

# SUFFICIENTLY GENERIC ORTHOGONAL GRASSMANNIANS

NIKITA A. KARPENKO

ABSTRACT. We prove the following conjecture due to Bryant Mathews (2008). Let  $Q$  be the orthogonal grassmannian of totally isotropic  $i$ -planes of a non-degenerate quadratic form  $q$  over an arbitrary field (where  $i$  is an integer satisfying  $1 \leq i \leq (\dim q)/2$ ). Assume that  $q$  is *sufficiently generic* in the following sense: the degree of each closed point on  $Q$  is divisible by  $2^i$  and the Witt index of  $q$  over the function field of  $Q$  is equal to  $i$ . Then the variety  $Q$  is *2-incompressible*.

Assuming that  $q$  is sufficiently generic in a different sense, we prove a stronger property of  $Q$  saying that its Chow motive with coefficients in  $\mathbb{F}_2$  (the finite field of 2 elements) is indecomposable. That is quite unexpected, especially after recent results of [20] on decomposability of the motives of incompressible *twisted grassmannians*.

The proofs are based on the theory of upper motives. The results allows one to compute the canonical 2-dimension of any projective homogeneous variety (i.e., orthogonal flag variety) associated to the generic quadratic form.

This paper is an extended version of [12] including the results of [6].

## 1. INTRODUCTION

This paper deals with motives of certain smooth projective varieties associated to quadratic forms over fields of arbitrary characteristic. We refer to [4] for notation and basic results concerning quadratic forms. By motives, we mean the Grothendieck Chow motives with coefficients in the finite field  $\mathbb{F}_2$  as introduced in [4]. We are using the theory of upper motives conceived in [7] and [11].

Let  $q$  be a non-zero non-degenerate quadratic form over a field  $F$  (which may have characteristic 2). For any integer  $i$  with  $0 \leq i \leq (\dim q)/2$  we write  $Q_i$  for the variety of  $i$ -dimensional totally isotropic subspaces of  $q$ .

For any  $i$ , the variety  $Q_i$  is smooth and projective. It is geometrically connected if and only if  $i \neq (\dim q)/2$ . In particular,  $Q_i$  is connected for any  $i$  if  $\dim q$  is odd. For even-dimensional  $q$  and  $i = (\dim q)/2$ , the variety  $Q_i$  is connected if and only the discriminant of  $q$  is non-trivial.

If a variety  $Q_i$  is not connected, it has two connected components and they are isomorphic. In particular, the dimension of  $Q_i$  is always the dimension of any connected component of  $Q_i$ . Here is a formula for the dimension, where  $n := \dim q$  (see, e.g., [2]):

$$\dim Q_i = i(i-1)/2 + i(n-2i).$$

---

*Date:* 30 December 2011.

*Key words and phrases.* Algebraic groups, quadratic forms, projective homogeneous varieties, Chow groups and motives. *Mathematical Subject Classification (2010):* 14L17; 14C25.

In the case where the quadratic form  $q$  is “sufficiently generic” (the precise condition can be formulated in terms of the variety  $Q_i$  with maximal  $i$  or in terms of the  $J$ -invariant of  $q$ ), we are going to show (see Theorems 3.1, 4.1, and 5.1) that the motives of all  $Q_i$  are indecomposable, if we stay away from the two exceptional cases described below (where the motive evidently decomposes).

Each of the both exceptional cases arises only if the dimension of  $q$  is even and the discriminant of  $q$  is trivial. The first case is the case of  $i = (\dim q)/2$ , where the variety  $Q_i$  has two connected components. Our assumption on  $q$  ensures that the motive of each component of  $Q_i$  is indecomposable.

The second case is the case of  $i = (\dim q)/2 - 1$ , where the variety  $Q_i$  is a rank  $i$  projective bundle over a component of  $Q_{i+1}$  (this statement is proved in the proof of Theorem 4.1). Therefore, the motive of  $Q_i$  is a sum of shifts of  $i+1$  copies of the motive of a component of  $Q_{i+1}$ , and this is a *complete* motivic decomposition of  $Q_i$  (where *complete* means that the summands of this decomposition are indecomposable).

We recall that a connected smooth projective variety  $X$  is called *2-incompressible*, if its *canonical 2-dimension*, as defined in [4, §90] (see also §6 here), takes its maximal value  $\dim X$ . This in particular implies that any rational map  $X \dashrightarrow X$  is dominant, i.e., that  $X$  is *incompressible*.

Any projective homogeneous variety  $X$  having indecomposable motive, is 2-incompressible, [7, §2e]. Therefore our indecomposability results imply 2-incompressibility of the corresponding varieties.

Let us point out that our incompressibility results compute the canonical 2-dimension of any projective homogeneous variety (i.e., orthogonal flag variety) associated to a sufficiently generic quadratic form. This is so indeed because for an arbitrary non-degenerate quadratic form  $q$  and an arbitrary sequence of integers  $i_1, \dots, i_k$  with  $0 \leq i_1 < \dots < i_k \leq (\dim q)/2$  we have an orthogonal flag variety  $Q_{i_1, \dots, i_k}$ , the variety of flags of totally isotropic subspaces of  $q$  of dimensions  $i_1, \dots, i_k$ , and the canonical 2-dimension (of a component) of this variety coincides with the canonical 2-dimension of (a component of)  $Q_{i_k}$ .

The motivic indecomposability of the varieties  $Q_i$  contrasts with a recent result of M. Zhykhovich [20] (see [19] for an extended version) saying that for any prime  $p$ , any central division  $F$ -algebra  $D$  of degree  $p^n$  for some  $n$ , and any  $i$  with  $0 < i < n$  (and  $i \neq 1$  if  $p = 2$ ), the motive with coefficients in  $\mathbb{F}_p$  of the variety of the right ideals of reduced dimension  $p^i$  in  $D$  (this variety is known to be  $p$ -incompressible and is a twisted form of the grassmannian of  $p^i$ -dimensional subspaces in a  $p^n$ -dimensional vector space) is decomposable.

We have to point out that the case of quadrics (i.e., of  $Q_1$ ) was done by A. Vishik (unpublished); his proof is much different.

The paper is organized as follows. In §2 we recall (and partially develop) the necessary aspects of the theory of upper motives (with an arbitrary prime integer  $p$  in place of 2). In the next three Sections we establish our main result: in §3 for odd-dimensional forms (Theorem 3.1), in §4 for even-dimensional forms of trivial discriminant (Theorem 4.1), and finally in §5 for even-dimensional forms of non-trivial discriminant (Theorem 5.1).

In §7, we prove a conjecture due to Bryant Mathews (see Theorem 7.1), another main result of the paper. It gives 2-incompressibility of orthogonal grassmannians under the

assumption that the quadratic form is sufficiently generic in a different from the above sense. In the preceding §6 we recall (and develop) the necessary aspects of the theory of canonical dimension (with an arbitrary prime integer  $p$  in place of 2).

None of the two appeared notions of sufficiently generic is weaker (or stronger) than the other. According to this, none of the two main results implies the other. Of course, a generic quadratic form (constructed in the beginning of Sections 3, 4, and 5) is sufficiently generic in both senses so that both main results apply to it.

## 2. UPPER MOTIVES

Let us fix a prime integer  $p$  and consider Chow motives with coefficients in the prime field  $\mathbb{F}_p$ . We write  $\text{Ch}$  for the Chow groups with coefficients in  $\mathbb{F}_p$ .

Let  $F$  be a field and  $K/F$  a finite separable field extension. Given a projective homogeneous (under an action of a semisimple affine algebraic group over  $K$ )  $K$ -variety  $X$ , we consider  $X$  as an  $F$ -variety via the composition  $X \rightarrow \text{Spec } K \rightarrow \text{Spec } F$ .

We recall that the following *Krull-Schmidt principle* holds: any summand of the motive  $M(X)$  of  $X$  decomposes and in a unique way in a finite direct sum of indecomposable motives, [3] or [11, Corollary 2.2].

We define the *upper motive*  $U(X)$  of  $X$  as an indecomposable summand of the motive  $M(X)$  of  $X$  with the property that the Chow group  $\text{Ch}^0 U(X)$  is non-zero (or, equivalently, the property that  $U(X)$  over an algebraic closure of  $F$  contains the Tate summand  $\mathbb{F}_p$ ). Since the Chow group  $\text{Ch}^0 X$  is a 1-dimensional vector space (over  $\mathbb{F}_2$ ), any given complete motivic decomposition of  $X$  contains precisely one upper summand. It follows by the Krull-Schmidt principle that the isomorphism class of  $U(X)$  is uniquely determined by  $X$ . (See [7, Corollary 2.15] for a more direct proof not relying on the Krull-Schmidt principle.)

Given smooth complete irreducible  $F$ -varieties  $X_1$  and  $X_2$ , we say that  $X_1$  *dominates*  $X_2$ , if there exists multiplicity 1 correspondence  $X_1 \rightsquigarrow X_2$ . (Our correspondences are with coefficients in  $\mathbb{F}_p$ , their multiplicities are elements of  $\mathbb{F}_p$ .) We say that  $X_1$  and  $X_2$  are *equivalent*, if they dominate each other.

Given projective homogeneous varieties  $X_1$  and  $X_2$  (under possibly different algebraic groups) over finite separable field extensions  $K_1$  and  $K_2$  of  $F$ , the upper motives of the  $F$ -varieties  $X_1$  and  $X_2$  satisfy the following isomorphism criterion:

**Lemma 2.1** ([7, Corollary 2.15]). *The upper motives  $U(X_1)$  and  $U(X_2)$  are isomorphic if and only if the  $F$ -varieties  $X_1$  and  $X_2$  are equivalent.*

Now let  $G$  be a semisimple affine algebraic group over  $F$ . The minimal (in a fixed separable closure of  $F$ ) field extension  $E/F$  such that  $G_E$  is of inner type, is finite and galois (it corresponds to the kernel of the action of the absolute galois group of  $F$  on the Dynkin diagram of  $G$ ). Assuming that  $[E : F]$  is a power of  $p$  (the possibility  $E = F$  is included), we have the following basic result of the upper motive theory:

**Theorem 2.2** ([11, Theorem 1.1]). *Let  $X$  be a projective  $G$ -homogeneous  $F$ -variety. Any indecomposable summand of the motive of  $X$  is a shift of the upper motive  $U(Y)$  of some projective  $G_K$ -homogeneous variety  $Y$  dominating  $X$ , where  $K$  is some intermediate field of the extension  $E/F$ .*

Apparently, the above theorem, showing that the upper motives are important, does not say anything about their structure. However, it makes it possible to prove the following structure result, which means that the upper motive possesses the same kind of symmetry as the whole motive of a variety. Let us define *dimension*  $\dim U(X)$  of  $U(X)$  as the biggest integer  $i$  such that the Tate motive  $\mathbb{F}_p(i)$  is a summand of the motive  $U(X)$  over an algebraic closure of  $F$ . (Putting  $M(X)$  in place of  $U(X)$  in the definition given, we will get the usual dimension of the variety  $X$ .) We write  $U(X)^*$  for the *dual* of the motive  $U(X)$ , [4, §65].

**Proposition 2.3** ([9, Proposition 5.2]).  $U(X) \simeq U(X)^*(\dim U(X))$ .

Another drawback of Theorem 2.2 is absence of a precise indication concerning the varieties  $Y$  whose upper motives do really appear in the complete motivic decomposition of  $X$ . Although this drawback can be recovered in many particular cases, there is no recipe to recover it in general. But the information on possible  $Y$  given in Theorem 2.2 can be made more precise using the following argument.

Let  $X$  and  $Y$  (and  $K$ ) be as in Theorem 2.2. We assume that  $U(Y)$  is, up to a shift, a motivic summand of  $X$  and we want to say something more on  $Y$  besides the fact that it dominates  $X$  claimed in Theorem 2.2. Let  $X'$  be a projective  $G$ -homogeneous  $F$ -variety dominated by  $X$  (for instance,  $X' = X$ ). Let us consider the complete motivic decomposition of  $X_{F(X')}$ . (If  $X'$  is non-trivial, the algebraic group acting on  $X$  is isotropic over the field  $F(X')$ . Using the motivic decompositions of projective homogeneous varieties under *isotropic* groups constructed in [1], we may know the complete motivic decomposition of  $X_{F(X')}$  by induction on the rank of the group.) The complete decomposition of  $U(Y)_{F(X')}$  is (up to a shift) a part of this decomposition. It follows that there exist an intermediate field  $K'$  of the extension  $E/F$  and a projective  $G_{K'(X')}$ -homogeneous variety  $Y'$  such that  $U(Y')$  (without shift) is a summand of  $U(Y)_{F(X')}$ .

**Proposition 2.4.** *The  $F(X')$ -varieties  $Y_{F(X')}$  and  $Y'$  are equivalent. This property determines the equivalence class of the  $F$ -variety  $Y$ . The motive  $U(Y')(\dim U(Y) - \dim U(Y'))$  is also a summand of  $U(Y)_{F(X')}$ .*

*Proof.* The upper motives of the  $F(X')$ -varieties  $Y_{F(X')}$  and  $Y'$  are isomorphic, therefore the varieties are equivalent by Lemma 2.1.

To prove the second statement, let us take one more intermediate field  $L$  of the extension  $E/F$  and a dominating  $X$  projective  $G_L$ -homogeneous variety  $Z$  such that the  $F(X')$ -varieties  $Y_{F(X')}$  and  $Z_{F(X')}$  are equivalent. Since  $Y_{F(X')}$  dominates  $Z_{F(X')}$ , the  $F(X')(Y) = F(X' \times Y)$ -variety  $Z_{F(X' \times Y)}$  has a closed point of coprime with  $p$  degree (a  *$p$ -coprime closed point* for short). Since  $Y$  dominates  $X$  which dominates  $X'$ ,  $Y$  dominates  $X'$  and it follows that  $X'_{F(Y)}$  has a  $p$ -coprime closed point. Let  $F'$  be the residue field of such a point. The tensor product  $F'' := F' \otimes_{F(Y)} F(X' \times Y)$  is a field containing  $F(X' \times Y)$ . Moreover, the field extension  $F''/F'$  is isomorphic to the function field of the projective homogeneous  $F'$ -variety  $X'_{F'}$  with rational point. Therefore the field extension  $F''/F'$  is purely transcendental. It follows by [4, Lemma 75.2] that  $Z_{F'}$  has a  $p$ -coprime closed point. Since  $p$  does not divide  $[F' : F(Y)]$ ,  $Z_{F(Y)}$  has a  $p$ -coprime closed point and therefore  $Y$  dominates  $Z$ .

Exchanging the roles of  $Y$  and  $Z$ , we get that  $Z$  dominates  $Y$  as well so that  $Y$  and  $Z$  are equivalent. This finishes the proof of the second statement of Proposition 2.4.

Finally, since  $U(Y')$  is a summand of  $U(Y)_{F(X')}$ ,  $U(Y')^*$  is a summand of  $U(Y)_{F(X')}^*$ . Applying Proposition 2.3, we get the third statement of Proposition 2.4.  $\square$

**Example 2.5.** Let  $p = 2$  and  $G = O^+(q)$  (in notation of [14, §23]) for a non-degenerate quadratic form  $q$  over  $F$ . We have  $E = F$  if  $\dim q$  is odd or  $\text{disc } q$  is trivial. Otherwise  $E/F$  is the quadratic galois field extension given by the discriminant of  $q$ . We set  $n := \dim q$ . For any integer  $i$  with  $0 \leq i < n/2$ , let  $Q_i$  be the variety (orthogonal grassmannian) of  $i$ -dimensional totally isotropic subspaces in  $q$ . (Note that we do not consider the variety  $Q_{n/2}$  here.) In particular,  $Q_0 = \text{Spec } F$  and  $Q_1$  is the projective quadric of  $q$ . The varieties  $Q_i$  are projective  $G$ -homogeneous (while the projective variety  $Q_{n/2}$  is never homogeneous) and form a complete system of representatives of the equivalence classes of all projective  $G$ -homogeneous varieties. (If  $n$  is even and  $\text{disc } q$  trivial, a component of  $Q_{n/2}$  is a projective  $G$ -homogeneous variety equivalent to  $Q_{n/2-1}$ .) Moreover, for  $i \geq j$ ,  $Q_i$  dominates  $Q_j$  so that if  $Q_j$  also dominates  $Q_i$  then  $Q_j$  is equivalent to  $Q_i$ . Therefore, by Theorem 2.2, any indecomposable summand of the motive of  $Q_i$  is a shift of the upper motive  $U(Q_j)$  or  $U(Q_{jE})$  for some  $j \geq i$ .

In order to apply Proposition 2.4, we will use the following motivic decomposition of  $Q_{iF(Q_1)}$  obtained in [1] (see also [8]). We assume that  $n \geq 3$  and  $i \geq 1$ . Let  $q'$  be an  $(n-2)$ -dimensional quadratic form over the field  $F(Q_1)$  Witt-equivalent to  $q_{F(Q_1)}$ . Write  $Q'_j$  for the orthogonal grassmannians of  $q'$ . Then

$$(2.6) \quad M(Q_{iF(Q_1)}) \simeq M(Q'_{i-1}) \oplus M(Q'_i)((\dim Q_i - \dim Q'_i)/2) \oplus M(Q'_{i-1})(\dim Q_i - \dim Q'_{i-1}).$$

### 3. ODD-DIMENSIONAL QUADRATIC FORMS

Let  $F$  be a field,  $m$  an integer  $\geq 0$ ,  $n := 2m + 1$ ,  $q$  a non-degenerate  $n$ -dimensional quadratic form over  $F$ . We assume that the following equivalent conditions hold:

- (1) the variety  $Q_m$  is 2-incompressible;
- (2) the Chow motive (with coefficients in  $\mathbb{F}_2$ ) of the variety  $Q_m$  is indecomposable;
- (3) the  $J$ -invariant of the quadratic form  $q$  (as defined in [4, §88]) takes its maximal value:

$$J(q) = \{1, 2, \dots, m\}.$$

The equivalence (1)  $\Leftrightarrow$  (3) is proved in [4, §90]. The equivalence (2)  $\Leftrightarrow$  (3) follows from [16]. Also, the equivalence (1)  $\Leftrightarrow$  (2) is a consequence of Theorem 2.2 and [9, Theorem 5.1].

By [4, Corollary 88.12], the conditions (1–3) are satisfied if the degree of any closed point on  $Q_m$  is divisible by  $2^m$ . The condition on the closed points is satisfied if the even Clifford algebra of  $q$  is a division algebra. Finally, the condition on the even Clifford algebra is satisfied if  $F = k(t_0, \dots, t_{2m})$ , where  $k$  is a field and  $t_0, \dots, t_{2m}$  are variables, and  $q = \langle t_0 \rangle \perp [t_1, t_2] \perp \dots \perp [t_{2m-1}, t_{2m}]$  (a sort of *generic*  $n$ -dimensional quadratic form).

**Theorem 3.1.** *Let  $q$  be a non-degenerate  $(2m + 1)$ -dimensional quadratic form over a field  $F$  such that the variety  $Q_m$  is 2-incompressible. Then for any  $i$  with  $0 \leq i \leq m$ , the motive of the variety  $Q_i$  is indecomposable. In particular, all  $Q_i$  are 2-incompressible.*

*Proof.* We induct on  $m$ . The induction base is the trivial case of  $m = 0$ . Now we assume that  $m \geq 1$ .

We do a descending induction on  $i$ . The induction base is the case of  $i = m$  which is served by our assumption on  $q$ . Now we assume that  $i < m$ . Since the case of  $i = 0$  is trivial, we may assume that  $i > 0$ .

We are using notation of Example 2.5. According to [4, §88], the assumption on the  $J$ -invariant (assumption (3)) still holds for the quadratic form  $q'$ . By the induction hypothesis, each of the three summands of the decomposition (2.6) is indecomposable.

It follows by Proposition 2.4 that if the motive of  $Q_i$  (over  $F$ ) is decomposable, then it has an indecomposable summand  $M$  such that  $M_L$ , where  $L := F(Q_1)$ , is a shift of  $M(Q'_i) = U(Q'_i)$ . Note that the varieties  $Q'_i$  and  $Q_{i+1L}$  are equivalent. By Proposition 2.4,  $M \simeq U(Q_{i+1})$ , that is,  $U(Q_{i+1})_L \simeq M(Q'_i)$ . By the induction hypothesis, the motive of  $Q_{i+1}$  is indecomposable, i.e.,  $U(Q_{i+1}) = M(Q_{i+1})$ . Therefore we have an isomorphism  $M(Q_{i+1})_L \simeq M(Q'_i)$  and, in particular,  $\dim Q_{i+1} = \dim Q'_i$ . However

$$\dim Q_{i+1} - \dim Q'_i = n - i - 2 = 2m - i - 1 > m - 1 \geq 0. \quad \square$$

#### 4. EVEN-DIMENSIONAL QUADRATIC FORMS OF TRIVIAL DISCRIMINANT

Let  $F$  be a field,  $m$  an integer  $\geq 1$ ,  $n := 2m$ ,  $q$  a non-degenerate  $n$ -dimensional quadratic form over  $F$  of trivial discriminant. In this case the variety  $Q_m$  (of totally isotropic  $m$ -dimensional subspaces in  $q$ ) has two (isomorphic) connected components, and we write  $Q_m^+$  for a component of the variety  $Q_m$ .

We assume that the following equivalent conditions hold:

- (1) the variety  $Q_m^+$  is 2-incompressible;
- (2) the Chow motive (with coefficients in  $\mathbb{F}_2$ ) of the variety  $Q_m^+$  is indecomposable;
- (3) the  $J$ -invariant of the quadratic form  $q$  takes its maximal value:

$$J(q) = \{1, 2, \dots, m - 1\}.$$

By [4, §88], the conditions (1–3) are satisfied if the degree of any closed point on  $Q_m$  is divisible by  $2^{m-1}$ . The condition on the closed points is satisfied if the even Clifford algebra of  $q$  is the direct product of two copies of a division algebra. Finally, the condition on the even Clifford algebra is satisfied if  $F$  is the discriminant quadratic extension over  $k(t_1, \dots, t_n)$  of the quadratic form  $[t_1, t_2] \perp \dots \perp [t_{n-1}, t_n]$ , where  $k$  is a field and  $t_1, \dots, t_n$  are variables, and  $q = ([t_1, t_2] \perp \dots \perp [t_{n-1}, t_n])_F$  (a sort of *generic*  $n$ -dimensional quadratic form of trivial discriminant).

**Theorem 4.1.** *Let  $q$  be a non-degenerate  $(2m)$ -dimensional quadratic form over a field  $F$  such that the discriminant of  $q$  is trivial and a component of the variety  $Q_m$  is 2-incompressible. Then for any  $i$  with  $0 \leq i \leq m - 2$ , the motive of the variety  $Q_i$  is indecomposable. In particular,  $Q_i$  is 2-incompressible for such  $i$ .*

*Proof.* We induct on  $m$ . The induction base is the vacuous case of  $m = 1$ . Now we assume that  $m \geq 2$ .

We do a descending induction on  $i \leq m - 2$ . Since the case of  $i = 0$  is trivial, we may assume that  $i > 0$  (and, in particular,  $m \geq 3$ ).

We are using notation of Example 2.5. The discriminant of the quadratic form  $q'$  is also trivial. According to [4, §88], the assumption on the  $J$ -invariant holds for  $q'$ . By the induction hypothesis, the motive  $M(Q'_{i-1})$  appearing in the decomposition (2.6) is indecomposable. However the motive  $M(Q'_i)$  – the middle summand of the decomposition (up to a shift) – is indecomposable if  $i \neq m - 2$ . Let us treat the case  $i = m - 2$  first.

The variety  $Q'_{m-2}$  is a rank  $m - 2$  projective bundle over  $Q'^+_{m-1}$  (a component of  $Q'_{m-1}$ ). Indeed, any totally isotropic  $(m - 2)$ -dimensional subspace in  $q'$  is contained in a unique totally isotropic  $(m - 1)$ -dimensional subspace in  $q'$  lying on  $Q'^+_{m-1}$ . This provides us with a morphism  $Q'_{m-2} \rightarrow Q'^+_{m-1}$  which is a projective bundle: the fiber over a point of  $Q'^+_{m-1}$  given by a space  $W$  is given by all 1-codimensional subspaces in  $W$  and is the dual projective space of  $W$ . Therefore the complete decomposition of  $M(Q'_{m-2})$  looks as follows:

$$M(Q'_{m-2}) \simeq M(Q'^+_{m-1}) \oplus M(Q'^+_{m-1})(1) \oplus \cdots \oplus M(Q'^+_{m-1})(m - 2).$$

It follows that  $U(Q_{m-1}) = U(Q_m^+) = M(Q_m^+)$  is not a shift of a summand of  $M(Q_{m-2})$ . Indeed, otherwise

$$M(Q_m^+)_L \simeq M(Q'^+_{m-1}) \oplus M(Q'^+_{m-1})(m - 1),$$

where  $L := F(Q_1)$ , would be a shift of a summand of  $M(Q_{m-2})_L$  but it is not because the motives  $M(Q'_{m-3}) = U(Q'_{m-3})$  and  $U(Q'^+_{m-1}) = M(Q'^+_{m-1})$  are not isomorphic (e.g., by dimension reason).

It follows that the motive of  $Q_{m-2}$  is indecomposable. This is the base case of our descending induction on  $i$ . Below we assume that  $i < m - 2$ .

Now each of the three summands of the decomposition of  $M(Q_i)_L$ , given in (2.6), is indecomposable. It follows by Proposition 2.4 that if the motive of  $Q_i$  (over  $F$ ) is decomposable, then it has an indecomposable summand  $M$  with  $M_L$  isomorphic to a shift of  $M(Q'_i)$ . By Proposition 2.4,  $M$  (over  $F$ ) is isomorphic to a shift of  $U(Q_{i+1})$ , so that  $U(Q_{i+1})_L \simeq M(Q'_i)$ . By the induction hypothesis, the motive of  $Q_{i+1}$  is indecomposable, i.e.,  $U(Q_{i+1}) = M(Q_{i+1})$ . Therefore we have an isomorphism  $M(Q_{i+1})_L \simeq M(Q'_i)$  and, in particular,  $\dim Q_{i+1} = \dim Q'_i$ . However

$$\dim Q_{i+1} - \dim Q'_i = n - i - 2 = 2m - i - 2 > m \geq 3. \quad \square$$

## 5. EVEN-DIMENSIONAL QUADRATIC FORMS OF NON-TRIVIAL DISCRIMINANT

Let  $F$  be a field,  $m$  an integer  $\geq 1$ ,  $n := 2m$ ,  $q$  a non-degenerate  $n$ -dimensional quadratic form over  $F$  of non-trivial discriminant.

We assume that the following equivalent conditions hold:

- (1) the variety  $Q_m$  is 2-incompressible;
- (2) the Chow motive (with coefficients in  $\mathbb{F}_2$ ) of the variety  $Q_m$  is indecomposable;
- (3) the  $J$ -invariant of the quadratic form  $q$  takes its maximal value:

$$J(q) = \{0, 1, 2, \dots, m - 1\}.$$

By [4, §88], the above conditions are satisfied if the degree of any closed point on  $Q_m$  is divisible by  $2^m$ . The condition on the closed points is satisfied if the even Clifford algebra of  $q$  is a division algebra (whose center is the discriminant quadratic extension). Finally, the condition on the even Clifford algebra is satisfied if  $F = k(t_1, \dots, t_n)$ , where  $k$  is a field and  $t_1, \dots, t_n$  are variables, and  $q = [t_1, t_2] \perp \dots \perp [t_{n-1}, t_n]$  (a sort of *generic*  $n$ -dimensional quadratic form).

**Theorem 5.1.** *Let  $q$  be a non-degenerate  $(2m)$ -dimensional quadratic form over a field  $F$  such that the discriminant of  $q$  is non-trivial and the (therefore connected) variety  $Q_m$  is 2-incompressible. For any  $i$  with  $0 \leq i \leq m$ , the motive of the variety  $Q_i$  is indecomposable. In particular, all  $Q_i$  are 2-incompressible.*

*Proof.* We induct on  $m$ . The induction base is the trivial case of  $m = 1$ . Now we assume that  $m \geq 2$ .

We do a descending induction on  $i \leq m$ . The induction base  $i = m$  holds by our assumption on  $q$ . Below we are assuming that  $i < m$ . Since the case of  $i = 0$  is trivial, we may assume that  $i > 0$ .

We are using notation of Example 2.5 and set  $L := F(Q_1)$ . Since  $F$  is algebraically closed in  $L$ , the discriminant of the quadratic form  $q'$  is non-trivial. Moreover, according to [4, §88], the assumption on the  $J$ -invariant holds for  $q'$ . By the induction hypothesis, each of the three summands of the decomposition (2.6) is indecomposable.

(We apologize for repeating almost word by word the end of the proof of Theorem 3.1 below.) It follows by Proposition 2.4 that if the motive of  $Q_i$  (over  $F$ ) is decomposable, then it has an indecomposable summand  $M$  with  $M_L$  isomorphic to a shift of  $M(Q'_i)$ . By Proposition 2.4,  $M$  (over  $F$ ) is isomorphic to a shift of  $U(Q_{i+1})$ , so that  $U(Q_{i+1})_L \simeq M(Q'_i)$ . By the induction hypothesis, the motive of  $Q_{i+1}$  is indecomposable, i.e.,  $U(Q_{i+1}) = M(Q_{i+1})$ . Therefore we have an isomorphism  $M(Q_{i+1})_L \simeq M(Q'_i)$  and, in particular,  $\dim Q_{i+1} = \dim Q'_i$ . However  $\dim Q_{i+1} - \dim Q'_i = 2m - i - 2 > m - 2 \geq 0$ .  $\square$

## 6. CANONICAL DIMENSION

In this Section, we make some development of the theory of canonical dimension of general projective homogeneous varieties which might be of independent interest and which will be used in the next Section. We fix a prime  $p$ . Let  $G$  be a semisimple affine algebraic group over a field  $F$  such that  $G_E$  is of inner type for some finite galois field extension  $E/F$  of degree a power of  $p$  ( $E = F$  is allowed). Let  $X$  be a projective  $G$ -homogeneous  $F$ -variety. We refer to [9] for a definition and discussion of the notion of *canonical  $p$ -dimension*  $\text{cdim}_p X$  of  $X$ . Actually, canonical  $p$ -dimension is defined in the context of more general algebraic varieties. For any irreducible smooth projective variety  $X$ ,  $\text{cdim}_p X$  is the minimal dimension of a closed subvariety  $Y \subset X$  with a 0-cycle of  $p$ -coprime degree on  $Y_{F(X)}$ . Recall that a smooth projective  $X$  is  *$p$ -incompressible*, if it is irreducible and  $\text{cdim}_p X = \dim X$ . We write CH for the Chow groups with integer coefficients.

**Proposition 6.1.** *Let  $X$  be a projective  $G$ -homogeneous variety with  $G$  as above. For  $d := \text{cdim}_p X$ , there exist a cycle class  $\alpha \in \text{CH}^d X_{F(X)}$  (over  $F(X)$ ) of codimension  $d$*

and a cycle class  $\beta \in \text{Ch}_d X$  (over  $F$ ) of dimension  $d$  such that the degree of the product  $\beta_{F(X)} \cdot \alpha$  is not divisible by  $p$ .

*Proof.* We use Chow motives with coefficients in  $\mathbb{F}_p := \mathbb{Z}/p\mathbb{Z}$ . By [9, Theorem 5.1 and Proposition 5.2], the upper motive  $U(X)$  of  $X$  (being by definition a direct summand of  $M(X)$ ) is also a direct summand of  $M(X)(d - m)$ , where  $m := \dim X$ . The composition

$$M(X) \rightarrow U(X) \rightarrow M(X)(d - m)$$

is given by a correspondence  $f \in \text{Ch}^d(X \times X)$ ; the composition

$$M(X)(d - m) \rightarrow U(X) \rightarrow M(X)$$

is given by a correspondence  $g \in \text{Ch}_d(X \times X)$ . The composition of correspondences  $g \circ f \in \text{Ch}_m(X \times X)$  is a projector on  $X$  such that  $U(X) = (X, g \circ f)$ . In particular, the multiplicity  $\text{mult}(g \circ f)$  of the correspondence  $g \circ f$  is  $1 \in \mathbb{F}_p$ . Taking for  $\alpha$  an integral representative of the pull-back of  $f$  with respect to the morphism

$$\text{Spec } F(X) \times X \rightarrow X \times X$$

induced by the generic point of the first factor, and taking for  $\beta$  an integral representative of the push-forward of  $g$  with respect to the projection of  $X \times X$  onto the first factor, we get that

$$\deg(\beta_{F(X)} \cdot \alpha) \pmod{p} = \text{mult}(g \circ f) = 1 \in \mathbb{F}_p. \quad \square$$

**Corollary 6.2.** *The canonical  $p$ -dimension  $\text{cdim}_p X$  of  $X$  is the minimal integer  $d$  such that there exists a cycle class  $\alpha \in \text{Ch}^d X_{F(X)}$  and a cycle class  $\beta \in \text{Ch}_d X$  with  $\deg(\beta_{F(X)} \cdot \alpha) = 1 \in \mathbb{F}_p$ .*

*Proof.* We only need to show that  $\text{cdim}_p X \leq d$ . The proof is similar to [13, Proof of  $\leq$  in Theorem 5.8]. Since  $\deg(\beta_{F(X)} \cdot \alpha) = 1 \in \mathbb{F}_p$  for some  $\beta \in \text{Ch}_d X$  (and some  $\alpha$ ), there exists a closed irreducible  $d$ -dimensional subvariety  $Y \subset X$  such that  $\deg([Y]_{F(X)} \cdot \alpha) \neq 0 \in \mathbb{F}_p$  (with the same  $\alpha$ ). Since the product  $[Y]_{F(X)} \cdot \alpha$  can be represented by a cycle on  $Y_{F(X)}$ , the variety  $Y_{F(X)}$  has a 0-cycle of  $p$ -coprime degree. Therefore  $\text{cdim}_p X \leq \dim Y = d$ .  $\square$

**Corollary 6.3.** *In the situation of Proposition 6.1, for any field extension  $L/F$ , the change of field homomorphism  $\text{Ch}_d X \rightarrow \text{Ch}_d X_L$  is non-zero.*

*Proof.* The image of  $\beta \pmod{p} \in \text{Ch}_d X$  in  $\text{Ch}_d X_L$  is non-zero because  $\deg(\beta_{L(X)} \cdot \alpha_{L(X)}) \not\equiv 0 \pmod{p}$ .  $\square$

**Remark 6.4.** If the variety  $X$  is *generically split* (meaning that the motive of  $X_{F(X)}$  is a sum of Tate motives (this implies that the adjoint algebraic group acting on  $X$  is of inner type)), then [13, Theorem 5.8] says that  $\text{cdim}_p X$  is the minimal  $d$  with non-zero  $\text{Ch}_d X \rightarrow \text{Ch}_d X_L$  for any  $L$ . Corollary 6.3 can be considered as a generalization of a part of [13, Theorem 5.8] to the case of a projective  $G$ -homogeneous variety  $X$  which is not necessarily generically split with  $G$  not necessarily of inner type. Note that the statement of [13, Theorem 5.8] in whole fails in such generality. Corollary 6.2 is its correct replacement (giving the original statement in the case of generically split  $X$ ).

**Lemma 6.5.** *In the situation of Proposition 6.1, let  $\alpha, \alpha' \in \text{Ch}^d X_{F(X)}$  and  $\beta, \beta' \in \text{Ch}_d X$  be cycle classes with  $\deg(\beta_{F(X)} \cdot \alpha) = 1 = \deg(\beta'_{F(X)} \cdot \alpha')$ . Then*

$$\deg(\beta_{F(X)} \cdot \alpha') \neq 0 \neq \deg(\beta'_{F(X)} \cdot \alpha).$$

*Proof.* We fix an algebraically closed field containing  $F(X)$  and write  $\bar{\cdot}$  when considering a variety or a cycle class over that field. The surjectivity of the pull-back with respect to the flat morphism  $\text{Spec } F(X) \times X \rightarrow X \times X$  induced by the generic point of the first factor of the product  $X \times X$ , tells us that the group  $\text{Ch}^d(\bar{X} \times \bar{X})$  contains a *rational* (i.e., coming from  $\text{Ch}^d(X \times X)$ ) cycle class of the form  $[\bar{X}] \times \bar{\alpha} + \dots + \bar{\gamma} \times [\bar{X}]$  with some  $\gamma \in \text{Ch}^d X_{F(X)}$ , where  $\dots$  is in the sum of products  $\text{Ch}^i \bar{X} \otimes \text{Ch}^j \bar{X}$  with  $0 < i, j < d$  and  $i + j = d$ . Multiplying by  $[\bar{X}] \times \bar{\beta}$ , we get a rational cycle class of the form  $[\bar{X}] \times [\mathbf{pt}] + \dots + \bar{\gamma} \times \bar{\beta}$ , where  $\mathbf{pt}$  is a rational point on  $\bar{X}$  and  $\dots$  is now in the sum of  $\text{Ch}^i \bar{X} \otimes \text{Ch}_i \bar{X}$  with  $0 < i < d$ . The composition of the obtained correspondence with itself equals  $[\bar{X}] \times [\mathbf{pt}] + \dots + \deg(\gamma \cdot \beta)(\bar{\gamma} \times \bar{\beta})$ . Since an appropriate power of this correspondence is a multiplicity 1 *projector* (cf. [10, Corollary 3.2] or [7]) and  $d = \text{cdim}_p X$ , it follows by [9, Theorem 5.1] that  $\deg(\gamma \cdot \beta) \neq 0$ . Now multiplying  $[\bar{X}] \times \bar{\alpha} + \dots + \bar{\gamma} \times [\bar{X}]$  by  $\bar{\beta} \times [\bar{X}]$ , transposing, and raising to  $(p-1)$ th power (by means of composition of correspondences), we get a rational cycle of the form  $[\bar{X}] \times [\mathbf{pt}] + \dots + \bar{\alpha} \times \bar{\beta}$ .

Similarly, there is a rational cycle of the form  $[\bar{X}] \times [\mathbf{pt}] + \dots + \bar{\alpha}' \times \bar{\beta}'$ . One of its compositions with the previous one produces  $[\bar{X}] \times [\mathbf{pt}] + \dots + \deg(\beta' \cdot \alpha)(\bar{\alpha}' \times \bar{\beta})$ , therefore  $\deg(\beta' \cdot \alpha) \neq 0 \in \mathbb{F}_p$ . The other composition produces  $[\bar{X}] \times [\mathbf{pt}] + \dots + \deg(\beta \cdot \alpha')(\bar{\alpha} \times \bar{\beta}')$ , so that  $\deg(\beta \cdot \alpha') \neq 0$ .  $\square$

Below we are using notation of Example 2.5:

**Corollary 6.6.** *If  $\text{cdim}_2 Q_i = \text{cdim}_2 Q'_{i-1} = \dim Q'_{i-1}$  for some  $i$  with  $0 < i < n/2$ , then  $\deg \text{CH } Q_i \ni 2^{i-1}$ .*

*Proof.* The statement being trivial for  $i = 1$ , we may assume that  $i \geq 2$ .

For  $d := \text{cdim}_2 Q_i$ , using Proposition 6.1, we find some  $\alpha \in \text{CH}^d Q_{iF(Q_1)}$  and  $\beta \in \text{CH}_d Q_i$  with odd  $\deg(\beta_{F(Q_1)} \cdot \alpha)$ . Note that  $\text{cdim}_2 Q_{iF(Q_1)} = \text{cdim}_2 Q'_{i-1} = d$  (because the varieties  $Q_{iF(Q_1)}$  and  $Q'_{i-1}$  are equivalent). We construct some special  $\alpha' \in \text{CH}^d Q_{iF(Q_1)(Q_i)}$  and  $\beta' \in \text{CH}_d Q_{iF(Q_1)}$  with  $\deg(\beta'_{F(Q_1)(Q_i)} \cdot \alpha') = 1$  as follows. Let  $Q_{1\subset i}$  be the variety of  $(1, i)$ -flags of totally isotropic subspaces in  $q$  together with the evident projections  $Q_{1\subset i} \rightarrow Q_1, Q_i$ . We define  $\beta'$  as the pull-back via  $Q_{1\subset iF(Q_1)} \rightarrow Q_{1F(Q_1)}$  followed by the push-forward via  $Q_{1\subset iF(Q_1)} \rightarrow Q_{iF(Q_1)}$  of the rational point class  $l_0$  on  $Q_{1F(Q_1)}$ . We define  $\alpha'$  as the product  $z_{i-1} \dots z_1$ , where  $z_j$  is the pull-back via  $Q_{1\subset iF(Q_1)(Q_i)} \rightarrow Q_{1F(Q_1)(Q_i)}$  followed by the push-forward via  $Q_{1\subset iF(Q_1)(Q_i)} \rightarrow Q_{iF(Q_1)(Q_i)}$  of the class  $l_j$  of a  $j$ -dimensional projective subspace on  $Q_{1F(Q_1)(Q_i)}$ . The degree condition on  $\alpha'$  and  $\beta'$  is satisfied by [18, Statement 2.15]. Fixing an algebraically closed field containing  $F(Q_1)(Q_i)$ , we see by Lemma 6.5 that the product  $\bar{\beta} \cdot \bar{\alpha}'$  is an odd degree 0-cycle class on  $\bar{Q}_i$ . Moreover, the class  $\bar{\beta}$  is rational. Since  $2\bar{z}_j$  is rational for every  $j < (n-2)/2$  (by the reason that  $2l_j$  is rational), the class  $2^{i-1}\bar{\beta}\bar{\alpha}$  is also rational and it follows that  $2^{i-1} \in \deg \text{CH } Q_i$ .  $\square$

## 7. MATHEWS' CONJECTURE

Theorem 7.1, proved below, has been conjectured in [15]. It is known for  $i = 1$  by [5] (the proof is essentially contained already in [17]; the characteristic 2 case has been treated later on in [4]). The case of maximal  $i$ , i.e., of  $i = \lfloor n/2 \rfloor$ , is also known and is discussed in the beginning of the proof. For  $i = 2$  and odd-dimensional  $q$ , Theorem 7.1 has been proved in [15] (the proof for  $i = 2$  given here is different; in particular, it does not make use of the motivic decompositions of [3] for products of projective homogeneous varieties).

**Theorem 7.1.** *Let  $q$  be a non-degenerate quadratic form over a field  $F$ . Let  $i$  be an integer satisfying  $1 \leq i \leq (\dim q)/2$ . If the degree of every closed point on  $Q_i$  is divisible by  $2^i$  and the Witt index of the quadratic form  $q_{F(Q_i)}$  equals  $i$ , then the variety  $Q_i$  is 2-incompressible (i.e.,  $\text{cdim}_2 Q_i = \dim Q_i$ ).*

*Proof.* We set  $n := \dim q$ . Note that for  $i = n/2$  (and even  $n$ ) the condition on closed points on  $Q_{n/2}$  ensures that  $\text{disc } q$  is non-trivial. In particular,  $Q_{n/2}$  is irreducible. Moreover, the variety  $Q_{n/2}$  is isomorphic to the orthogonal grassmannian of totally isotropic  $(n/2 - 1)$ -planes of  $q^1$  considered as a variety over  $F$ , where  $q^1$  is any 1-codimensional non-degenerate subform in  $q_E$ , and  $E/F$  is the quadratic field extension given by the discriminant of  $q$ . Therefore the statement of Theorem 7.1 for  $i = n/2$  follows from the statement for  $i = (n - 1)/2$ . By this reason, we do not consider the case of  $i = n/2$  below. In particular,  $Q_i$  below is a projective  $G$ -homogeneous variety.

We induct on  $n$ . There is nothing to prove for  $n < 3$ . Below we are assuming that  $n \geq 3$ .

Over the field  $F(Q_1)$ , the motive of  $Q_{iF(Q_1)}$  decomposes as in (2.6). Since  $n' := \dim q' = n - 2 < n$ , the variety  $Q'_{i-1}$  is 2-incompressible by the induction hypothesis (more precisely, the induction hypothesis is applied if  $i \geq 2$ , for  $i = 1$  the statement is trivial). Indeed, since the extension  $F(Q_1)/F$  is a tower of a purely transcendental extension followed by a quadratic one, the degree of any closed point on  $Q'_{i-1}$  is divisible by  $2^{i-1}$ ; the Witt index of  $q'_{F(Q_1)(Q'_{i-1})}$  is  $i - 1$ , that is, the Witt index of  $q_{F(Q_1)(Q'_{i-1})}$  is  $i$  because the field extension  $F(Q_1)(Q'_{i-1})(Q_i)/F(Q_i)$  is purely transcendental.

By Theorem 2.2 and Example 2.5, the motive of  $Q'_{i-1}$  decomposes in a direct sum of one copy of  $U(Q'_{i-1})$ , shifts of  $U(Q'_j)$  with various  $j \geq i$ , and (in the case of even  $n$  and non-trivial  $\text{disc } q$ ) shifts of  $U(Q'_{jE})$  with  $j \geq i - 1$  (where  $E/F$  is the quadratic field extension corresponding to  $\text{disc } q$ ). The motive of  $Q'_i$  decomposes in a direct sum of shifts of  $U(Q'_j)$  and (in the case of even  $n$  and non-trivial  $\text{disc } q$ ) shifts of  $U(Q'_{jE})$  with various  $j \geq i$ . Note that the condition on the Witt index of the form  $q_{F(Q_i)}$  ensures that for any  $j \geq i$  the motive  $U(Q'_{i-1})$  is not isomorphic to  $U(Q'_j)$  (and  $U(Q'_{i-1}) \not\cong U(Q'_{jE})$  anyway). Therefore the complete motivic decomposition of  $Q_{iF(Q_1)}$  contains one copy of  $U(Q'_{i-1})$ , one copy of  $U(Q'_{i-1})(\dim Q_i - \dim Q'_{i-1})$  and no other shifts of  $U(Q'_{i-1})$ .

The complete decomposition of  $U(Q_{iF(Q_1)})$  contains the summand  $U(Q'_{i-1})$ . If it also contains the second (shifted) copy of  $U(Q'_{i-1})$ , then  $\text{cdim}_2 Q_i = \dim Q_i$  by [9, Theorem 5.1], and we are done. Otherwise, by Proposition 2.4,  $\text{cdim}_2 Q_i = \text{cdim}_2 Q'_{i-1} = \dim Q'_{i-1}$ , and we get by Corollary 6.6 that  $Q_i$  has a closed point of degree not divisible by  $2^i$ .  $\square$

## REFERENCES

- [1] BROSNAN, P. On motivic decompositions arising from the method of Białynicki-Birula. *Invent. Math.* 161, 1 (2005), 91–111.
- [2] BUCH, A. S., KRESCH, A., AND TAMVAKIS, H. Quantum Pieri rules for isotropic Grassmannians. *Invent. Math.* 178, 2 (2009), 345–405.
- [3] CHERNOUSOV, V., AND MERKURJEV, A. Motivic decomposition of projective homogeneous varieties and the Krull-Schmidt theorem. *Transform. Groups* 11, 3 (2006), 371–386.
- [4] ELMAN, R., KARPENKO, N., AND MERKURJEV, A. *The algebraic and geometric theory of quadratic forms*, vol. 56 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2008.
- [5] KARPENKO, N., AND MERKURJEV, A. Essential dimension of quadrics. *Invent. Math.* 153, 2 (2003), 361–372.
- [6] KARPENKO, N. A. Incompressibility of generic orthogonal grassmannians. *Linear Algebraic Groups and Related Structures* (preprint server) 409 (2010, Nov 23), 7 pages.
- [7] KARPENKO, N. A. Upper motives of algebraic groups and incompressibility of Severi-Brauer varieties. *Linear Algebraic Groups and Related Structures* (preprint server) 333 (2009, Apr 3, revised: 2009, Apr 24), 18 pages. *J. Reine Angew. Math.*, to appear.
- [8] KARPENKO, N. A. Cohomology of relative cellular spaces and of isotropic flag varieties. *Algebra i Analiz* 12, 1 (2000), 3–69.
- [9] KARPENKO, N. A. Canonical dimension. In *Proceedings of the International Congress of Mathematicians. Volume II* (New Delhi, 2010), Hindustan Book Agency, pp. 146–161.
- [10] KARPENKO, N. A. Hyperbolicity of orthogonal involutions. *Doc. Math. Extra Volume: Andrei A. Suslin's Sixtieth Birthday* (2010), 371–392 (electronic). With an Appendix by Jean-Pierre Tignol.
- [11] KARPENKO, N. A. Upper motives of outer algebraic groups. In *Quadratic forms, linear algebraic groups, and cohomology*, vol. 18 of *Dev. Math.* Springer, New York, 2010, pp. 249–258.
- [12] KARPENKO, N. A. Incompressibility of orthogonal grassmannians. *C. R. Math. Acad. Sci. Paris* 349, 21-22 (2011), 1131–1134.
- [13] KARPENKO, N. A., AND MERKURJEV, A. S. Canonical  $p$ -dimension of algebraic groups. *Adv. Math.* 205, 2 (2006), 410–433.
- [14] KNUS, M.-A., MERKURJEV, A., ROST, M., AND TIGNOL, J.-P. *The book of involutions*, vol. 44 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 1998. With a preface in French by J. Tits.
- [15] MATHEWS, B. G. Incompressibility of orthogonal Grassmannians of rank 2. Preprint (October 2008), 14 pages. To appear in *J. Algebra*.
- [16] PETROV, V., SEMENOV, N., AND ZAINOULLINE, K.  $J$ -invariant of linear algebraic groups. *Ann. Sci. École Norm. Sup. (4)* 41 (2008), 1023–1053.
- [17] VISHIK, A. Direct summands in the motives of quadrics. Preprint, 1999, 13 pages. Available on the web page of the author.
- [18] VISHIK, A. Fields of  $u$ -invariant  $2^r + 1$ . In *Algebra, arithmetic, and geometry: in honor of Yu. I. Manin. Vol. II*, vol. 270 of *Progr. Math.* Birkhäuser Boston Inc., Boston, MA, 2009, pp. 661–685.
- [19] ZHYKHOVICH, M. Decompositions of motives of generalized Severi-Brauer varieties. arXiv:1110.1023v2 [math.AG] (5 Dec 2011), 13 pages. To appear in *Doc. Math.*
- [20] ZHYKHOVICH, M. Motivic decomposability of generalized Severi-Brauer varieties. *C. R. Math. Acad. Sci. Paris* 348, 17-18 (2010), 989–992.

UNIVERSITÉ PIERRE ET MARIE CURIE, INSTITUT DE MATHÉMATIQUES DE JUSSIEU, PARIS, FRANCE

Web page: [www.math.jussieu.fr/~karpenko](http://www.math.jussieu.fr/~karpenko)

E-mail address: [karpenko at math.jussieu.fr](mailto:karpenko@math.jussieu.fr)