in the r_j's; we assume that H is analytic, so that the rest is a convergent power series whose terms are of degree at least $s+1$ in x_j, y_j. We suppose that $s > 4$, so $H^{(s)}$ has the form

$$\begin{aligned} H^{(s)}(r) &= \sum_j \alpha_j r_j + \frac{1}{2} \sum_{i,j} \alpha_{ij} r_i r_j + O(\|r\|^3) \\ &= \alpha \cdot r + \frac{1}{2}(Ar \cdot r) + O(\|r\|^3), \end{aligned} \tag{8}$$

where $A = (\alpha_{ij})$ is a symmetric matrix. This way of writing determines the sign of the α_j's; note that if they are all of the same sign, the stability problem is immediately settled (positively), because the origin is a local maximum (or minimum) of the Hamiltonian. Here we shall work again under the convexity assumption that A is a, say, positive matrix and we let $m > 0$

(respectively, $M \geqslant m$) be its smallest (respectively, largest) eigenvalue. Under these assumptions we prove the following theorem.

Theorem 4. *Consider a trajectory $z(t)$ under the evolution governed by $H(z)$ (see (7) and (8)). There is a constant $\nu > 0$ such that if $z = z(0)$ is small enough and satisfies*

$$r_j = \frac{1}{2}(x_j^2 + y_j^2) \geq \nu \|z\|^{2+\frac{1}{n+2}}, \quad j = 1, \ldots, n, \tag{9}$$

($r_j = r_j(0)$, and so on), then one has

$$\|r_j(t) - r_j\| \leq \frac{\nu}{2}\|z\|^{2+\frac{1}{n+2}}, \quad j = 1, \ldots, n,$$

provided that t satisfies

$$|t| \leq \mathcal{T} \exp\left(\|z\|^{-\frac{1}{n+2}}\right),$$

where $\mathcal{T}$ is some strictly positive constant.

Before making some comments, we show how this is an easy consequence of Theorem 3; as the reader will see, we prove in fact a more precise and slightly stronger statement. First, introduce the usual symplectic polar coordinates (r, θ) defined as

$$x_j = \sqrt{2r_j}\cos\theta_j, \quad y_j = \sqrt{2r_j}\sin\theta_j.$$

We fix $z = z(0)$ and set $\varepsilon = \Sigma_j r_j = (1/2)\|z\|^2$. Then perform the scaling $r = \varepsilon\rho$, $H = \varepsilon K$, which multiplies the symplectic form by the factor ε and leaves the equations invariant. Then

$$K(\rho, \theta) = \alpha \cdot \rho + \frac{1}{2}\varepsilon(A\rho \cdot \rho) + \varepsilon^2 f(\sqrt{\varepsilon\rho}, \theta),$$

where we write "componentwise" $\sqrt{r} = (\sqrt{r_1}, \ldots, \sqrt{r_n}) \in \mathbb{R}_+^n$. The function f is analytic, periodic in θ, and we let σ be its analyticity width in θ.

We may now apply Theorem 3, provided that we keep away from the singularities at $\rho_j = r_j = 0$, $j = 1, \ldots, n$. Now recall that in Theorem 2 (or even Theorem 1), the analyticity width in the action variables need not be of order 1, but only at least equal to the confinement radius. This is quite an important feature in the present context, because it says how close we may approach the singularities. From Theorem 3 (or rather Theorem 2), we compute the confinement radius:

$$\|\rho_j(t) - \rho_j\| \leq 10^{-2}\frac{\sigma}{M}\varepsilon^a = 10^{-2}\frac{\sigma}{M}\left(\frac{1}{2}\|z\|^2\right)^a < 10^{-2}\frac{\sigma}{M}\|z\|^{2a}.$$

Here $a = \dfrac{1}{2n+4}$ and we set $\nu = 10^{-2}\dfrac{\sigma}{M}$. We thus get the inequality on the drift of the action variables, with a time of validity

$$\mathcal{T}(z) = \mathcal{T}\exp(\varepsilon^{-a}) > \mathcal{T}\exp(\|z\|^{-2a}), \quad \mathcal{T} = 1.5 \cdot 10^{-2}\frac{\sigma}{\|\alpha\|}.$$

All this is valid provided that the inequalities $\rho_j(t) > \nu\varepsilon^a$ keep holding during the time $\mathcal{T}(z)$, which is guaranteed by inequality (9). We have thus proved the theorem and computed the quantities ν and $\mathcal{T}$. The threshold of validity, that is, the maximal possible value of $\|z\|$, could of course also be computed explicitly, using Theorems 2 and 3. □

We add some short comments about this result. First, a similar, slightly weaker estimate holds with $s \geqslant 4$ (the minimal order of a possible resonance), which is also the condition under which KAM theory applies. We took $s > 4$ for convenience only.

Second, one may improve on this result if s is really larger, by performing some steps of the Birkhoff normal form algorithm and applying this type of reasoning afterwards. In fact, this strategy may also be used in the contexts of Theorems 2 and 3, at least under certain circumstances. This is a combination of the usual method and the closed orbit method we put forward in this paper, and this may be useful in trying to improve the estimates, possibly in a computer assisted way.

Third, if the matrix A has no definite sign, steepness cannot be decided from the knowledge of α and A alone and one must compute more Birkhoff invariants (hence s must be larger, at least $\geqslant 6$); then, in the steep case, it would in principle be possible to apply a variant of the strategy of [43] to prove a result of the type of Theorem 4, but again this looks very cumbersome indeed.

Fourth, we have not proved an exponential "exit time" estimate because of the seemingly artificial and spurious requirement (9) on the initial conditions. This stems from the fact that we had to use the action-angle variables (r, θ), which present singularities on the coordinate planes $r_j = 0$. Exactly the same difficulty is encountered (and left unsolved) in KAM theory (see, for example, [46], last paragraph). We do not know if and how it may be overcome and accordingly we have had to leave out small cusp-shaped regions with vertices at the fixed point.

Fifth, there are some obvious generalizations which may be useful. For example, one may require *quasi*-convexity only: in this context, it means that the quadratic form $Ar \cdot r$ has to be of definite sign, but only when restricted to the plane $\alpha \cdot r = 0$. Alternatively, one may consider *periodic* perturbations: A must have definite sign but the Hamiltonian may depend periodically on time.

Lastly, the same comment is in order concerning Gevrey type estimates, as was discussed in connection with Theorem 3 (we again refer the reader to Appendix 2). If α is a Diophantine vector, that is, if it satisfies inequalities (6) (with α in place of ω_0), one obtains exponential stability estimates in an elementary, purely algebraic way, by controlling the growth of the Birkhoff series (see, for example [24] and [25]). Again, these estimates depend strongly on the arithmetics of α. We note that if one wants to derive Nekhoroshev type estimates, as we did, one cannot use the usual complex coordinates ($w = x+iy$ and the complex conjugate vector) because these are not directly related to the action-angle variables of the unperturbed part, unless the latter is taken to be *linear*, as is the case in Gevrey type estimates.

In some sense, case iii) mentioned at the beginning of this section, namely the neighbourhood of an invariant Lagrangian torus, is easier to disentangle. To start with, by the symplectic tubular neighbourhood theorem, one may symplectically describe the neighbourhood of the torus as $\mathbb{T}^n \times B_\delta$, where B_δ is again the open ball of radius δ centred at the origin in $\mathbb{R}^n$. We still denote the corresponding coordinates as $(\theta, r) \in \mathbb{T}^n \times B_\delta$; the invariant torus has the equation $r = 0$, and after conjugation the flow on it is linear with vector $\alpha \in \mathbb{R}^n$. The crux of the matter is that if α is *not* Diophantine, the situation is structurally unstable and it seems quite hard to say anything at all. Indeed normal theory *at first order* already requires that α be strongly irrational.

Suppose now that α is indeed Diophantine; as a side remark we note that this implies, under weak regularity assumptions, that the torus is Lagrangian, so this need not be part of the hypothesis any more. Then one is reduced to a situation very similar to that of the elliptic point, namely, after a change of coordinates, to the Hamiltonian

$$
\begin{aligned}
H(\theta,r) &= H^{(s)}(r) + O(\|r\|^{s+1}),\\
H^{(s)}(r) &= \sum_j \alpha_j r_j + \frac{1}{2}\sum_{i,j} \alpha_{ij} r_i r_j + O(\|r\|^3) \qquad (10)\\
&= \alpha\cdot r + \frac{1}{2}(Ar\ r) + O(\|r\|^3).
\end{aligned}
$$

Here s is arbitrary and $r \in B_\delta$ runs through a neighbourhood of the origin. No singularities occur and one may derive, without any convexity or steepness assumptions, Gevrey type estimates, because α is highly non-resonant. This is done as in the case of the elliptic fixed point, except that here complex coordinates cannot be used ((θ, r) do not arise as polar coordinates); this situation is similar to the case of the elliptic fixed point, with the latter "blown-up". These estimates seem not to have yet been written out in detail, although they describe in particular the time needed to move away from a Kolmogorov invariant torus. Note that the latter is a problem with *two* small parameters: ε describing the perturbation from integrability and $\|r\|$ measuring the distance from the torus.

§3. Systems with (infinitely) many degrees of freedom

Corollaries 2 and 3 of Chapter III are perhaps of great relevance to a class of problems with a large—possibly infinite—number of degrees of freedom. Here we are thinking of simple statistical models, such as spin lattices, chains, crystals, and so on, as well as some particular PDE's, mainly in one space dimension. These problems have been studied during the past few years with varied successes and a special emphasis on KAM theorem; the bibliography of [48] contains some of the important references on the subject.

Rather than being too vague or abstract, it is perhaps best to consider a simple example which displays the main features and difficulties: a one-dimensional chain of rotators with nearest neighbours interactions. We thus look at the Hamiltonian

$$H(p,q) = \sum_{i=1}^{N}\Big(\frac{1}{2}p_i^2 + \varepsilon V(q_{i+1} - q_i)\Big).$$

Here V is a potential with a critical point at some value $a > 0$ ($V'(a) = 0$) representing the average distance between two free rotators. If N is finite, one should add boundary conditions (for example, periodicity, say $q_{N+1} = q_1$) and then look for results which do not depend on N, at least asymptotically when this tends to infinity (thermodynamical limit). Alternatively, one may set $N = \infty$ from the start, with a suitable mathematical setting.

Now, the point we want to make in this short section is that *localization is resonance* and that, by the results of Chapter III, convexity and resonance together imply stability, because of a local abundance of periodic orbits. From this, it should be possible to derive strong "non-linear localization results". Indeed, suppose that at time $t = 0$ one jiggles d of the N rotators (assume that N is finite and impose periodic boundary conditions for simplicity), that is, we have the following initial conditions:

$$p_i(0) \text{ arbitrary}, \quad i = 1,\ldots,d; \quad p_i(0) = 0, \ i = d+1,\ldots,N;$$
$$q_i(0) \text{ arbitrary}, \quad i = 1,\ldots,N.$$

This is a resonant situation, since the frequency vector is none other than

$$\omega(0) = (p_1(0),\ldots,p_d(0),0,\ldots,0),$$

so we start on a d-dimensional resonant surface. Now apply Corollary 2 of Chapter III and conclude that the action variables are stable over an interval of time essentially of the order of $\exp(\varepsilon^{-1/(2d)})$ for ε small enough, independently of the number N of degrees of freedom. Corollary 3 adds the important flexibility that one may even allow for some energy to be fed into the remaining $N-d$ rotators, still getting essentially the same result.

But all this is cheating, of course! What is it that is lacking? Not much really; only the fact that the number of degrees of freedom is buried in the definition of the norms we use, for example, to measure the strength of the perturbation. These do not take advantage of the fact that the problem displays some locality property in *real* space, to wit that the interaction involves nearest neighbours only. Pöschel, starting from the work of various authors (including himself) has abstracted a general scheme to deal with these *local structures* (see [48]). We hope that combining this with the principles of the present paper could lead to interesting results, of the kind that were carelessly stated above.

Again convexity is here an essential ingredient. In particular, chains of perturbed harmonic oscillators could be treated, using Theorem 3 in §2, only insofar as the non-linear perturbation presents some convexity property *in action-angle variables* ("normal modes" coordinates). Unfortunately, this is a rather unnatural requirement in this context (compare the Fermi–Pasta–Ulam model).

We recall also that KAM theory has been recently extended to some classes of infinite-dimensional systems; under some technical assumptions, one proves the existence of either *finite*-dimensional invariant tori (Kuksin, Pöschel, Wayne, et al.) that is, quasi-periodic motions with finitely many frequencies and/or *infinite*-dimensional invariant tori (Vittot, Pöschel, et al.). We refer to [48] and [35] for a bibliography. Our last remark is that, among other conditions, KAM theory requires a priori some form of non-degeneracy condition, as is usual, but that in the context of statistical mechanics, this essentially *implies* convexity. This stems from the fact that the unperturbed integrable system is usually assumed to be an ensemble of non-interacting *identical* objects. Non-degeneracy means that each of the microscopic entities is "truly non-linear" (for example, a rotator, rather than a harmonic oscillator). But then the Hessian matrix of the unperturbed Hamiltonian will obviously be diagonal with identical non-zero entries, which implies convexity.

§4. Steepness, quasi-convexity, and closed orbits

We have repeatedly emphasized that the closed orbit method we use in the present paper is restricted to the quasi-convex case and that stability in the general steep situation is just not amenable to it. Maybe this could provoke a renewal of interest for the latter case, which has been very little investigated? All the more since in the analytic framework, taking advantage of the rigidity of analytic objects, Il'yashenko has given a completely algebraic characterization of steepness (in [30]), which was originally introduced as a C^∞-notion. It would thus be quite interesting to rewrite Nekhoroshev's proof ([43], [44]), trying to clarify the relationship with geometry and singularity theory, from which, incidentally, steepness originally emerged. One could also try to isolate interesting subclasses of steep functions, beyond the quasi-convex one, which, we recall, is the only one where steepness can be read off the 2-jet of the function.

On the other hand, quasi-convexity has been recognized, in the past few years, to imply very specific properties, and from this standpoint the stability properties explored in the present paper fit well into the picture. It may thus be useful to mention some of these features. (Quasi-)convexity is naturally appealing first of all, because the kinetic energies which one comes across in physics usually enjoy this property. This is also linked to the fact that even in a non-perturbative framework, Hamiltonians are usually derived from Lagrangians, and that convexity with respect to the action variables goes along with the existence and nice properties of the Legendre transform.

Then, convexity is also the natural and simplest framework of variational methods, for example if one tries to prove the existence of closed orbits "in the large", that is, for arbitrary Hamiltonians with compact energy surfaces. The assumption that the energy surface is convex enormously simplifies the problem and much more precise results are known than in the general case.

Some rather subtle specificities of convexity have been revealed recently. Let us consider, as in §2 (formula (1)), a globally canonical integrable map of the annulus:

$$(\theta, r) \to (\theta + \omega(r) \bmod \mathbb{Z}^n, r), \quad (\theta, r) \in \mathbb{A}^n = \mathbb{T}^n \times \mathbb{R}^n, \quad \omega = \nabla h.$$

Here we shall need only a finite order of differentiability, so that everything is really local in the r variables. One considers a globally canonical perturbation of the above (see (2) in §2). If $n = 1$, under the twist condition $\omega'(r_0) \neq 0$, Birkhoff showed that any invariant curve Γ close to the circle $r = r_0$ is the graph of a continuous function, that is, there exists $\psi \in C^0(\mathbb{T}^1, \mathbb{R})$ such that $\Gamma = \Gamma_\psi = \{(\theta, \psi(\theta)), \theta \in \mathbb{T}^1\}$. Moreover, ψ is in fact Lipschitz, and its derivative (which exists almost everywhere) satisfies an a priori estimate; Birkhoff's theory is in fact *not* of perturbative nature, but we restrict ourselves to this case for simplicity.

Now, if $n > 1$, Herman ([28]) shows that various pathologies may arise, unless one restricts consideration to *Lagrangian* tori homotopic to $r = 0$ *and* if one assumes *monotone twisting* (convexity), that is, that the matrix $A(r) = \partial\omega/\partial r = \nabla^2 h$ has a definite sign. Only in that case can Birkhoff's regularity theory be generalized to more than one dimension, at least in a perturbative way.

We shall devote the end of this section to a brief discussion of the existence of (exact) closed orbits for near integrable Hamiltonians; we first recall the old perturbative result, essentially due to Poincaré ([45], Chapters III and IV), and emphasize how the specificity of quasi-convexity is already quite visible at this level, something which never seems to be pointed out in the literature.

Let us go back to the setting of Chapter II. Let $H = h(p) + \varepsilon f(p, q)$ be a perturbed Hamiltonian; ε is written explicitly and there is no loss of generality in assuming that it is $\geqslant 0$. When $\varepsilon = 0$, $p = 0$ is an invariant periodic torus of period T and rational frequency $\omega(0) = \omega_0$. We do *not* assume quasi-convexity for the moment, only non-degeneracy: $\det A_0 \neq 0$ ($A(p) = \nabla^2 h$, $A_0 = A(0)$). As in Chapter II, if $g(q)$ is a function on $\mathbb{T}^n$, $\langle g \rangle$ denotes its average along ω_0. Let $\langle f \rangle(q) = \langle f \rangle(0, q)$ be the average of the perturbation of the torus $p = 0$. It is constant on the orbits of the linear flow along ω_0 and can be thought of as a function on the space of orbits $\mathcal{O} = \mathbb{T}^{n-1}$. We suppose that it is a Morse function on this space. Viewed on the torus $\mathbb{T}^n$, it means that the Hessian matrix

$F_0 = \dfrac{\partial^2}{\partial q^2}\langle f\rangle(0, q^{(0)})$ at a critical point $q^{(0)}$ has a *one-dimensional* kernel spanned by ω_0 (critical points are in fact critical orbits). The following assertion holds.

Theorem. *Let $H(p, q) = h(p)+\varepsilon f(p, q)$ ($\varepsilon \geqslant 0$) be such that $p = 0$ is, for $\varepsilon = 0$, a periodic torus of frequency ω_0 and period T. Assume that h is non-degenerate at $p = 0$ ($\det \nabla^2 h(0) \neq 0$) and that the average $\langle f\rangle(0,q)$ has a one-dimensional null space (spanned by ω_0) at its critical points.*

Then for $\varepsilon > 0$ small enough there exist, in an $O(\varepsilon)$ neighbourhood of $p = 0$, at least 2^{n-1} orbits of period T, including multiplicity, of which at least n are geometrically distinct.

Moreover, if h is quasi-convex, one may specify the spectral type of these orbits and assert that there are at least $\binom{n-1}{k}$ k-hyperbolic orbits, $k = 0, 1, \ldots, n-1$.

Here we call an orbit k-hyperbolic if it has k pairs of Floquet exponents which are not purely imaginary; recall that μ is a Floquet exponent of some orbit of period T if $\lambda = e^{\mu T}$ is an eigenvalue of the linearized return map. The last item thus says that *in the quasi-convex case* one may predict the linear stability of the orbits which are born from a periodic torus. For instance, there will arise at least one linearly stable (that is 0-hyperbolic or elliptic) orbit. This comes from the fact that the Floquet exponents, which are paired in pairs of opposite signs, may be expanded in powers of $\sqrt{\varepsilon}$ (two of them vanish); at first order, they are of the form $\pm(\varepsilon\Omega_j)^{1/2}$, $j = 1, \ldots, n$, where the Ω_j's are the eigenvalues of the matrix $-A_0F_0\left(A_0 = \nabla^2 h(0), F_0 = \dfrac{\partial^2}{\partial q^2}\langle f\rangle(0, q^{(0)})\right)$. In the non-degenerate case one uses Morse inequalities to specify the number and spectral type of the critical points (or rather orbits of $\langle f\rangle$, that is, the spectral type of F_0. Then, adding the assumption of quasi-convexity, one may use the following elementary assertion.

Proposition. *Let A and B be two symmetric matrices, $A > 0$. Then, the spectrum of the product AB is real and of the same type as that of B, that is, it contains the same number of positive, zero and negative eigenvalues (including multiplicity).*

Indeed, if $P^2 = A$, $P > 0$, the spectrum of AB coincides with that of PBP, which is symmetric, and subspaces over which B is > 0 (respectively, $= 0$, < 0) are carried over by P into corresponding subspaces of PBP. □

To apply this proposition, one considers the orthogonal complement of ω_0, so only *quasi*-convexity is required. The upshot is that even at the perturbative level, only in the quasi-convex situation can one predict the stability of (at least some of) the periodic orbits which are born from a periodic torus.

The theorem above applies when $0 < \varepsilon \leqslant \varepsilon_0 = \varepsilon_0(h, T)$. It took a century to prove that one may at least partially remove the dependence of ε_0 on T, and this again in the quasi-convex situation only. Loosely speaking, Bernstein and Katok proved (in [8]) that *if h is quasi-convex*, for ε small enough, *independently of T*, there survive at least n closed orbits in a $O(\varepsilon^{1/3})$ neighbourhood of a torus of period T. This is a deeper result, strongly connected with the multidimensional version of "Poincaré's last geometric theorem", as proved by Conley and Zehnder (in [14]). This is also the first step in trying to understand what happens under perturbation, when a sequence of rational tori accumulates to a given limiting torus (in the unperturbed situation), that is, in trying to generalize the Aubry–Mather theory of "cantori" to higher dimensions. Once more, all this requires quasi-convexity, not only because the methods are often variational, but also because many "wild" phenomena seem to be liable to occur otherwise (see again [28]). Of course, in order to prove stability results over exponentially long times, we only had to use some simple arithmetics related to these results; we shall go deeper into the arithmetics in the next section (Chapter V, §1).

CHAPTER V

ROBUST TORI; ARNOL'D DIFFUSION

§1. Robust tori and "renormalization"

We do not know precisely how simultaneous approximation can be used to prove KAM type results, although this is certainly possible. Note that such a method would be closer to the ideas of "renormalization" and especially the original intuition of Greene in [26]. This could be useful in several respects, for example, for the study of lower-dimensional invariant tori. The only thing we wish to mention in this direction is a simple proposition which makes more precise the convergence of the time averages to the space average for linear flows on the torus. It should probably be used, in some form at least, on the way towards KAM type results via simultaneous approximation.

Let $\mathcal{A}_\rho$ denote the space of functions on $\mathbb{T}^n$ which extend analytically to the strip $|\mathrm{Im}\, q| < \rho$ and are continuous at the boundary; $\mathcal{A}_\rho$ is provided with the norm $\|\cdot\|_\rho$ of the maximum over the closed strip. On the other hand, let $\omega \in \mathbb{R}^n$, let $(T_j)_{j\geqslant 0}$ be the sequence of its period, and $(\omega_j)_{j\geqslant 0}$ the corresponding sequence of best approximations (ω_j has period T_j; see Appendix 1). We denote $\eta_j = \|\omega_j - \omega\|$ and one has the estimate

$$(1) \qquad \eta_j \le \frac{\sqrt{n}}{T_j T_{j+1}^{\frac{1}{n}}} < \frac{\sqrt{n}}{T_j^{1+\frac{1}{n}}}.$$

Lastly, we introduce the operators M_j of time average along ω_j and M_∞ the space average; for a function $g(q)$ on the torus

$$M_j(g) = \frac{1}{T_j}\int_0^{T_j} g(q+\omega_j t)dt; \quad M_\infty(g) = \int_{\mathbb{T}^n} g(q)dq.$$

Then the following statement holds.

Proposition. *Assume that ω is Diophantine, more precisely:*

$$\text{(2)} \qquad \begin{aligned} &\exists \tau > n-1, \quad \gamma > 0 \quad \textit{such that} \quad \forall k \in \mathbb{Z}^n \backslash \{0\} \\ &|\,\omega\cdot k\,| \geq \gamma |k|^{-\tau}, \ \textit{where} \ |k| = \sum_i |k_i|. \end{aligned}$$

For any $g \in \mathcal{A}_\rho$ and any δ, $0 < \delta \leqslant \rho$,

$$\text{(3)} \qquad \left\| M_j(g) - M_\infty(g) \right\|_{\rho-\delta} \leq 4^n \|g\|_\rho \delta^{-n} \exp\left[-\frac{\delta}{2}\left(\frac{\gamma}{\eta_j}\right)^{\frac{1}{\tau+1}}\right].$$

Note that the right-hand side may be estimated in terms of T_i only, using (1). To prove (3), let $g \in L^2(\mathbb{T}^n)$ with Fourier coefficients g_k, $k \in \mathbb{Z}^n$; $M_j(g)$ is the function whose only non-zero Fourier coefficients are equal to g_k, for the values of k satisfying $\omega_j \cdot k = 0$. This last relation implies that

$$\gamma|k|^{-\tau} \leq |\,\omega\cdot k\,| = \Big|\,(\omega-\omega_j)\cdot k\,\Big| \leq \|\omega-\omega_j\| \cdot \|k\| = \eta_j \|k\|.$$

Hence

$$\text{(4)} \qquad |k| \geq K_j \overset{\text{def}}{=} \left(\frac{\gamma}{\eta_j}\right)^{\frac{1}{\tau+1}},$$

where the inequality $|k| \geqslant \|k\|$ (Euclidean norm) has been used. On the other hand, the fact that $g \in \mathcal{A}_\rho$ provides the estimate

$$|g_k| \leq \|g\|_\rho e^{-2\pi\rho|k|},$$

which enables us to evaluate the tail of the Fourier series. Namely, if $K \in \mathbb{Z}_+$, one sets

$$g^{\geq K(q)} = \sum_{k,|k|\geq K} g_k e^{2\pi i(k,q)}.$$

Then (see [6], for example):

$$\text{(5)} \qquad \|g^{\geq K}\|_{\rho-\delta} \leq \|g\|_\rho \sum_{|k|\geq K} e^{-2\pi\delta|k|} \leq c_n \|g\|_\rho \delta^{-n} \exp\left(-\frac{\delta K}{2}\right);$$

one can take $c_n = 4^n$ (see [7]). Letting $K = K_j$, this implies (3), in view of (4). □

This rather elementary proposition is interesting in itself. It is however unsatisfactory because it uses an arithmetical condition of linear type (here (2), which could be generalized) whereas one would like to start from a hypothesis