# ISOTROPY OF SYMPLECTIC INVOLUTIONS

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Abstract. We prove the symplectic analogue of the isotropy theorem for orthogonal involutions [6, Theorem 1]. We apply (a modification of) the method of [8] originally applied to prove the symplectic analogue of the hyperbolicity theorem for orthogonal involutions [5, Theorem 1.1].

**Isotropie d'involutions symplectiques.** Nous démontrons l'analogue symplectique du théorème d'isotropie des involutions orthogonales [6, Theorem 1]. Nous utilisons (une modification de) la méthode de [8] initialement utilisée pour démontrer l'analogue symplectique du théorème d'hyperbolicité des involutions orthogonales [5, Theorem 1.1].

We refer to [7] for terminology and basic facts concerning involutions on central simple algebras. Below, we'll meet myriads of finite odd degree field extensions; we simply call them *odd* for short.

In this note we prove

**Theorem 1.** *Let $F$ be a field of characteristic not 2, $A$ a central simple $F$-algebra, $\sigma$ a symplectic involution on $A$. The following two conditions are equivalent:*

(1) *$\sigma$ becomes isotropic over any field extension $E/F$ such that* $\operatorname{ind} A_E = 2$*;*
(2) *$\sigma$ becomes isotropic over some odd extension of $F$.*

(We recall that $\sigma$ is always isotropic and, moreover, hyperbolic as far as $\operatorname{ind} A = 1$.)

Theorem 1 is the symplectic analogue of the following result on orthogonal involutions:

**Theorem 2** ([6, Theorem 1])**.** *Let $F$ be a field of characteristic not 2, $A$ a central simple $F$-algebra, $\tau$ an orthogonal involution on $A$. The following two conditions are equivalent:*

(1) *$\tau$ becomes isotropic over any field extension $E/F$ such that* $\operatorname{ind} A_E = 1$*;*
(2) *$\tau$ becomes isotropic over some odd extension of $F$.*

The symplectic analogue of an earlier and weaker than Theorem 2 result [5, Theorem 1.1] concerning hyperbolicity of orthogonal involution has been obtained by J.-P. Tignol in [8, Theorem 1]. We prove (the "difficult" part (1) $\Rightarrow$ (2) of) Theorem 1 by a slight modification of Tignol's method. The necessity of modification comes from the presence of odd extensions in the "isotropy business" and from their absence (due to [7, Corollary 6.16]) in the "hyperbolicity business". Note that a different modification, making use of valuations on quaternion skew fields, has been suggested by J.-P. Tignol himself. In contrast to this, our modification makes use of valuations on fields and is contained in Corollary 7, a statement on a field of Laurent series which has nothing to do with central simple algebras or involutions.

Let us explain the characteristic assumption $\operatorname{char} F \neq 2$. Deducing Theorem 1 from Theorem 2, we need the characteristic assumption in order to reduce to a perfect base field,

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the need of a perfect field coming from Remark 8. Recall that anyway, the characteristic assumption is needed in the proof of Theorem 2 itself, because it exploits the Steenrod operations on the Chow groups modulo 2 which (the operations) are not available in characteristic 2.

We are going to use several lemmas. The first one is elementary and easy, the others come from the classical theory of complete discrete valuation fields.

**Lemma 3** ([4, Lemma 3.3])**.** *Let $F$ be a field, $K$ an odd extension of $F$, and $E$ an arbitrary field extension of $F$. Then there exists an odd extension $L/E$ and an $F$-embedding $K \hookrightarrow L$.*

A *coefficient field* of a discrete valuation field $L$ is a subfield of the valuation ring of $L$ mapped under the residue map onto the residue field of $L$.

**Lemma 4.** *Let $L$ be a complete discrete valuation field with characteristic* 0 *residue field, and let $F$ be a subfield of the valuation ring of $L$. Then $L$ has a coefficient field containing $F$.*

*Proof.* Since the characteristic of the residue field is 0 (and $L$ is complete), any maximal subfield of the valuation ring of $L$ is a coefficient field, [3, Proof of Propositions (5.2)Ch.II]. Therefore we can simply take a maximal subfield containing $F$. □

**Lemma 5.** *Let $L$ be a complete discrete valuation field and assume that $p := \operatorname{char} L$ is a prime. Then*

(1) *$L$ has a coefficient field;*
(2) *any coefficient field contains any perfect subfield of the valuation ring;*
(3) *if the residue of an element of the valuation ring is not a pth power, then this element is contained in some coefficient field.*

*Proof.* (1) is [3, Proposition (5.4)Ch.II].

(2) is similar to [1, Theorem 10(c)]. I order to prove (2), let us fix some coefficient field. Let $a$ be an element of $F$, $b$ its image under the residue map, and $c$ the element of the coefficient field mapped to $b$. Since $F$ is perfect, $a$ and $c$ are *multiplicative representatives* (also called *Teichmüller representatives*) of $b$, [3, definition in (7.1)Ch.I] (this notion makes sense only if the characteristic of the residue field is positive). Therefore $a = c$ by the uniqueness of the multiplicative representatives, [3, Proposition (7.1)Ch.I].

To prove (3), let $b$ be an element of the residue field. If $b$ is not a $p$th power, it can be included in a *p-basis*, [3, definition in (5.3)Ch.II], of the residue field. Therefore, for any representative $a$ of $b$ (in the valuation ring), there exists a coefficient field containing $a$, [3, Proof of Proposition (5.4)Ch.II]. □

**Corollary 6.** *Let $F$ be a perfect field, $x, t$ variables, and $\hat{L}$ an odd extension of the field $F((x))$. Then there exist a subfield $L \subset \hat{L}$ containing $F$ and odd over $F$, and an $L$-identification $L((t)) = \hat{L}$ such that the product $xt$ is a square in $\hat{L}$.*

*Proof.* We supply the filed $\hat{L}$ with the (unique) extension $v$ of the $x$-adic valuation on $F((x))$. We are identifying the totally ordered group $v(\hat{L}^\times)$ with $\mathbb{Z}$. The discrete valuation field $\hat{L}$ is complete, [3, Theorem (2.5)Ch.II]. Let $L'$ be its residue field. Then $L'$ is a finite extension of $F$, moreover $[L' : F] \cdot v(x) = [\hat{L} : F((x))]$, [3, Theorem (2.5)Ch.II]. In particular, the integers $[L' : F]$ and $v(x)$ are odd.

By Lemmas 4 and 5, $\hat{L}$ has a coefficient field $L$ containing $F$. One can $L$-identify $\hat{L}$ with the field of Laurent series over $L$ in one variable corresponding to any given uniformizing element in $\hat{L}$ (that is, any element in $\hat{L}$ of valuation 1), [3, Corollary (5.2)Ch.I].

Let $s$ be a uniformizing element in $\hat{L}$ and set $t := s^{v(x)+1}/x$. Then $t$ is also a uniformizing element, and $xt$ is a square in $\hat{L}$. □

**Corollary 7.** *Let $F$ be a perfect field, $x, y, t_x, t_y$ variables, and $\hat{L}$ an odd extension of the field $F((x))((y))$. Then there exist a subfield $L \subset \hat{L}$ containing $F$ and odd over $F$, and an $L$-identification $L((t_x))((t_y)) = \hat{L}$ such that the products $xt_x$ and $yt_y$ are squares in $\hat{L}$.*

*Proof.* We first consider the case where $\operatorname{char} F = 0$. In this case we simply apply Corollary 6 twice. Applying it first to the (perfect) field $F((x))$ and the odd extension $\hat{L}/F((x))((y))$, we get a subfield $\check{L} \subset \hat{L}$ containing $F((x))$ and odd over $F((x))$, and an $\check{L}$-identification $\check{L}((t_y)) = \hat{L}$ such that $yt_y$ is a square in $\hat{L}$. Then we apply Corollary 6 for the second time, now to the field $F$ and the odd extension $\check{L}/F((x))$, getting this time a subfield $L \subset \check{L}$ containing $F$ and odd over $F$, and an $L$-identification $L((t_x)) = \check{L}$ such that $xt_x$ is a square in $\check{L}$. Substituting, we get a required $L$-identification $L((t_x))((t_y)) = \hat{L}$.

Now we assume that $p := \operatorname{char} F > 0$. Since the field $F((x))$ is no longer perfect, the above procedure has to be modified. The filed $\hat{L}$ is complete with respect to the (unique) extension $v$ of the $y$-adic valuation on $F((x))((y))$. Let $\check{L}'$ be its residue field. Then $\check{L}'$ is a finite extension of $F((x))$ and $[\check{L}' : F((x))] \cdot v(y) = [\hat{L} : F((x))((y))]$. In particular, the integers $[\check{L}' : F((x))]$ and $v(y)$ are odd.

Applying Corollary 6 to the perfect field $F$ and the odd extension $\check{L}'/F((x))$, we find a subfield $L' \subset \check{L}'$ containing $F$ and odd over $F$, and an $L'$-identification $L'((t'_x))) = \check{L}'$ such that $xt'_x$ is a square in $\check{L}'$: $xt'_x = b^2$ for some $b \in \check{L}'$. Since $t'_x$ is not a $p$th power in $L'((t'_x))$, for an arbitrary chosen representative $t_x$ of $t'_x$ in the valuation ring of $\hat{L}$ we can find by Lemma 5 a coefficient field of $\hat{L}$ containing $t_x$. Let $a$ be a representative of $b$. We choose $t_x := a^2/x$ and write $\check{L}$ for a coefficient field containing this $t_x$. So, $\check{L}$ is a subfield of $\hat{L}$, and we can find an $\check{L}$-identification $\check{L}((t_y)) = \hat{L}$ such that $yt_y$ is a square. Let $L$ be the subfield of the coefficient field $\check{L}$ corresponding to the subfield $L'$ of the residue field $\check{L}'$ of $\check{L}$. The field $L$ contains $F$ and is $F$-isomorphic to $L'$; in particular, $L/F$ is odd. Furthermore, $\check{L} = L((t_x))$. Substituting, we get the identification $L((t_x))((t_y)) = \hat{L}$. The product $xt_x$ is the square of $a \in \hat{L}$. □

**Remark 8.** The statements of Corollaries 6 and 7 fail for general (imperfect) $F$.

*Proof of Theorem 1.* The implication (2) ⇒ (1) is an easy consequence of the classical Springer theorem on quadratic forms, [2, Corollary 18.5]. Assume that we are given an odd extension $L/F$ such that $\sigma_L$ is isotropic and a field extension $E/F$ such that $\operatorname{ind} A_E = 2$. By Lemma 3, there exists an odd extension $EL$ of $E$ containing $L$. Let $Q$ be a quaternion $E$-algebra Brauer-equivalent to $A_E$. We can find a right $Q$-module $V$, an isomorphism of $E$-algebras $\operatorname{End}_Q V \simeq A_E$, and a hermitian (with respect to the canonical involution on $Q$) form $h$ on $V$ such that the involution $\sigma_E$ is adjoint to $h$. Note that for any $v \in V$, the element $h(v, v) \in Q$ is symmetric and therefore lies in $E$, [7, Proposition (2.6)]. Let $q$ be the quadratic form on the vector $E$-space $V$ defined by $q(v) = h(v, v)$.

We get the following chain of implications: $\sigma_L$ is isotropic $\Rightarrow$ $\sigma_{EL}$ is isotropic $\Rightarrow$ $h_{EL}$ is isotropic $\Rightarrow$ $q_{EL}$ is isotropic $\Rightarrow$ (by the Springer theorem) $q$ is isotropic $\Rightarrow$ $h$ is isotropic $\Rightarrow$ $\sigma_E$ is isotropic.

The implication (1) $\Rightarrow$ (2) is proved by the method of [8] and with a help of Corollary 7. Since $\operatorname{char} F \neq 2$, we may assume that $F$ is perfect (replacing an imperfect $F$ by its perfect closure). Let $\tilde{F} := F(x, y)$ be the field of rational functions in variables $x$ and $y$ over $F$. Let now $Q$ be the quaternion $\tilde{F}$-algebra $(x, y)_{\tilde{F}}$. Let $\tilde{A}$ be the tensor product of the $\tilde{F}$-algebras $A_{\tilde{F}}$ and $Q$ endowed with the (orthogonal) involution $\tilde{\sigma}$ defined as the tensor product of $\sigma_{\tilde{F}}$ by the canonical involution on $Q$.

Let $\tilde{E}$ be the function field of the Severi-Brauer variety of $\tilde{A}$. Since the algebra $\tilde{A}_{\tilde{E}}$ is split, the algebra $A_{\tilde{E}}$ is Brauer-equivalent to the quaternion algebra $Q_{\tilde{E}}$. In particular, $\operatorname{ind} A_{\tilde{E}}$ divides 2. It follows by (1) that the involution $\sigma_{\tilde{E}}$ is isotropic, i.e., $\sigma_{\tilde{E}}(a) \cdot a = 0$ for some non-zero element $a \in A_{\tilde{E}}$. The element $b := a \otimes 1 \in \tilde{A}_{\tilde{E}}$ is also non-zero and satisfies $\tilde{\sigma}_{\tilde{E}}(b) \cdot b = 0$. Therefore the orthogonal involution $\tilde{\sigma}_{\tilde{E}}$ is isotropic. Applying Theorem 2, we get an odd extension $\tilde{L}/\tilde{F}$ such that the involution $\tilde{\sigma}_{\tilde{L}}$ is isotropic.

The field $\tilde{F}$ is a subfield of the field $\hat{F} := F((x))((y))$. By Lemma 3, there exists an odd extension $\hat{L}$ of $\hat{F}$ containing $\tilde{L}$. The involution $\tilde{\sigma}_{\hat{L}}$ is isotropic for such $\hat{L}$. We apply Corollary 7, find the odd field extension $L/F$ and the identification $\hat{L} = L((t_x))((t_y))$. We note that the quaternion algebra $Q_{\hat{L}} = (x, y)_{\hat{L}}$ is isomorphic to $(t_x, t_y)_{\hat{L}}$ because $xt_x$ and $yt_y$ are squares. Now [8, Proposition 1] affirms that $\sigma_L$ is isotropic. This finishes the proof of Theorem 1. □

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