The Fundamental Groups at Infinity of the Moduli Spaces of Curves

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Abstract

In this note we explicit and prove some assertions contained in the *Esquisse*, concerning the fundamental groups "at infinity" (see precise definitions below) of the fine moduli spaces of non singular pointed curves over the complex numbers. This is achieved essentially by connecting these assertions to some known results, obtained by topological methods. We also discuss related assertions for the coarse moduli spaces.

§1. Introduction

We shall here be mainly concerned with one page (p.7) of the *Esquisse*, dealing with the fundamental group of the moduli spaces of pointed curves in an analytic context (i.e. over $\mathbb{C}$). Let us first quote the key sentence (starting at the bottom of p.6): "Ce principe de construction de la tour de Teichmüller n'est pas démontré à l'heure actuelle – mais je n'ai aucun doute qu'il ne soit valable. Il résulterait [...] d'une propriété extrêmement plausible des multiplicités modulaires ouvertes $M_{g,\nu}$, dans le contexte analytique complexe, à savoir que pour une dimension modulaire $N \geq 3$, le groupe fondamental de $M_{g,\nu}$ (i.e. le groupe de Teichmüller habituel $T_{g,\nu}$) est isomorphe au 'groupe fondamental à l'infini' i.e. celui d'un 'voisinage tubulaire de l'infini' ".

Let us first recall a few standard definitions and introduce some notation (which differs slightly from Grothendieck's). But first a warning to algebraic-geometers; here we shall actually be concerned only with the *analytic* part of the theory ("la théorie transcendante" as Grothendieck puts it on p.7 of the *Esquisse*), and indeed mostly with topological or at most real analytic properties. In particular, we shall have at our disposal the Teichmüller spaces, which are typically analytic objects. We shall accordingly recall mainly the analytic definitions and only quickly mention (with references) the connection with algebraic geometry, postponing some remarks until §5. For background information on the analytic treatment, we refer to the several books which are available, in particular [IT] and [T].

So let $S_{g,n}$ be *the* topological surface of genus g with n *numbered* marked points $(p_1, \ldots, p_n)$; we fix an orientation and a differentiable structure and there is one and only one class of such data, up to diffeomorphism. Next, a Riemann surface X of type (g, n) is a compact Riemann surface of genus

g with n numbered marked points, $(x_1, \ldots, x_n)$. Finally, a marked surface of type (g,n) is defined as a pair (X, f) where X is a Riemann surface of type (g,n) and f is a (marking) diffeomorphism $f : S_{g,n} \to X$ such that $f(p_i) = x_i$ for $i = 1, \ldots, n$. One introduces an equivalence relation on the set of marked surfaces by declaring (X, f) and (X', f') equivalent if there exists a diffeomorphism h of S onto itself, *isotopic to the identity map*, preserving the marked points, and such that the map $f' \circ h \circ f^{-1}$ is a biholomorphic map from X to X'.

From now on, we drop the subscript (g,n) when the context is clear; also, in this note, we shall always deal with surfaces of hyperbolic type or, equivalently, of strictly negative Euler characteristic; that is, we assume throughout that $2g - 2 + n > 0$.

The Teichmüller space $\mathcal{T}(S_{g,n}) = \mathcal{T}_{g,n} = \mathcal{T}$ parametrizes the equivalence classes of *marked* surfaces of type (g,n). The point is that it is not hard to coordinatize this set of classes, using for instance the so-called Fenchel-Nielsen coordinates, and to endow $\mathcal{T}$ with a real analytic structure; it is a classical result (essentially due to Teichmüller, and the main motivation for introducing *marked* surfaces) that $\mathcal{T}_{g,n}$ is a contractible space of real dimension $6g - 6 + 2n$. It can actually also be given a *complex* analytic structure, making it into a contractible analytic manifold of complex dimension $3g - 3 + n$ (see e.g. [IT]).

The moduli space $\mathcal{M}_{g,n}$ is obtained by "forgetting the marking": namely let $\mathrm{Diff}^+(S)$ be the group of orientation preserving diffeomorphisms of the surface $S = S_{g,n}$ fixing the points p_i, and let $\mathrm{Diff}_0(S)$ be the connected component of the identity map, i.e. the subgroup of such diffeomorphisms isotopic to the identity. Let $\Gamma = \Gamma_{g,n} = \pi_0(\mathrm{Diff}^+(S)) = \mathrm{Diff}^+(S)/\mathrm{Diff}_0(S)$ be the group of the connected components of $\mathrm{Diff}^+(S)$; the group structure is inherited from that of $\mathrm{Diff}^+(S)$. This group $\Gamma_{g,n}$ is the one which Grothendieck denotes $T_{g,\nu}$ $(\nu = n)$ in the quotation from the *Esquisse* given above. This mapping class group or Teichmüller modular group Γ acts on $\mathcal{T}$: an element $h \in \Gamma$ sends $(X, f) \in \mathcal{T}$ to $(X, f \circ h)$; here we do not notationally distinguish a diffeomorphism of S and its image in Γ, because there is no need to. The moduli space $\mathcal{M} = \mathcal{M}_{g,n}$ is defined as the quotient $\mathcal{T}/\Gamma$. We denote by p the quotient map $\mathcal{T} \to \mathcal{M}$.

It is easy to see that this really amounts to "forgetting the marking". Indeed, say that two (*un*marked) Riemann surfaces X and X' (of type (g,n)) are equivalent, simply if they are isomorphic as marked Riemann surfaces, i.e. if it exists a biholomorphic map ϕ between X and X' respecting the marked points. If X and X' come from marked surfaces (X, f) and (X', f'), the map ϕ will give rise to a diffeomorphism $h = f'^{-1} \circ \phi \circ f$ from S to itself, but it will not in general be isotopic to the identity; the quotient map

p thus appears as a "forgetful" (with respect to marking) surjective map between $\mathcal{T}$ and $\mathcal{M}$.

Let us add a word about marked points. In the above we have used "marked points" which are considered as part of the (differentiable or Riemann) surface. We could have used "punctures" instead, obtained by deleting the points from the surfaces. In some sense, it is more natural to use marked points when dealing with conformal structures, and punctures when dealing with hyperbolic structures. We shall sometimes not make the choice explicit, and just speak of surfaces – or other objects – "of type (g, n)". The point is that we always consider surfaces *without* boundary, which from the point of view of topology means that the Dehn twists around simple closed curves encircling the removed points vanish. Also, we have assumed above that the points are numbered and that the various maps do not permute them; in §4, we shall need the extension to the case where punctures can be permuted, and we therefore include a short discussion of this point at the beginning of §4.

Returning to the main stream, an important (not so easy; see e.g. [IT]) result asserts that the action of Γ on $\mathcal{T}$ is proper and discontinuous; in particular, the stabilizers have finite order: they correspond in fact to the automorphism groups of the underlying Riemann surfaces, which are well-known to be finite in the hyperbolic case $2g + n \geq 3$. The space $\mathcal{M}$ can now be considered as a – possibly singular – variety given as the quotient space $\mathcal{T}/\Gamma$. From the point of view of algebraic geometry, $\mathcal{M}$ viewed this way solves the *coarse* moduli problem for complex non singular algebraic curves of type (g, n) (alias smooth Riemann surfaces of type (g, n)). For a general discussion of the moduli problem, we refer to §5.1 of [MFK]; a coarse moduli scheme is defined in §5.2 of that book (Definition 5.6) in terms of representability of a functor. Roughly speaking, $\mathcal{M}$ meets the requirements because its points (recall we are working over $\mathbb{C}$ and "point" means "geometric point") are in one-to-one correspondence with the equivalence classes of curves, as briefly noted above (this is (i) in Definition 5.6 of [MFK]); the universality requirement ((ii) of Definition 5.6) can also be shown to hold true.

It is crucial to note however that in the quotation from the *Esquisse* given above, Grothendieck has in mind the *fine* moduli spaces. Rougly speaking, from the point of view of topology or analysis, it amounts to viewing the spaces $\mathcal{M}_{g,n}$ as orbifolds, retaining the information encoded in the finite groups defined at every point of $\mathcal{M}$; that is, to a point of $\mathcal{M}$ representing the Riemann surface X, one associates its finite group of analytic automorphisms $Aut(X)$. We postpone to the beginning of §3 a more detailed discussion of this important point, stressing that all in all we shall only make use of the definition-proposition 3 (in §3) which the reader may take as a working definition if she or he wishes to. We shall need nothing from Thurston's theory of orbifolds and shall only briefly sketch the connection with algebraic geometry i.e. with "multiplicities" (to use Grothendieck's word) or "stacks". Also, all the fundamental groups we consider are topological, not algebraic, i.e. they are discrete and not profinite groups.

So in this introduction, we just note that the $\mathcal{M}_{g,n}$'s can be regarded as open (noncompact) orbifolds and that it is natural to look at their fundamental groups in the orbifold category. On the other hand, define the fundamental group "at infinity" of a space M to be the inverse limit of the fundamental groups of the subspaces $M \setminus K$ where K runs over compact sets of M (see §2 for details). With these definitions in mind, Grothendieck's assertion (see the quotation from the *Esquisse* given above, and the more precise statement $(*)$ in §3 following definition-proposition 3) makes sense in the analytic setting; this will be elaborated further in §3. Before that, we discuss in §2 some geometric properties of the moduli and Teichmüller spaces which are essential for understanding and proving Grothendieck's assertion. We have also included for completeness a fourth section which deals with the fundamental groups of the coarse moduli spaces, that is, as topological spaces, forgetting the orbifold structure. Finally, the last section consists of a short informal discussion of the context of Grothendieck's assertion and possible consequences.

In closing this introduction, it is a pleasure to thank X.Buff, A.Douady, I.Faucheux, J.Fehrenbach and L.Schneps for numerous discussions on this and related topics. I also wish to warmly thank G.Maltsiniotis, H.Nakamura and F.Oort for carefully reading a first version of the manuscript and for suggesting changes and improvements. In particular, H.Nakamura pointed out the paper of D.Patterson ([P]) to me.

§2. Some geometry at infinity of the moduli spaces

We fix a type (g, n) of hyperbolic surfaces (i.e. we assume that $2g+n > 2$) and let $\mathcal{T}$ and $\mathcal{M} = \mathcal{T}/\Gamma$ be the associated Teichmüller and moduli spaces as above, with projection $p : \mathcal{T} \to \mathcal{M}$. A point of $\mathcal{M}$ corresponds to a Riemann surface X which is canonically endowed with a Poincaré metric of constant curvature -1. For a given closed curve γ on X, we let $\ell(\gamma)$ denote the length of the unique geodesic curve which is freely homotopic to γ. It exists and is simple if γ is simple. Now, for $\varepsilon > 0$, we define $\mathcal{M}^\varepsilon$ as the set of Riemann surfaces X such that there exists on X a simple closed curve γ with $\ell(\gamma) < \varepsilon$; we define $\mathcal{T}^\varepsilon = p^{-1}(\mathcal{M}^\varepsilon)$. These are open sets in $\mathcal{M}$ and $\mathcal{T}$ respectively.

Note that $\mathcal{M}^\varepsilon$ is indeed a neighbourhood of infinity in $\mathcal{M}$ (see below for more details) whereas the analog is not true for $\mathcal{T}^\varepsilon$, which for any ε extends "well inside" $\mathcal{T}$; the part at infinity of the Teichmüller space $\mathcal{T}$ will not play any role here. We also recall that on $\mathcal{T}$, but *not* on $\mathcal{M}$, there is a well-defined *length function.* Precisely, we may consider C, a simple closed curve on the reference surface S, or rather a free homotopy class of closed curves, containing a simple representative (we shall sometimes simply write "a curve C"). Then for any $(X, f) \in \mathcal{T}$, we use the marking f to determine the homotopy class $f_*(C)$ on the Riemann surface X, and define $\ell_C(X, f) = \ell(f_*(C))$. With this in mind one may redefine $\mathcal{T}^\varepsilon$ as the set of $(X, f) \in \mathcal{T}$ such that $\ell_C(X, f) < \varepsilon$ for *some* simple closed curve C, and then $\mathcal{M}^\varepsilon = p(\mathcal{T}^\varepsilon)$.

We now recall some facts from hyperbolic geometry, which *in fine* (see corollary below) will enable us to relate the fundamental group of $\mathcal{M}$ at infinity to the fundamental group of $\mathcal{M}^\varepsilon$ for ε small enough. First $\mathcal{M} \setminus \mathcal{M}^\varepsilon$ is compact for any ε, and the family $(\mathcal{M} \setminus \mathcal{M}^\varepsilon)_{\varepsilon>0}$ is *cofinal* in the partially ordered (by inclusion) family of all compact subsets of $\mathcal{M}$. Indeed, a classical theorem of Mumford (cf. e.g. [T], Lemma 3.2.2 and Appendix C) asserts that the lengths of the simple closed geodesics of a compact family K of Riemann surfaces are bounded from below, which precisely means that $K \subset \mathcal{M} \setminus \mathcal{M}^\varepsilon$ for some $\varepsilon > 0$.

Before we formally define the fundamental group of a space M "at infinity", we briefly discuss the problem of base points in this setting. We assume that M is a countable union of compact sets; the compact subsets K of M, partially ordered by inclusion, define an inverse sytem of sets: if $K \subset K'$, we simply consider the inclusion $M \setminus K' \subset M \setminus K$. A *base point at infinity* (denoted $*$) is given by an open part $U \subset M$ such that for any compact set K, there exists a compact set K' with $K \subset K'$ and $U \setminus K'$ non empty and *simply* connected. Let now π_1 denote the fundamental group, either topological, or as an orbifold, for the time being; the two cases for $M = \mathcal{M}$ will be discussed in the next two sections. We set:

Definition. Let M be a space which is a countable union of compact sets and assume there exists a base point at infinity $*$ for M, defined by an open set U. We define the *fundamental group at infinity of M, based at $*$* as:

$$\pi_1^\infty(M, *) = \varprojlim \pi_1(M \setminus K, U \setminus K),$$

where the inverse limit is over the cofinal family of compact subsets K of M such that $U \setminus K$ is simply connected, partially ordered by inclusion and using the natural induced maps on the fundamental groups.

We want of course to apply this to the case $M = \mathcal{M}$, which necessitates the construction of – at least one – base point at infinity. We shall be rather sketchy at this point as in this note we just need the existence of such an object: in order to construct it, one can start from a pants decomposition (maximal multicurve) of the topological surface S, which defines a system of Fenchel-Nielsen coordinates on the Teichmüller space $\mathcal{T}$. These coordinates can be written as $(r_i, t_i) \in \mathbb{R}^{+*} \times \mathbb{R}$, $i = 1, \ldots, 3g-3+n$, where the pairs are indexed by the simple closed curves appearing in the decomposition; the r_i's denote the lengths of the curves, and the t_i's are the associated twist parameters. Now, if we consider the region defined by $t_i \notin 1 + 2\mathbb{Z}$ for all i, and let $U \subset \mathcal{M}$ be the projection of this region to $\mathcal{M}$, U defines a base point at infinity $*$. This being said, we shall now for simplicity drop the mention of base points (finite as well as at infinity) from the notation. The above should make what we write clear.

Now, from the discussion at the beginning of this section, we conclude that in both cases (topological and orbifold fundamental groups), $\pi_1^\infty(\mathcal{M}) = \lim\ \pi_1(\mathcal{M}^\varepsilon)$ as $\varepsilon \to 0$, where there are natural maps $\pi_1(\mathcal{M}^\varepsilon) \to \pi_1(\mathcal{M}^{\varepsilon'})$ when $0 < \varepsilon < \varepsilon'$, induced by the inclusion $\mathcal{M}^\varepsilon \subset \mathcal{M}^{\varepsilon'}$. Another important phenomenon then comes in; namely there exists an absolute constant μ (actually $\mu = \ln(1+\sqrt{2})$ will do and is optimal) such that on any hyperbolic Riemann surface, two simple closed geodesics of lengths $< \mu$ do not intersect. This is a straightforward consequence of the so-called "collar lemma" (cf. [T], Lemma 3.2.1 and Appendix D) and $\mathcal{M}^\mu$ (resp. $\mathcal{T}^\mu$) can be called the *thin* part of the moduli (resp. Teichmüller) space. Using this, we can now state and prove (essentially following [Ha] §3.2) a proposition which is the first important step in the justification of Grothendieck's assertion.

Proposition 1. *Let ε and ε' be given, with $0 < \varepsilon < \varepsilon' < \frac{\mu}{3}$; then:*
i) $\mathcal{T} \setminus \overline{\mathcal{T}}^\varepsilon \simeq \mathcal{T} \setminus \overline{\mathcal{T}}^{\varepsilon'} \simeq \mathcal{T}$, where "$\simeq$" means "diffeomorphic to"; moreover the diffeomorphisms can be chosen equivariant with respect to the action of the Teichmüller modular group Γ;
ii) $\mathcal{T}^{\varepsilon'} \simeq \mathcal{T}^\varepsilon$, and the implied diffeomorphism can be chosen Γ-equivariant;
iii) $\mathcal{T}^{\varepsilon'}$ is a strong deformation retract of $\partial\overline{\mathcal{T}}^\varepsilon$; in particular, these two spaces have the same homotopy type. Again the retraction can be chosen Γ-equivariant.

Note that in i), which will not be used in the sequel, the assertions for $\mathcal{T}^\varepsilon$ and $\mathcal{T}^{\varepsilon'}$ are not visibly related and we simply have to prove the property for any $\varepsilon < \frac{\mu}{3}$. Here $\overline{\mathcal{T}}^\varepsilon$ denotes the closure of $\mathcal{T}^\varepsilon$, which consists of the marked surfaces (X, f) with $\ell_{C_0}(X, f) \leq \varepsilon$ for some curve C_0. The open set $\mathcal{T} \setminus \overline{\mathcal{T}}^\varepsilon$ thus consists of the marked surfaces such that $\ell_C(X, f) > \varepsilon$ for all curves C. The fact that this set is open, or equivalently that the description of $\overline{\mathcal{T}}^{\varepsilon}$ given above is correct, that is describes indeed the closure of $\mathcal{T}^{\varepsilon}$, is a consequence of the discreteness of the length spectrum of any Riemann surface: when verifying the above conditions, one has in fact to deal with a *finite* number of short curves only, which implies that the condition is indeed open, since for a given curve C, the length function ℓ_C is continuous on $\mathcal{T}$.

As for iii), note that $\mathcal{T}^{\varepsilon} \subset \mathcal{T}^{\varepsilon'}$ and that $\partial\overline{\mathcal{T}}^{\varepsilon}$ denotes the boundary of $\overline{\mathcal{T}}^{\varepsilon}$, comprising the marked surfaces such that $\ell_C(X, f) \geq \varepsilon$ for all curves C and $\ell_{C_0}(X, f) = \varepsilon$ for at least one curve C_0. The complement $K^{\varepsilon} = \mathcal{T} \setminus \mathcal{T}^{\varepsilon}$ is a compact set and $\partial\overline{\mathcal{T}}^{\varepsilon} = \partial K^{\varepsilon}$. The image $p(\partial\overline{\mathcal{T}}^{\varepsilon})$ of this set in $\mathcal{M}$ is simply the set of Riemann surfaces which have all their simple closed geodesics of length $\geq \varepsilon$, with at least one of length exactly ε .

Proposition 1 is an easy consequence of the following lemma, due to S.Wolpert (see [Ha], Lemma 3.7).

Lemma 2. (S. Wolpert) *Let $\phi : [0, \infty) \to [0, 1]$ be a C^{∞} function on the positive real axis, with support in $[0, \mu)$. Then there exists a Γ-equivariant vector field $W = W_{\phi}$ on $\mathcal{T}$ with the following property: let Φ^t be the flow generated by W, and C be any curve on S, defining a length function ℓ_C on $\mathcal{T}$; then one has:*

$$\frac{d}{dt}(\Phi^t \circ \ell_C) = \phi \circ \ell_C.$$

Here the surprise, if any, consists in the possibility of dealing with all the (classes of simple closed) curves simultaneously; it is indeed easy, using Fenchel-Nielsen coordinates, to produce a vector field with the above property for the curves of a given "pants decomposition" of the surface S, but here one achieves much more. This statement reflects the convexity properties of the length fonctions; the proof is not difficult, given some geometric work of S.Wolpert. Note that W actually vanishes outside the thin part $\mathcal{T}^{\mu}$ of $\mathcal{T}$. The equivariance of W means, as usual, that for any $\gamma \in \Gamma$, one has $W \circ \gamma = \gamma_* W$; thus W descends to a vector field on $\mathcal{M}$, vanishing outside the thin part $\mathcal{M}^{\mu}$ lying "at infinity".

Let us now turn to the proof of proposition 1, granting lemma 2. To prove i) for $\mathcal{T}^{\varepsilon}$, we choose (as in [Ha]) a function ϕ which is decreasing, and such that $\phi = 1$ on $[0, \varepsilon]$ and $\phi = 0$ on $[2\varepsilon, \infty)$. Consider the vector field $W = W_{\phi}$ whose existence is asserted by lemma 2 and the corresponding time ε diffeomorphism Φ^{ε}. It is easy to see that Φ^{ε} maps $\mathcal{T}$ diffeomorphically onto $\mathcal{T} \setminus \overline{\mathcal{T}}^{\varepsilon}$. Since W actually vanishes outside $\mathcal{T}^{2\varepsilon}$, the set $\mathcal{T} \setminus \overline{\mathcal{T}}^{2\varepsilon}$ is kept fixed pointwise by the diffeomorphism.

In order to prove ii), we choose another function ϕ. The support of ϕ is now contained in $[0,2\varepsilon']$ and we require that $\phi(0)=0$ and $\phi=1$ over the interval $[\varepsilon,\varepsilon']$. The corresponding time $(\varepsilon'-\varepsilon)$ diffeomorphism, i.e. $\Phi^{\varepsilon'-\varepsilon}$, then maps $\mathcal{T}^{\varepsilon}$ onto $\mathcal{T}^{\varepsilon'}$.

To prove iii), we construct the required retraction Ψ as follows: on $\overline{\mathcal{T}}^{\varepsilon}$ we use the same function ϕ as in i), only with ε replaced with ε', and the corresponding vector field W and flow Φ^t; we flow the points forward, until they cross the boundary $\partial\overline{\mathcal{T}}^{\varepsilon}$. That is, if $\tau=(X,f)\in\overline{\mathcal{T}}^{\varepsilon}$, we set $\Psi(\tau)=\Phi^{t_*}(\tau)$ where $t_*\geq 0$ is the first (and actually only) instant t when $\Phi^t(\tau)\in\partial\overline{\mathcal{T}}^{\varepsilon}$. It is easy to see from i) that t_* exists, with $t_*\leq\varepsilon'$. Now, on $\mathcal{T}^{\varepsilon'}\setminus\mathcal{T}^{\varepsilon}$ we use the same function ϕ as in ii) and define the retraction by flowing the points backward until they reach $\partial\overline{\mathcal{T}}^{\varepsilon}$. This completes the proof of proposition 1.

We record for future use a corollary to proposition 1 (more precisely assertions ii) and iii)), which comes from the fact that ε and ε' are arbitrary, subject only to $0<\varepsilon<\varepsilon'<\frac{\mu}{3}$.

Corollary. *For any ε with $0<\varepsilon<\frac{\mu}{3}$, we have $\pi_1^{\infty}(\mathcal{M})=\pi_1(\mathcal{M}^{\varepsilon})$, and $\mathcal{T}^{\varepsilon}$ has the homotopy type of $\partial\overline{\mathcal{T}}^{\varepsilon}$.*

§3. The orbifold fundamental groups of the moduli spaces

In this section, we prove Grothendieck's assertion, using known topological results. We postpone to §5 below a short discussion of the context. Here, as mentioned in §1, we have to deal with the *fine* moduli spaces $\mathcal{M}_{g,n}$ and we proceed to say a few words about it; there are however several versions of the theory, with a more or less abstract and algebraic flavour. Again, all we need here is the (essentially topological) definition-proposition 3 below and the coming discussion has been added only for the sake of completeness.

The basic "defect" of the *coarse* moduli spaces is that they do not carry a *universal family*. Using the language of analytic geometry, recall that a family Z of curves of type (g,n) parametrized by an analytic space S is given by a smooth and proper map $Z\to S$ whose fibers (i.e. the preimages of –geometric– points) are curves of type (g,n). By definition, such a family $\mathcal{C}_{g,n}$, parametrized by $\mathcal{M}_{g,n}$, is universal if any family $Z\to S$ can be viewed (in a unique way) as the pull-back of $\mathcal{C}_{g,n}\to\mathcal{M}_{g,n}$ via a map $\phi:S\to\mathcal{M}_{g,n}$. Such a universal family (or "universal curve") does *not* exist in the category of analytic spaces, but it does exist in the category of –analytic– orbifolds and this is why these are important in this contexts. The nonexistence of a universal family in the context of analytic spaces is due to the existence of nontrivial automorphisms of some curves (or Riemann surfaces). And, as noted very briefly in §1, orbifolds are made to keep track of these automorphisms. More precisely, an orbifold is given by the data of suitably compatible charts such that in each chart the orbifold is modelled on a quotient D/G where D is an open set (a polydisk) of $\mathbb{C}^d$ and G is a finite group acting on it; it is important that the various groups G are considered as part of the data. One may endow the spaces $\mathcal{M}_{g,n}$ with orbifold structures, where the groups are the automorphism groups of the underlying (isomorphism classes of) Riemann surfaces. This is carried out in [Mu, §2] (in the case $n = 0$). From the analytic viewpoint, these are the objects we actually have to deal with. Concerning the fundamental group of an orbifold, again, we shall only make use of the definition-proposition 3 below; we refer the reader to [HQ] for a concrete topological approach in terms of loops and equivariant homotopies. In this section we always denote the orbifold fundamental group simply as "π_1".

We note that there is an algebraic way to deal with the above, namely using algebraic "stacks", which were introduced in [DM, §§ 4,5] in this context; the fact that they have thus defined a nice fine moduli space is given in their proposition §5.1, which says that $\mathcal{M}_{g,0}$ (they deal with the case $n = 0$ only), which is a priori defined via a representability property, is in effect a separated algebraic stack of finite type over Spec($\mathbb{Z}$). One then proceeds to develop the theory of the algebraic fundamental groups of stacks, which is the subject matter of T.Oda's contribution to this volume, to which we refer the interested reader (who will also find there a quick introduction to stacks in general).

As for now, we return to more topological matters and state a simple

Definition-Proposition 3. *Let T be a simply connected non singular manifold on which the discrete group G acts properly discontinuously. Then the quotient space $M = T/G$ has a natural orbifold structure and the orbifold fundamental group of M (with respect to any base point) is isomorphic to G: $\pi_1(M, *) \simeq G$.*

This applies to the case when $T = \mathcal{T}$, $G = \Gamma$ and $M = \mathcal{M}$, once one knows that the action of Γ is proper and discontinuous, a classical result. Grothendieck's assertion now reads:

$$\pi_1^\infty(\mathcal{M}_{g,n}) = \pi_1(\mathcal{M}_{g,n}) = \Gamma_{g,n} \text{ if and only if } d = 3g - 3 + n > 2. \qquad (*)$$

Now, the action of Γ on $\mathcal{T}$ obviously restricts to a still proper and discontinous action on $\mathcal{T}^\varepsilon$, simply because this is an open domain of $\mathcal{T}$, invariant under the action of Γ. Moreover, one has $\mathcal{M}^\varepsilon = \mathcal{T}^\varepsilon/\Gamma$ by definition. Picking any ε with $0 < \varepsilon < \frac{\mu}{3}$, and refering to the first assertion of the corollary in the last section, we see that, according to definition-proposition 3 above, assertion $(*)$ is implied by (actually equivalent to) the following result:

Proposition 4. *For $\varepsilon > 0$ small enough (actually for $\varepsilon < \frac{\mu}{3}$), $\mathcal{T}^{\varepsilon}_{g,n}$ is simply connected if and only if $d = 3g - 3 + n > 2$.*

The next reduction comes by using the second assertion of the corollary, which asserts that $\mathcal{T}^{\varepsilon}$ has the same homotopy type as $\partial\overline{\mathcal{T}}^{\varepsilon}$. So in order to prove the result, one needs only determine the homotopy type of this last space.

In order to state the next result, one has to introduce complexes of curves, which have played a prominent role in the recent topological investigations of the Teichmüller modular groups (see [Ha], [I] and references therein). In fact, we need only define the complex $\mathcal{Z} = \mathcal{Z}_{g,n}$ which was originally introduced by Harvey in his short seminal paper ([H]). It is a simplicial complex whose dimension k simplexes are defined to be the isotopy classes of families $\{C_0, C_1, \ldots, C_k\}$ of disjoint simple closed curves on the reference topological surface $S = S_{g,n}$. Vertices (zero simplexes) are given by (the class of) one such curve and the complex has dimension $3g-4+n$, a simplex of maximal dimension being associated to a "pants decomposition" of S (a maximal multicurve), comprising $3g - 3 + n$ curves and defining a system of Fenchel-Nielsen coordinates on $\mathcal{T}$. To finish with the definition of $\mathcal{Z}_{g,n}$ one has to describe the face relations, and this is done quite naturally, by defining the k-simplex associated to the family $\{C'_0, C'_1, \ldots, C'_j\}$ to be a face of the one associated to $\{C_0, C_1, \ldots, C_k\}$ $(j \leq k)$ if the (unordered) family $\{C'_r\}$ is a subset of $\{C_s\}$ (as isotopy classes of curves). The relevance of this object is obvious from the following

Lemma 5. *$\partial\overline{\mathcal{T}}^{\varepsilon}_{g,n}$ has the homotopy type of $\mathcal{Z}_{g,n}$.*

This is actually an easy lemma, which is proven by describing $\partial\overline{\mathcal{T}}^{\varepsilon}_{g,n}$ "explicitly" (see [Ha]); tracing through the definitions, it emerges that this is built up from $\mathcal{Z}$ by replacing cells by Teichmüller spaces of the right dimensions or the products of such spaces with Euclidean spaces. In both cases, these pieces are contractible and the homotopy type reduces to that of the complex $\mathcal{Z}$; we refer again to [Ha] for a brief account, as well as to the references there (and to [I]) for more details. This description also leads to the construction of a bordification of the Teichmüller space $\mathcal{T}$, which was the initial motivation of Harvey, when introducing $\mathcal{Z}$.

Lemma 5 reduces the proof of proposition 4 (hence also of $(*)$) to investigating the homotopy type of $\mathcal{Z}$. This is settled by the following

Theorem 6. *The complex $\mathcal{Z}_{g,n}$ is homotopically equivalent to a wedge of k-spheres, where $k = 2g - 2$ for $g > 0$, $n = 0$, whereas $k = 2g - 3 + n$ for $g > 0$, $n > 0$, and $k = n - 4$ when $g = 0$ $(n \geq 4)$.*

This finishes the proof of $(*)$, since in all cases, $k \geq 2$ if and only if $d = 3g-3+n > 2$, implying that $\mathcal{Z}_{g,n}$ is simply connected. The above theorem is stated in this form in [Ha, §4], with a sketch of proof. One can find a variety of such statements in the literature as it gradually emerged, starting with [H] and [HT], that many properties of the Teichmüller modular groups $\Gamma_{g,n}$ can be traced to homotopical properties of complexes of curves. Various such complexes have been introduced, and many variants and connections are recorded in [I], including the connection between $\mathcal{Z}$, as introduced in [H], and the – at first sight less natural – complex which enabled Hatcher and Thurston to prove (in [HT]) that the $\Gamma_{g,n}$'s are finitely presented (see also [W], [G]).

§4. The topological fundamental groups of the moduli spaces

In this section, we shall be concerned with the fundamental groups of the *coarse* moduli spaces, for which we retain the notation $\mathcal{M}_{g,n}$, "forgetting" however about the orbifold structure. These – discrete – fundamental groups will be denoted as "π_1^{top}". We have added the discussion below partly in order to emphasize the contrast with the previous section. Results are not quite complete and we shall have to leave some assertions as they are, hoping that they sound plausible enough and that some readers might want to try and provide the missing proofs (assuming they do not already exist in the literature, which we cannot of course guarantee).

Actually, for reasons which will appear clearly below, in this section, we have to include the possibility of permuting the marked points, and must accordingly slightly redefine the moduli spaces and their fundamental groups (the Teichmüller spaces remain unchanged). The permutation group $\mathcal{S}_n$ acts on the moduli space $\mathcal{M}_{g,n}$; actually the action is faithful except when $(g,n) = (0,4)$ (in which case it factors through $\mathcal{S}_3$), and $\mathcal{S}_n = Aut(\mathcal{M}_{g,n})$, the group of analytic automorphisms of the space $\mathcal{M}_{g,n}$, except again when $(g,n) = (0,4)$ (since $Aut(\mathcal{M}_{0,4}) = \mathcal{S}_3$). So we now consider the quotient space $\mathcal{M}_{g,[n]} = \mathcal{M}_{g,n}/\mathcal{S}_n$. Another way to put it is that $\mathcal{M}_{g,[n]}$ is obtained as a quotient of $\mathcal{T}_{g,n}$ by allowing maps which fix the set of marked points but can permute them; the notation $[n]$ is intended to suggest that the marked points are then considered setwise, not individually. The Teichmüller modular groups are modified accordingly, introducing $\Gamma_{g,[n]}$ as the orbifold fundamental group of $\mathcal{M}_{g,[n]}$, with the tautological exact sequence:

$$1 \to \Gamma_{g,n} \to \Gamma_{g,[n]} \to \mathcal{S}_n \to 1.$$

As a matter of terminology, the group $\Gamma_{g,[n]}$ is often called the *full* mapping class group ($\Gamma_{g,n}$ being the *pure* subgroup) and classifies the connected components of the group of orientable diffeomorphisms of the surface $S_{g,n}$, preserving the marked points setwise. Note the obvious equalities: $\Gamma_{g,[0]} = \Gamma_{g,0} = \Gamma_g$ and $\Gamma_{g,[1]} = \Gamma_{g,1}$. Here again, we shall sometimes drop the subscript $(g,[n])$ from the notation.

The starting point for investigating the fundamental groups of the coarse moduli spaces is a simple and useful result, contained in [A], which is to be contrasted with definition-proposition 3.

Proposition 7. *Let T be a simply connected non singular manifold on which the discrete group G acts properly discontinously. Let G^f be the subgroup of G generated by those elements in G which have fixed points (when acting on T). Then the fundamental group of the (possibly singular) quotient space $M = T/G$ is isomorphic to G/G^f, i.e. $\pi_1^{top}(M,*) \simeq G/G^f$.*

Note that the subgroup G^f is normal, because for any $(g,g') \in G$ and $\tau \in T$, one has $g'gg'^{-1}(g'\tau) = g'\tau$ if $g\tau = \tau$. In the case we are interested in, namely again $T = \mathcal{T}$, $G = \Gamma$ and $M = \mathcal{M}$, the subgroup G^f coincides with the group generated by the elements of finite order in Γ. This comes from a classical result (due to Nielsen), which says that any element of finite order in Γ has a fixed point in $\mathcal{T}$. So in our case, $G^f = \Gamma^t$, the subgroup of the Teichmüller modular group Γ generated by the elements of finite order.

We thus obtain the equality: $\pi_1^{top}(\mathcal{M}_{g,[n]},*) \simeq \Gamma_{g,[n]}/\Gamma^t_{g,[n]}$, where $*$ denotes any given base point. This last group was computed by D.Patterson in [P] in the general case, with the same purpose as ours. The final result reads:

Theorem 8. *The coarse moduli spaces $\mathcal{M}_{g,[n]}$ are simply connected, i.e. we have $\pi_1^{top}(\mathcal{M}_{g,[n]},*) = \{1\}$, except when $g = 2$ and $n = 4$ mod 5, in which cases $\pi_1^{top}(\mathcal{M}_{2,5k+4},*) = \mathbb{Z}/5\mathbb{Z}$.*

The result for $n = 0$ and any genus had been previously obtained by C.MacLachlan in [M], again as a consequence of proposition 7 and in conjunction with a note of J.Birman ([B]) in which she discusses certain explicit generators for the groups Γ_g. The existence of a series of exceptional cases in theorem 8 looks odd and we have no general explanation to offer for this phenomenon, which perhaps deserves further investigation. Note that it is at this point that one has to use the *full* modular groups, and consequently to allow permutation of the marked points in the definition of the moduli spaces. Indeed, whereas $\Gamma_{0,[n]}$ is generated by its torsion elements, $\Gamma_{0,n}$ is torsion free.

We turn briefly to the topological fundamental groups at infinity of the coarse moduli spaces. The definition of §2 is still valid in this setting, except that, for consistency, we have to use here the unfortunately heavy notation "$\pi_1^{\infty,top}$". We shall see that the equality of the fundamental group and the fundamental group at infinity for spaces of dimensions $d > 2$, is intimately connected with the distribution of Riemann surfaces with nontrivial symmetry groups, a theme which also appears in the *Esquisse*, as Grothendieck suggests to use these exceptional surfaces as base points for the fundamental Teichmüller groupoid. In order to investigate these topological fundamental groups at infinity, we start again from the corollary in §2. That is, we still have $\pi_1^{\infty,top}(\mathcal{M}) = \pi_1^{top}(\mathcal{M}^\varepsilon)$ for ε small enough ($\varepsilon < \frac{\mu}{3}$). Moreover, we may again use proposition 4 in conjunction with proposition 7. We conclude that for $d > 2$ (and only then), the thin part $\mathcal{T}_{g,n}^\varepsilon$ of Teichmüller space is simply connected (recall that $\mathcal{T}_{g,[n]}^\varepsilon = \mathcal{T}_{g,n}^\varepsilon$) and $\pi_1^{top}(\mathcal{M}_{g,[n]}^\varepsilon)$ is then determined by the – proper and discontinous – action of $\Gamma_{g,[n]}$ on this space.

We thus find that for $d > 2$, we have $\pi_1^{\infty,top}(\mathcal{M}_{g,[n]}) = \Gamma_{g,[n]}/\Gamma_{g,[n]}^f$, where $\Gamma_{g,[n]}^f$ is the subgroup of $\Gamma_{g,[n]}$ generated by the elements which have a fixed point when acting on $\mathcal{T}_{g,n}^\varepsilon$ ($\varepsilon < \frac{\mu}{3}$). We shall now elucidate the nature of this subgroup, making it plausible that it is indeed independent of ε and actually equal to $\Gamma_{g,[n]}^t$. So we state as a

Plausible assertion. *One has* $\pi_1^{\infty,top}(\mathcal{M}_{g,[n]}) = \pi_1^{top}(\mathcal{M}_{g,[n]})$ *if (and only if)* $d = 3g - 3 + n > 2$.

Again, compatible base points and base point are infinity (as explained in §2) are implied. Thus $\mathcal{M}_{g,[n]}$ would be "simply connected at infinity", except along the mysterious exceptional series which appears in theorem 8.

This assertion is of course a strengthening of theorem 8 since the torsion elements of $\Gamma_{g,n}$ are exactly those which have a fixed point when acting on the whole Teichmüller space $\mathcal{T}_{g,n}$. Also, in theorem 8, the cases of dimensions 1 and 2 ("les deux premiers étages" of the *Esquisse*) can be dealt with "by inspection". The relationship between the fixed points of the action of the Teichmüller modular groups and the Riemann surfaces with non trivial symmetries is classical and has been developped by several authors. Here we just recall a few relevant facts, refering to [GH] for more details and references. Fix some type (g, n) of surfaces, and let $H \subset \mathrm{Diff}^+(S)$ ($S = S_{g,n}$ etc.) be a finite group of of diffeomorphisms of the reference surface, such that no two disctinct elements are isotopic (permutations of marked points allowed). We also view H as a finite subgroup of the mapping class group Γ, by considering the classes of the elements of H modulo isotopy. Define now $\mathcal{T}(H)$ as the set of marked surfaces (X, f) such that there is a subgroup H_X of the group of the – conformal – automorphisms of X which is conjugate to H via the marking f; that is, $H_X \subset Aut(X)$ satisfies $H_X = f \circ H \circ f^{-1}$, where the elements of H_X are viewed as diffeomorphisms (and X as a topological surface). We shall identify H_X with H, via f. Finally, we let $\mathcal{M}(H) = p(\mathcal{T}(H))$ where p denotes as usual the projection from $\mathcal{T}$ onto $\mathcal{M}$. It is natural to call $\mathcal{M}(H)$ the set of – isomorphism classes of – Riemann surfaces with H-symmetry. By definition, if H and S are fixed as above, and if $X \in \mathcal{M}(H)$, there is a subgroup $H_X \subset Aut(X)$, isomorphic to H, and such that for any marking g, i.e. any diffeomorphism $g : S \to X$, the finite group $f^{-1} \circ H_X \circ f$ is *conjugate to* H in $\mathrm{Diff}^+(S)$. Warning: it may be that (X, g) is not in $\mathcal{T}(H)$, because the modular group Γ permutes its finite subgroups by conjugation: if $(X, f) \in \mathcal{T}(H)$ for some $H \subset \mathrm{Diff}^+(S)$, one has $(X, g) \in \mathcal{T}(H)$ if and only if $f^{-1} \circ g$ normalizes H in Γ.

There is another viewpoint on $\mathcal{T}(H)$, the equivalence being the subject matter of classical theorems (cf. Theorem A in [GH]). Namely, start from any finite subgroup of the mapping class group Γ, and let it act on $\mathcal{T}$ via the natural action of Γ: the fixed point set is precisely $\mathcal{T}(H)$. Moreover, by the positive answer to the Nielsen realization problem, given by Kerchkhoff (in [K]), $\mathcal{T}(H)$ defined this way is not empty, whatever the finite group $H \subset \Gamma$ (at least in the compact case, i.e. when $n = 0$). In particular, we can always start from such a subgroup and lift it to $\mathrm{Diff}^+(S)$ if necessary (as in the first description of $\mathcal{T}(H)$), because given an element (X, f) of $\mathcal{T}(H)$, the subgroup H_X of $Aut(X)$ will define such a lift; here one uses an old result, due to Hurwitz, which says that a nontrivial automorphism of a Riemann surface is not isotopic to identity. This sets up the connection between surfaces with symmetries and fixed points of the action of Γ on $\mathcal{T}$.

The space $\mathcal{T}(H)$ parametrizes the marked surfaces (X, f) with H-symmetry; now we may view H as a subgroup of $\mathrm{Diff}^+(S)$ and look at the quotient surface $R = S/H$, the projection map $S \to R$ being ramified at some points $p_i \in R, i = 1, \ldots, r$. We also introduce R^*, which is obtained by puncturing R at the points p_i, and we let (γ, ν) denote the type of the surface R^*. For any (X, f), the marking f induces a diffeomorphism from R^* to Y^*, obtained analogously to R^*, by puncturing $Y = X/H$ at the ramification points of the cover $X \to Y$. This being said, it should not come as a surprise that this construction induces a biholomorphic equivalence between $\mathcal{T}(H)$ and the Teichmüller space $\mathcal{T}_{\gamma,\nu}$ of surfaces of type (γ, ν) (cf. Theorem B in [GH]). We shall call a finite subgroup H of Γ *maximal* if $\mathcal{T}(H)$ (as well as $\mathcal{M}(H)$) has dimension 0, which by the above is tantamount to saying that R^* is a thrice punctured sphere. We shall say that $\mathcal{M}(H)$ *extends to infinity* if $\mathcal{M}(H) \cap \mathcal{M}^\varepsilon$ is not empty, for any $\varepsilon > 0$. We hope the following statements will now sound natural (perhaps tantalizing) to the reader:

i) For any (g, n), and any *non* maximal finite subgroup $H \subset \Gamma_{g,[n]}$, the subvariety $\mathcal{M}_{g,[n]}(H)$ extends to infinity;

ii) If $d = 3g - 3 + n > 2$, $\Gamma^t_{g,[n]}$ is actually generated by torsion elements which generate *non* maximal – cyclic – subgroups.

By the above, these two statements would imply the "plausible assertion". Statement ii) is purely group theoretic and could hopefully be extracted from a careful reading of [P]. Statement i) is geometric and is interesting in its own right. It can be rephrased in terms of the Teichmüller space $\mathcal{T}(H)$ and the thin part $\mathcal{T}^\varepsilon$ and can be further reduced as follows: going to infinity in the moduli spaces is done by pinching simple closed geodesics of the corresponding Riemann surfaces; so, all one has to do in order to prove i) is to construct such a geodesic, in the situation at hand. More precisely, let $X \in \mathcal{M}(H)$ be a surface with H-symmetry, with $H_X \subset Aut(X)$ the corresponding symmetry group; set again $Y = X/H_X$. Thinking in hyperbolic terms, X is endowed with its canonical Poincaré metric μ (constant curvature -1), which descends to a metric $\bar{\mu}$ on Y (since the elements of H_X act isometrically on X), which is however singular at the ramification points of the covering map $X \to Y$. It does not of course coincide with the Poincaré metric of the punctured surface Y^*. Now, assertion i) above would be a direct consequence of the following:

iii) Under the above assumptions (in particular H non maximal), there exists on Y a simple closed geodesic for the quotient metric $\bar{\mu}$, which does not contain a ramification point of the covering map $X \to Y$.

We shall end at this point our discussion of the fundamental groups at infinity of the coarse moduli spaces, hoping to have somehow convinced the reader that the analogs of Grothendieck's assertion in this situation also give rise to interesting problems, that moreover seem to be "within reach".

§5. Informal discussion

This volume is mainly intended as a guide to certain themes which appear more or less explicitly in the *Esquisse.* We thus feel it may be useful to devote this last section to some remarks which may help the "nonexpert" reader put the above in a more general context. The groups $\Gamma_{g,n}$ appear in two different contexts, namely topology, where they go under the name "mapping class groups" ($\Gamma_{g,n} = \pi_0(\mathrm{Diff}^+(S_{g,n}))$ and analytic or algebraic geometry where they are often called "Teichmüller modular groups" ($\Gamma_{g,n} = \pi_1(\mathcal{M}_{g,n})$). It is of course easy to recognize that these define one and the same family of groups, but the flavour and the methods of investigation remain quite distinct. There are many reasons why these groups have attracted a lot of attention since almost a century ago, which can often be traced to the fact that they embody an alternative (w.r.t. to the classical arithmetic groups) generalization of $SL(2,\mathbb{Z})$ (cf. *Esquisse*, p.6, the second underlined sentence). This is perhaps exemplified essentially by the $n = 0$ cases, while the braid groups of the sphere ($g = 0$) and their manifold generalizations trace a somewhat different story.

These groups $\Gamma_{g,n}$ have thus been the subject matter of a long series of investigations, conducted mainly via topological methods (see [B 1,2]). In particular, it took a lot of efforts to prove that they are finitely generated and a lot more time to show that they are actually finitely presented. This was achieved essentially by looking at their natural faithful action on the complexes of curves ([H] and [HT]), eventually leading to an explicit presentation ([W], [G]) and making it possible to extract information on their cohomology. Since the Teichmüller spaces are contractible, the rational group cohomology of the $\Gamma_{g,n}$ coincides with the rational cohomology of the spaces $\mathcal{M}_{g,n}$, an identity which is often exploited. In any case, the starting point of these – comparatively recent – studies often resides in statements which sound very much like theorem 6 in §3. Now, Grothendieck is of course on the algebraic-geometric side and he extracted a statement (formally $(*)$ in §3) which, as we have seen, can be proved by "translating" it into topological terms. In fact, Grothendieck, being apparently largely unaware of the topological ideas which were being developped almost simultaneously, had an insight which, among other things, would eventually lead to a proof that the $\Gamma_{g,n}$'s are finitely presented (see below for more details), a result people had been after for half a century.

This statement about the fundamental groups at infinity of the moduli spaces should of course not be taken as an isolated assertion, as we shall now try to make more precise. Let us say right away that it is not clear to the author what exactly could be done on this particular subject along the lines suggested by Grothendieck, which has not already been achieved by topological methods. But it is certainly a beautiful and compelling vision, which in any case deserves to be explored. We first note that Grothendieck does suggest a way of proving assertion $(*)$ by algebraic-geometric methods (p.7), and it would be interesting to pursue this suggestion. Then, this assertion should be seen as the first (or say the general) step of a "dévissage", justifying the important "two level reconstruction principle", stated on p.6 (first underlined sentence). Indeed this "example" was apparently the main incentive that lead Grothendieck to developping an algebraic theory of "stratified structures". This is explained in section 5 of the *Esquisse* (in particular on pp.35-36), and note 7 (p.56, related to p.34, bottom) is also quite suggestive in that respect. Two of the main tools appearing are the notion of "tubular neighbourhood" in an algebraic context and Van Kampen theorem with several base points (cf. bottom of p.5), i.e. in a groupoid version. We proceed to make this more precise in the context of moduli spaces, recalling on the way some "well-known"' facts and references about the divisors at infinity of their stable completions, which are actually useful to keep in mind when reading the *Esquisse*.

The moduli space $\mathcal{M}_{g,n}$ can be compactified by adding the moduli points of "stable curves", here Riemann surfaces with nodes (ordinary double points, when viewed as algebraic curves). This compactification $\overline{\mathcal{M}}_{g,n}$ was first studied by Deligne and Mumford in a more general setting, and the structure of the divisor at infinity $\mathcal{D}_{g,n} = \overline{\mathcal{M}}_{g,n} \setminus \mathcal{M}_{g,n}$ was studied by Knudsen in [Kn]. It is made of finitely many irreducible components, which may be enumerated as follows: Let $S_{g,n}$ be again the reference topological surface of type (g,n) and consider the simple closed curves on $S_{g,n}$, up to diffeomorphism of $S_{g,n}$. There are finitely many such classes; restricting from now on to the case $n=0$ for simplicity, they are as follows. First all non disconnecting curves (i.e. the curves γ such that S_g remains connected after cutting along γ) are equivalent; then if a curve disconnects S_g into two surfaces S_1 and S_2 of type $(g_1,1)$ and $(g_2,1)$ respectively, with $g_1+g_2=g$, the class of the curve depends only on the unordered pair (g_1,g_2). A component of the divisor at infinity $\mathcal{D}_g$ is associated to pinching along such a simple closed curve, again up to diffeomorphism of S_g (since we are dealing with the moduli space, surfaces are unmarked, and only these classes make sense anyway). So by the above there are $1+[g/2]$ such irreducible components (where $[a]$ denotes the integer part of a). Moreover, each component is itself "almost" isomorphic to a moduli space of lower dimension, or to a product of two such spaces. More precisely, assuming again that $n=0$, one sees that the nondisconnecting curves give rise to a component of type $\mathcal{M}_{g-1,2}$ (or rather the compactification of this space), whereas disconnecting curves are associated to the $[g/2]$ direct products $\mathcal{M}_{k,1}\times\mathcal{M}_{g-k,1}$ (or the product of the compactifications). Knudsen proved among other things that the natural "clutching morphism" (glueing the two surfaces "along the nodes") $\mathcal{M}_{k,1}\times\mathcal{M}_{g-k,1}\to\mathcal{M}_g$, is a closed immersion if $k\neq g-k$, whereas the map $\mathcal{M}_{g-1,2}\to\mathcal{M}_g$ is at most finite.

So, the complete space $\overline{\mathcal{M}}_{g,n}$ is indeed a beautiful stratified structure, with open highest dimensional stratum the open moduli space $\mathcal{M}_{g,n}$ and essentially copies of lower dimensional such spaces in the divisor at infinity $\mathcal{D}_{g,n}=\overline{\mathcal{M}}_{g,n}\setminus\mathcal{M}_{g,n}$. The lowest (zero) dimensional strata are obtained by pinching *all* the curves of a "pants decomposition" of the reference surface $S_{g,n}$, again up to diffeomorphisms; there are finitely many such decompositions, each consisting of $3g-3+n$ non intersecting simple curves (the cells of maximal dimension in the complex $\mathcal{Z}_{g,n}$ of §3), which in principle can be combinatorially enumerated. Finally, all the intersections are "as good as possible". More precisely, all intersecting pairs of irreducible components of the divisor at infinity have normal crossings: this reflects the simple fact that they correspond to Riemann surfaces with nodes at different places, and the intersections to surfaces with two nodes. Again, the intersections are essentially moduli spaces of lower dimensions and products thereof. The types can be determined simply by looking at the topology. So, the global picture, which however ignores some technical difficulties (as mentioned above, some maps may not be closed immersions), is that of an intersection graph for $\mathcal{D}_{g,n}$, whose structure can be combinatorially determined, with the vertices corresponding to the irreducible components and the edges to their intersections (including self-intersections); all the spaces appearing are – completed – moduli spaces or products of two copies of such spaces.

Returning to the quotation we started this note with (*Esquisse* p.7), we see that in dimensions strictly larger than 2, the – orbifold – fundamental group of $\mathcal{M}_{g,n}$ is indeed roughly the same as that of a tubular neighbourhood of $\mathcal{D}_{g,n}$ (where this divisor is of course *not* included), namely $\mathcal{M}^{\varepsilon}_{g,n}$ for ε small enough. Computing such a fundamental group is the subject matter of the Grothendieck-Murre theory, expounded in [GM], where however it is required that the irreducible components of the divisor with normal crossings be non singular, which here is not always the case. Recently, Nakamura was able to apply part of this theory in this setting, in order to obtain (in [N]) some arithmetic results on the Galois action on the –profinite– Teichmüller modular groups. Returning briefly to geometry, we note that the fundamental group of the tubular neighbourhood is generated by two kinds of loops: First those which come from $\mathcal{D}_{g,n}$; more precisely, one can shift the generators of the fundamental groups of the components of $\mathcal{D}_{g,n}$ off into the non singular part $\mathcal{M}_{g,n}$. Then one has to add the loops which arise when "going around" the components of $\mathcal{D}_{g,n}$ and correspond to Dehn twists performed along the curves which are pinched when "tending to infinity". This is made precise in [GM], again under some smoothness assumptions. Putting these data together in order to express the fundamental group $\Gamma_{g,n}$ is apparently what it means to apply Van Kampen theorem, with carefully selected base points. Since again all the spaces involved are moduli spaces, this procedure should be viewed as one step of a "dévissage", here a descending induction on the maximal dimension of the spaces involved, which ends when one reaches dimension 2 and the fundamental (dimensions 1 and 2) pieces of the "lego".

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