CODIMENSION 2 CYCLES ON QUADRATIC WEIL TRANSFER OF BIQUATERNIONIC SEVERI-BRAUER VARIETY

NIKITA A. KARPENKO

Abstract. Let F be a field, B a biquaternion F-algebra, L/F an étale quadratic extension, X the Weil transfer with respect to L/F of the Severi-Brauer variety of B_L. We show that the Chow group of codimension 2 cycle classes on X is torsion-free.

Our Chow groups are those with integral coefficients. The motives used in the proof are the Grothendieck Chow motives (still with the integral coefficients) as in [1, §64].

Theorem 1. *Let F be a field, B a biquaternion F-algebra, L/F an étale quadratic extension, X the Weil transfer with respect to L/F of the Severi-Brauer variety of B_L. Then the Chow group $\mathrm{CH}^2(X)$ is torsion-free.*

Proof. If L is not a field, then $L \simeq F \times F$ and $X \simeq \mathrm{SB}(B) \times \mathrm{SB}(B)$. Since X is a rank 3 projective bundle over $\mathrm{SB}(B)$, the Chow motive of X is then isomorphic to the sum $M \oplus M(1) \oplus M(2) \oplus M(3)$, where M is the motive of the variety $\mathrm{SB}(B)$. Therefore the total Chow group of X is a direct sum of four (shifted) copies of the total Chow group of $\mathrm{SB}(B)$. Since the total Chow group of $\mathrm{SB}(B)$ is torsion-free [5, Corollary 7], the total Chow group of X is also torsion-free; in particular, so is the group $\mathrm{CH}^2(X)$.

Now let us assume that L is a field and B_L is not a division algebra. In this case, by [8, Proposition 16.2], there exists a quaternion F-algebra Q such that the algebra B_L is isomorphic to the algebra of (2×2)-matrices over Q_L. Let now M be the motive (over L) of the Severi-Brauer variety of Q_L. Then the motive of $\mathrm{SB}(B_L)$ is isomorphic to $M \oplus M(2)$ [4, Theorem 1.3.1]. Therefore, by [3, Lemma 2.1], the motive (over F) of X is isomorphic to the sum $R_{L/F}M \oplus (R_{L/F}M)(4) \oplus \mathrm{cor}_{L/F}(M \otimes \sigma M)(2)$, where σ is the non-trivial automorphism of L/F, $R_{L/F}$ is the motivic Weil transfer functor of [6], and $\mathrm{cor}_{L/F}$ is the motivic corestriction functor introduced in [7]. The motive $R_{L/F}M$ is the motive of the variety $R_{L/F}\,\mathrm{SB}(Q_L)$ (abusing notation, we write $R_{L/F}$ also for the Weil transfer on varieties). This variety is a smooth projective quadric surface; its total Chow group is torsion-free. Concerning the remaining summand $\mathrm{cor}_{L/F}(M \otimes \sigma M)(2)$, we first note that the corestriction functor preserves the Chow group. The motive $M \otimes \sigma M$ is the motive of the variety $\mathrm{SB}(Q_L) \times \mathrm{SB}(Q_L)$ which is a rank 1 projective bundle over $\mathrm{SB}(Q_L)$. Since the total Chow group of $\mathrm{SB}(Q_L)$ is torsion-free, the total Chow group of $\mathrm{cor}_{L/F}(M \otimes \sigma M)$ is torsion-free and it follows that the total Chow group of X is torsion-free in this case also.

To finish the proof of Theorem 1, we consider the remaining case where B_L is a skew field (and L is a field). We consider the Grothendieck group $K(X)$ of classes of coherent $\mathcal{O}_X$-modules together with its topological filtration, and we are going to show that the

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total associated graded group is torsion-free. Since its codimension 2 graded piece is isomorphic to $\mathrm{CH}^2(X)$, this is sufficient for our purpose.

We start by describing the group $K(X)$. Note that the Grothendieck group (but not the topological filtration on it) is computed for any projective homogeneous variety in [9]. More precisely, for any projective homogeneous variety Y over a field l, an isomorphism $K_0(Y) \simeq K_0(A)$ with certain separable l-algebra A is constructed (and $K(Y) = K_0(Y)$). Moreover, if l/k is a finite separable field extension, then $K_0(R_{l/k}Y) \simeq K_0(N_{l/k}A)$, where $N_{l/k}A$ is the norm algebra.

We turn back to our particular variety X. According to [9] and since X is projective homogeneous, the restriction homomorphism $K(X) \to K(X_E)$ is injective for any field extension E/F. We will describe $K(X)$ as a subgroup of $K(X_E)$ with E/F such that X_E is "simple enough".

Let E/F be a maximal subfield of the F-algebra B. Since B_L is a skew field, $EL := E \otimes_F L$ is a field. The EL-variety X_{EL} is isomorphic to $\mathbb{P}^3 \times \mathbb{P}^3$. Let x be the class of $\mathcal{O}(-1)$ in $K(\mathbb{P}^3)$. The ring $K(\mathbb{P}^3)$ is generated by x with the only relation $(1-x)^4 = 0$. In particular, the group $K(\mathbb{P}^3)$ is freely generated by x^i, $i \in \{0, 1, 2, 3\}$. The ring $K(X_{EL}) = K(\mathbb{P}^3 \times \mathbb{P}^3)$ is isomorphic to the tensor product of 2 copies of the ring $K(\mathbb{P}^3) = \mathbb{Z}[x]/(1-x)^4$, the isomorphism is given by the exterior product. It follows that the group $K(X_{EL})$ is freely generated by $x^i \times x^j$ with $i, j \in \{0, 1, 2, 3\}$.

The non-trivial element of the Galois group of EL/E acts on $K(X_{EL})$ by exchanging the factors. The subgroup (subring) $K(X_E) \subset K(X_{EL})$ is the set of invariant elements; it is freely generated by $x^i \times x^i$, $i \in \{0, 1, 2, 3\}$ and $x^i \times x^j + x^j \times x^i$ with $i \neq j \in \{0, 1, 2, 3\}$.

The subgroup (subring) $K(X) \subset K(X_E)$ is generated by certain multiples of the generators of $K(X_E)$. Namely, by $x^i \times x^i$, $i \in \{0, 1, 2, 3\}$ (each of which comes with the coefficient $1 = \operatorname{ind} N_{L/F} B_L^{\otimes i}$), by $x^i \times x^j + x^j \times x^i$ with even $i+j$ (still with the coefficient $1 = \operatorname{ind} B_L^{\otimes(i+j)}$), and, finally, by $4(x^i \times x^j + x^j \times x^i)$ with odd $i+j$ (the coefficient 4 being the index of $B_L^{\otimes(i+j)}$). This is our desired description of $K(X)$ and now we come to the consideration of its topological filtration.

First of all, we know the topological filtration on $K(X_{EL})$. It is easily described in terms of $h^i \times h^j$, a different system of additive generators, where $h := 1 - x$ is the hyperplane class. In terms of these generators, the topological filtration is simply the degree filtration: the term of codimension d is generated by all $h^i \times h^j$ with $i + j \geq d$.

We claim that the topological filtration on $K(X_E)$ is induced by that on $K(X_{EL})$. In general, we only have the inclusion $K(X_E)^{(d)} \subset K(X_{EL})^{(d)}$ for any d. This gives a homomorphism of the associated graded groups. By a transfer argument, the kernel of this homomorphism consists of torsion. On the other hand, as we already know, the total Chow group of X_E has no torsion. Therefore, the graded group of the topological filtration on $K(X_E)$ is also torsion-free. It follows that the homomorphism of the graded groups is injective and this means that the inclusion $K(X_E)^{(d)} \subset K(X_{EL})^{(d)}$ is in fact the equality $K(X_E)^{(d)} = K(X_{EL})^{(d)} \cap K(X_E)$, that is, the filtration on $K(X_E)$ is the induced one.

We come to the crucial point of the proof. We claim that the graded group of $K(X)$ is also torsion-free. In other words, we claim that the filtration on $K(X)$ is induced by that on $K(X_E)$. We prove this by the method of [2, Lemma 2.1]. Since the index of $K(X)$ in

$K(X_E)$ is $4^4 = 2^8$, it suffices to show that the order of the cokernel of the homomorphism of the graded groups is at most 2^8.

The graded group of $K(X_E)$ is generated by the classes of the following elements:

$$\begin{array}{ll}
\text{codim} = 0: & 1 \times 1, \\
\text{codim} = 1: & 1 \times h + h \times 1, \\
\text{codim} = 2: & 1 \times h^2 + h^2 \times 1 \quad \text{and} \quad h \times h, \\
\text{codim} = 3: & 1 \times h^3 + h^3 \times 1 \quad \text{and} \quad h \times h^2 + h^2 \times h, \\
\text{codim} = 4: & h \times h^3 + h^3 \times h \quad \text{and} \quad h^2 \times h^2, \\
\text{codim} = 5: & h^2 \times h^3 + h^3 \times h^2, \\
\text{codim} = 6: & h^3 \times h^3.
\end{array}$$

The cokernel order in codimension 0 is 1.

The image in codimension 1 contains the class of the element $1 \times h + h \times 1$. Indeed,

$$K(X) \ni \alpha := 1 \times 1 - x \times x = 1 \times 1 - (1-h) \times (1-h) =$$
$$1 \times h + h \times 1 - h \times h \equiv 1 \times h + h \times 1 \pmod{K(X_E)^{(2)}}.$$

Since $\alpha \in K(X_E)^{(1)}$ and the quotient $K(X)^{(0)}/K(X)^{(1)}$ is torsion-free, $\alpha \in K(X)^{(1)}$. It follows that the cokernel order in codimension 1 is also 1.

The image in codimension 2 contains the class of $1 \times h^2 + h^2 \times 1 + 2(h \times h)$, as this is the square of the class of $1 \times h + h \times 1$. It also contains the class of $4(h \times h)$: by a transfer argument applied to the degree 4 field extension E/F, every element of the graded group of $K(X_E)$ multiplied by 4 is in the image. It follows that the cokernel order in codimension 2 is at most 4.

The crucial thing happens in codimension 3. First of all we have the class of

$$(1 \times h + h \times 1)^3 = (1 \times h^3 + h^3 \times 1) + 3(h \times h^2 + h^2 \times h)$$

in the image. On the other hand, we have a morphism $\mathrm{SB}(B) \to R_{L/F}\,\mathrm{SB}(B_L) = X$ becoming over L the diagonal morphism $\mathrm{SB}(B_L) \to \mathrm{SB}(B_L) \times \mathrm{SB}(B_L)$. This is the morphism corresponding to the identity under the canonical bijection

$$\mathrm{Mor}\big(\mathrm{SB}(B), R_{L/F}\,\mathrm{SB}(B_L)\big) = \mathrm{Mor}\big(\mathrm{SB}(B_L), \mathrm{SB}(B_L)\big)$$

which we have because the Weil restriction functor is right-adjoint to the scalar extension functor. Therefore, $\mathrm{SB}(B)$ is a closed subvariety of X whose class in $\mathrm{CH}^3(X_E) = K(X_E)^{(3)}/K(X_E)^{(4)}$ is equal to

$$(1 \times h^3 + h^3 \times 1) + (h \times h^2 + h^2 \times h).$$

It follows that the cokernel order in codimension 3 is at most $2 = \left|\det\begin{pmatrix} 1 & 1 \\ 3 & 1 \end{pmatrix}\right|$. We also see that the class of $2(h \times h^2 + h^2 \times h)$ is in the image.

In codimension 4 the image contains $2(h^2 \times h^2)$ as this is $(1 \times h + h \times 1)^4$ modulo 4. We also have $2(h \times h^3 + h^3 \times h)$ as the product of $2(h \times h^2 + h^2 \times h)$ by $1 \times h + h \times 1$ modulo 4. Therefore the cokernel order in codimension 4 is at most 4.

The doubled generator in codimension 5 is in the image as the product of $2(h \times h^3 + h^3 \times h)$ by $1 \times h + h \times 1$. So, the cokernel order in codimension 5 is at most 2.

Finally, the cokernel order in codimension 6 is at most 4.

Summarizing (multiplying) over all codimension, we see that the total cokernel order is at most $1 \cdot 1 \cdot 4 \cdot 2 \cdot 4 \cdot 2 \cdot 4 = 2^8$, as desired. □

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UPMC Univ Paris 06, Institut de Mathématiques de Jussieu, F-75252 Paris, FRANCE

Web page: www.math.jussieu.fr/~karpenko
E-mail address: karpenko *at* math.jussieu.fr