CYCLES OF CODIMENSION 3 ON A PROJECTIVE QUADRIC

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Let X be a nonsingular quadratic hypersurface in a projective space over an arbitrary field (of characteristic not two) and let CH^pX be a Chow group of codimension p, that is, a group of classes of codimension p cycles on X with respect to rational equivalency. It is proved that torsion in CH^3X is either trivial or is a second order group. Torsion in CH^pX, when $p \neq 3$, was studied earlier in RZhMat 1990, 9 A334 and 10 A389.

Let X be a nonsingular quadratic hypersurface in a projective space over field F and suppose that the characteristic of F is not equal to two; let CH^*X be a graded (with respect to the codimension of cycles) Chow ring of variety X [5]. This paper is devoted to proving the following theorem.

THEOREM. The torsion subgroup of group CH^3X is either trivial or isomorphic to $\mathbb{Z}/2$.

Thus, this paper lies within the framework of the intensive study of the Chow ring of a quadric undertaken in recent years. The impetus for this study was furnished by the calculation of the K-theory of quadrics carried out by Swan [10], which provided approaches to the calculation of the Chow ring in certain special cases and offered hope for a nice solution in the general case. Specific results have since been obtained by Merkur'ev, Rost, Swan, the author, and others [2], [4], [8], [11].

The problem of describing CH^*X is reducible to the problem of describing for every p a torsion subgroup $TCH^pX \subset CH^pX$. Simple arguments show that groups TCH^0X and TCH^1X are trivial. Theorem 1.2, proved in [2], gives a complete description of TCH^2X; in this case, it turns out, it is either isomorphic to $\mathbb{Z}/2$ or trivial. However, hopes for a nice solution in the general case have not been realized: as was shown by Merkur'ev and the author [4], for $p \geq 4$ a group TCH^pX, under a suitable choice of field F and quadric X over it, will contain arbitrarily many different second order elements. The final group for which the answer could still be simple is TCH^3X. The theorem proved here about CH^3X strengthens this last hope, though it must be said that it has not been possible to verify in the general case for which quadrics TCH^3X is trivial and for which it is not. This problem has been solved only for quadrics of dimension not greater than five [2].

The calculations in the Grothendieck ring of quadric X given in §2 are at the basis of the proof of the theorem about CH^3X. These calculations became possible, of course, only after the appearance of [10]. Theorems 2.13 and 2.15, the latter in a somewhat modified form, can also be found in [2], although with less detailed and less complete proofs.

Throughout the letter X denotes a projective quadric defined by a nonsingular quadratic form φ over F, that is, a hypersurface in a projective space over F given by the equation $\varphi = 0$. In the second part of §2 it is assumed that φ is anisotropic. The notation connected with the Grothendieck ring of X is introduced in §2. The standard notation and theorems from [6] are used for the quadratic forms (which in the text are sometimes simply referred to as "forms"). In particular, a form of type $\langle 1,-a_1\rangle \otimes \langle 1,-a_2\rangle \otimes \ldots \otimes \langle 1,-a_n\rangle$ is called a Pfister n-form and is denoted by $\langle\langle a_1, a_2, \ldots, a_n\rangle\rangle$. Forms φ_1 and φ_2 over F are called proportional if $\varphi_1 \simeq c\varphi_2$ for some $c \in F^*$. By $I^n(F)$ we denote the n-th degree of the ideal of even-dimensional forms I(F) in Witt ring W(F) of quadratic forms over F. By $i(\varphi)$ we denote the Witt index of φ. An important part is played by the invariant $s(\varphi)$ defined in 2.4. A form is called split (totally split) if it is isotropic (correspondingly, has maximal Witt index possible for its dimension). We say that extension E/F (totally) splits φ, where φ is a quadratic form over F, if form φ_E is (totally) split.

1. Necessary Information on the Chow Groups of a Projective Quadric. The proofs of both assertions of this section are found in [2].

Proposition 1.1. Let form φ be isotropic, $\varphi = \mathbb{H} \perp \psi$, and let Y be a projective quadric corresponding to form ψ (note that $\dim Y = \dim X - 2$). Then for $p = 1, 2, \ldots, \dim X - 1$ group CH^pX is canonically isomorphic to group $CH^{p-1}Y$.

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This proposition reduces the problem about the Chow group of an arbitrary quadric to the case of a quadric given in an anisotropic form.

THEOREM 1.2. If form φ has dimension greater than four and is proportional to the subform of any anisotropic Pfister 3-form, then $TCH^2X \simeq \mathbb{Z}/2$. Otherwise, $TCH^2X = 0$.

Note that the second order nontrivial element arising in CH^2X is constructed in [2].

2. The Grothendieck Ring. Let $K = K(X)$ be the Grothendieck group of quadric X. The variety X is nonsingular, therefore $K = K'_0(X) \simeq K_0(X)$ [7]. The tensor product of locally free sheaves (O_X-modules) induces on $K_0(X)$ a ring structure and group $K'_0(X)$ possesses the so-called topological filtration [7]. Therefore K is a ring with filtration, moreover we know that these structures are compatible. Let us introduce the notation for filtration: $0 = K^{(d+1)} \subset K^{(d)} \subset \ldots \subset K^{(0)} = K$ $(d = \dim X)$ where $K^{(p)} \subset K$ is a subgroup generated by sheaf classes the codimension of whose support is greater than or equal to p. In addition, let $K_{(p)} = K^{(d-p)}$, $G^pK = G_{d-p}K = K^{(p/p+1)}$ and suppose that $TG^pK \subset G^pK$ is a torsion subgroup.

Any simple cycle $Z \subset X$ defines an element in K(X) to be the class of structure sheaf O_E extended by zero on X. The arising mapping $CH^*X \longrightarrow G^*K(X)$ s, as is known, an epimorphism of a graded ring.

THEOREM 2.1 [2]. The kernel of epimorphism $CH^pX \longrightarrow G^pK(X)$ is contained in TCH^pX for any p and is trivial for $p \leq 3$.

The remainder of this section is devoted to the study of groups G^pK; the main result is formulated in Theorems 2.13 and 2.15.

K-theory of a quadric and, in particular, the Grothendieck group K(X) have been calculated by Swan [10]. He has constructed a canonic sheaf U on X: a locally free O_X-module with the structure of the right C_0-module, where $C_0 = C_0(\varphi)$ is the even part of Clifford algebra of φ, and proved

THEOREM 2.2 [10]. The group homomorphism $\mathbb{Z}^d \oplus K_0(C_0(\varphi)) \longrightarrow K(X)$, taking the standard generators of group $\mathbb{Z}^d$, respectively, into $[O_X], [O_X(-1)], \ldots, [O_X(-d+1)]$ and class $[M] \in K_0(C_0(\varphi))$ of left module M over algebra $C_0(\varphi)$ into the class of tensor product $\mathcal{U} \otimes_{C_0} M$ is an isomorphism.

Proposition 2.3 [1], [6]. If $\varphi \notin I^2(F)$, then C_0 is a simple F-algebra (central if $\varphi \notin I(F)$; if, however, $\varphi \in I^2(F)$, then $C_0 \simeq A \times A$, where A is a simple central F-algebra, moreover the complete Clifford algebra $C(\varphi)$ is isomorphic to $M_2(A)$.

Definition 2.4. Let us define the number $s = s(\varphi)$ in the following way. If $\varphi \notin I^2(F)$, then, according to 2.3, $C_0 \simeq M_{2^s}(\mathcal{D})$ for some division ring D and nonnegative integer s. If, however, $\varphi \in I^2(F)$, then let s be a number such that $A \simeq M_{2^s}(\mathcal{D})$, where $A \times A \simeq C_0$.

From Definition 2.4 and Proposition 2.3 there immediately follows

Proposition 2.5. There exist canonic isomorphisms $K_0(C_0) \simeq \mathbb{Z}$ if $\varphi \notin I^2(F)$, and $K_0(C_0) \simeq \mathbb{Z}^2$, if $\varphi \in I^2(F)$. Here the class of algebra C_0 in $K_0(C_0)$ is equal to 2^s in the first case and to $K_0(C_0)$ in the second.

Note that from Theorem 2.2 and Proposition 2.5 we obtain

COROLLARY 2.6. The group K(X) is torsion free.

Let us denote by h an element of $1 - [O_X(-1)] \in K^{(1)}(X)$ which is the class of the total section of X by a hypersurface in a projective space.

LEMMA 2.7 [2]. Let U be a Swan sheaf on X. Then $[\mathcal{U}(d)] = h^d + 2h^{d-1} + \ldots + 2^{d-1}h + 2^d$ in group K(X).

From Proposition 2.5, in particular, we can see that the class of the Swan sheaf in group K is divisible by 2^s. Consequently, the sum $h^d + 2h^{d-1} + \ldots + 2^d$ from Lemma 2.7 is also divisible by 2^s and, therefore, the following definition is correct.

Definition 2.8. For $i = 0, 1, \ldots, s-1$ we define elements $l_i \in K$ by equalities $l_i = \frac{1}{2^{i+1}}(h^d + 2h^{d-1} + \ldots + 2^i h^{d-i})$;

in addition, for convenience, we let $l_{-1} = 0$.

The geometrical meaning of l_i is as follows: if form φ defining X is totally split, then l_i is the class of any i-dimensional projective space $\mathbb{P}_F^i$ contained in X.

LEMMA 2.9. For all $i = 0, 1, \ldots, s-1$ the following equalities hold:

$$hl_i = l_{i-1} \quad ; \quad 2l_i - l_{i-1} = h^{d-i}.$$

Proof. It is obvious from the definition of l_i.

LEMMA 2.10. Let p_i be the dimension of element l_i, that is, a number such that $\ell_i \in K_{(p_i)} \setminus K_{(p_i-1)}$. Then $p_0 < p_1 < \ldots < p_{s-1}$ and if X has no rational points, then, in addition, $p_i > i$ for all i.

Proof. The dimension of l_{i-1} is less than that of l_i since $\ell_{i-1} = h\ell_i$, $h \in K^{(1)}$ and multiplication in K is compatible with filtration. If the quadric contains no rational points, then $\ell_0 \notin K_{(0)}$ [2], [9], [11]; hence $p_0 > 0$ and, consequently, by virtue of the first assertion of the lemma, $p_i > i$ for all i.

In the remainder of this section we assume that form φ defining X is anisotropic.

LEMMA 2.11. The element t_i has order two in group $G_{p_i}K$.

Proof. Obviously, $t_i \neq 0$. On the other hand, according to Lemma 2.9, element $2l_i$ is equal to $h^{d-i} + \ell_{i-1}$ and, consequently, lies in $K_{(p_i-1)}$ since the dimension of l_{i-1} is less than the dimension of l_i and the dimension of h^{d-i}, the number i, is less than p_i according to Lemma 2.10.

Definition 2.12. Let us define torsion of the first kind to be the graded subgroup $\underline{I}_* \subset TG_*K$ defined by the formula

$$\underline{I}_p = \begin{cases} (\mathbb{Z}/2)\cdot t_i\,, & \text{if } p = p_i\,, \text{where } i = 0, 1, \ldots, s-1; \\ 0 \text{ for the remaining } p\,. \end{cases}$$

Torsion of the second kind II_* is defined to be the factor-group TG_*K by I_*.

Let $H \subset K$ be a subring generated by element h. Clearly, $H = \mathbb{Z}[h]/(h^{d+1})$, moreover the filtration on H induced with K is a filtration with respect to the powers of h: $H^{(p)} = \mathbb{Z}\cdot h^p \oplus \mathbb{Z}\cdot h^{p+1} \oplus \ldots \oplus \mathbb{Z}\cdot h^d$. Since $h = 1 - [\mathcal{O}_X(-1)]$, subring H is also generated by element $[O_X(-1)]$ and contains $[O_X(n)]$ for all integer n.

THEOREM 2.13. If $\varphi \notin I^2(F)$, then $II_*=0$, that is, I_* coincides with TG_*K (groups I_* and II_* are defined in 2.12).

The following lemma is at the basis of the proof of Theorem 2.13.

LEMMA 2.14. If $\varphi \notin I^2(F)$, then K/H is a group of order 2^s.

Proof. Let us consider the composition

$$\mathbb{Z}^d \oplus K_0(C_0(\varphi)) \xrightarrow{\sim} K \xrightarrow[\sim]{[\mathcal{O}_X(d)]} K \longrightarrow\!\!\!\!\!\rightarrow K/H$$

of Swan isomorphism 2.2 of a d-fold twist and a natural mapping on a factor-group. The term $\mathbb{Z}^d$ lies in the kernel since $[\mathcal{O}_X(n)] \in H$. We obtain the epimorphism $\alpha: \mathbb{Z} \simeq K_0(C_0(\varphi)) \twoheadrightarrow K/H$. According to Lemma 2.7, $[\mathcal{U}(d)] \in H$; therefore $\alpha(2^s) = 0$ and, in addition, $\alpha(2^{s-1}) \neq 0$ since $[\mathcal{U}(d)] = h^d + 2h^{d-1} + \ldots + 2^d$ is not divisible by two in H. Consequently, $Ker\,\alpha = 2^s\mathbb{Z}$. The lemma is proved.

Proof of Theorem 2.13. Let us endow factor-group K/H induced with K with a filtration. Let $G_*(H)$ and $G_*(K/H)$ be associative graded groups. The exact sequence $0 \to H \to K \to K/H \to 0$, since the filtrations of the extreme terms are lowered from the middle one, induces the exact sequence of graded groups $0 \to G_*H \to G_*K \to G_*(K/H) \to 0$. The group G_*H is torsion free and $G_*(K/H)$, according to Lemma 2.14, is a group of order 2^s. Consequently, the order of torsion in G_*K divides 2^s. Since the constructed torsion subgroup of the first kind $\underline{I}_* \subset TG_*K$ has order 2^s, we conclude that $\underline{I}_* = TG_*K$.

THEOREM 2.15. Let $\varphi \in I^2(F)$. Then: 1) II_p is a cyclical 2-group for each p; $II_p=0$ for $p \leq m$ ($m = d/2$); 2) if $\underline{II}^0 = \underline{II}^1 = \ldots = \underline{II}^p = 0$ for some $p < m$, then $\underline{I}^0 = \underline{I}^1 = \ldots = \underline{I}^p = \underline{I}^{p+1} = 0$.

Proof. Let P be any (of the two nonisomorphic) simple $C_0(\varphi)$ modules; we call element $u = [\mathcal{U} \otimes_{C_0} P] \in K$ a Swan generator. Denote by L a subgroup of $H + \mathbb{Z}\cdot\frac{[\mathcal{U}]}{2^s} = H + \mathbb{Z}\cdot\ell_{s-1}$ in K with an induced filtration.

LEMMA 2.16. 1) The factor-group K/L is an infinite cyclical group generated by element u. 2) The associated graded group G_*L is the direct sum $G_*H \oplus \underline{I}_*$.

Proof. Assertion 1) follows from Swan's Theorem 2.2 and 2) is analogous to 2.13.

Let us continue proving Theorem 2.15. Let us endow factor-group K/L with an induced filtration and consider the associated graded group $G_*(K/L)$. From the exact sequence $0 \to G_*L \to G_*K \to G_*(K/L) \to 0$ we obtain isomorphism $G_*(K/L) \simeq G_*K/(G_*H \oplus \underline{I}_*)$. Consequently, torsion of the second kind II_* is isomorphic to the torsion subgroup in $G_*(K/L)$. Since K/L is cyclical, II_p is also a cyclical group for every p. The factor-group G_mK/G_mH contains an element of infinite order [2], [9]; in view of the existence of isomorphism $G_m(K/L) \simeq G_mK/(G_mH \oplus \underline{I}_m)$, we can also say the same of $G_m(K/L)$. Therefore, $G_m(K/L)$ is an infinite cyclical group and $G_p(K/L)$ is trivial for $p < m$. Consequently, $II_p = 0$ for $p \leq m$.

Let us now turn to the proof of the second assertion of the theorem. Let $\overline{X} = X_E$, where E/F is an extension totally splitting φ, and suppose that $l_m \in K_{(m)}(\overline{X})$ is a class of some m-dimensional projective space $\mathbb{P}^m_E$ contained in $\overline{X}$. Note that l_m, in contrast to l_i for $i < m$, is not a linear combination of the powers of h (with rational coefficients) and depends on the choice of $\mathbb{P}^m_E \subset \overline{X}$.

LEMMA 2.17. The product hl_m is equal to l_{m-1}.

Proof. Applying projection formula to closed inclusion $i : \mathbb{P}^m \hookrightarrow \overline{X}$ we obtain $hl_m = h \cdot i_*[\mathbb{P}^m] = i_*(i^*(h)\cdot[\mathbb{P}^m]) = i_*[\mathbb{P}^{m-1}]$. Since $i_*[\mathbb{P}^{m-1}] = l_{m-1}$ the lemma is proved.

Swan's Theorem 2.2 implies that $2^{m-s}K(\overline{X}) \subset K(X)$. Denote element $2^{m-s}l_m \in K$ by l.

LEMMA 2.18. The inequalities

$$\dim l \geq m \quad ; \quad \dim 2^s l = m \quad ; \quad \dim l_{s-1} < \dim l$$

hold in K.

Proof. Since $l_m \in G_m K(\overline{X})$ is an element of infinite order [2], [9], we have $\dim l \geq m$ and, moreover, $\dim cl \geq m$ for any integer c. If extension E/F, for which $\overline{X} = X_E$, is chosen so that its degree is divisible by 2^m, then applying norm mapping $N_{E/F} : K_{(m)}(\overline{X}) \longrightarrow K_{(m)}$ to element l_m we obtain $2^m l_m = 2^s l \in K_{(m)}$. Finally, $hl = h(2^{m-s}l_m) = 2^{m-s}l_{m-1} \equiv l_{s-1} \pmod{H_{(m-1)}}$, whence $\dim l_{s-1} < \dim l$. The lemma is proved.

Let I_p be a nontrivial component of torsion of the first kind with maximal dimension and suppose that $p \geq m$. To prove assertion 2) of Theorem 2.15, it is enough to find a number $q > p$ for which $TG_q K \neq 0$. From the definition of group I_* it is clear that $p = \dim l_{s-1}$. Let $q = \dim l$. Then $q > p$ according to Lemma 2.18 and $G_q K$ contains a nontrivial element l, moreover $2^s l = 0$, again according to the same lemma. Consequently, $TG_q K \neq 0$ and the proof of Theorem 2.15 is completed.

Remark 2.19. We can show under the hypotheses of Theorem 2.15 that the exact sequence $0 \longrightarrow I_* \longrightarrow TG_* K \longrightarrow II_* \longrightarrow 0$ splits.

3. The Main Theorem. THEOREM 3.1. For any projective quadric X the group TCH^3X is 0 or $\mathbb{Z}/2$.

We preface the proof of the theorem with several auxiliary lemmas.

LEMMA 3.2. For any six-dimensional anisitropic quadratic form ψ of determinant -1 the invariant $s(\psi)$ is equal to zero.

Proof. If $s(\psi) > 0$, then the index of the complete Clifford algebra $C(\psi)$ does not exceed one. Consequently, for some quadratic extension E/F algebra $C(\psi)_E \simeq C(\psi_E)$ is isomorphic to matrix algebra, that is $\psi_E \in I^3(E)$ [3]. Since $\dim \psi_E = 6 < 2^3$, the ψ_E is hyperbolic [6], whence $\psi \simeq \langle\langle a \rangle\rangle \otimes \rho$ for some $a \in F^*$ and a three-dimensional form ρ [6]. Calculating the determinants, we obtain $a = -\det\psi = 1$, which contradicts the anisotropic nature of ψ.

LEMMA 3.3. If $s(\psi) = 2$, where ψ is an eight-dimensional anisotropic form of determinant 1, then ψ contains a subform proportional to a Pfister 2-form.

Proof. Let E/F be a quadratic extension splitting ψ : $\psi_E \simeq H \perp \psi'$. Then $s(\psi') = s(\psi_E) - 1 \geq 1$ and, consequently, according to Lemma 3.2, ψ' is isotropic. Therefore $i(\psi_E) \geq 2$, whence ψ contains a subform proportional to a Pfister 2-form [6].

LEMMA 3.4. Under the hypotheses of the previous lemma, ψ is divisible by $\langle\langle a \rangle\rangle$ for some $a \in F^*$.

Proof. According to 3.3, $\psi \simeq f \cdot \langle\langle b_1, b_2 \rangle\rangle \perp g \cdot \langle\langle c_1, c_2 \rangle\rangle$, whence

$$[C(\psi)] = \left[\left(\frac{b_1, b_2}{F}\right) \otimes \left(\frac{c_1, c_2}{F}\right)\right] \in Br(F).$$

Since $s(\psi) = 2$, the above tensor product of quaternion algebras is not a division ring, that is, the form $\langle b_1, b_2, -b_1b_2, -c_1, -c_2, c_1c_2 \rangle$ is isotropic [6]. The latter means that there exists an element $a \in F^*$ which is the value of the pure subforms of both Pfister 2-forms. Therefore $\langle\langle b_1, b_2 \rangle\rangle$ and $\langle\langle c_1, c_2 \rangle\rangle$ are divisible by $\langle\langle a \rangle\rangle$ [6] and, consequently, ψ is divisible by $\langle\langle a \rangle\rangle$.

LEMMA 3.5. Let ψ be an eight-dimensional quadratic form over F and suppose that for some quadratic extension L/F form ψ_L is proportional to a Pfister 3-form. Then there exists a quadratic extension L'/F such that $i(\psi_{L'}) \geq 3$.

Proof. If ψ is isotropic, then ψ_L is hyperbolic and we can take L itself for L'. In the anisotropic case let ρ be some seven-dimensional subform of ψ and let $\psi' = \rho \perp \langle \det\rho \rangle$. If ψ' is isotropic, then L/F splits and, hence, totally splits and we again take $L' = L$. If, however, ψ' is anisotropic, then we use Lemma 3.4. Since $s(\psi') \geqslant 2$, we conclude that ψ' totally splits in some quadratic extension L'/F. Clearly, for this extension $i(\psi_{L'}) \geqslant 3$.

Proof of the Theorem. If form φ defining quadric X is isotropic, then, by virtue of Proposition 1.1, CH^3X is isomorphic to group CH^2 of some quadric and we can apply Theorem 1.2. Next we assume that φ is anisotropic.

According to 2.1, $CH^3X \simeq G^3K(X)$. If $\varphi \notin I^2(F)$, then according to Theorem 2.13 $TG^3K(X)$ is 0 or $\mathbb{Z}/2$. Next we assume that $\varphi \in I^2(F)$.

If the even number $\dim\varphi$ does not exceed eight, then, in view of 2.15, torsion of the second kind does not exist in codimension three; consequently, the theorem is also proved in this case. Next we assume that $\dim\varphi \geqslant 10$.

Groups $G^0K(X)$, $G^1K(X)$, $G^2K(X)$ are torsion free (the latter by Theorem 1.2); consequently, according to Theorem 2.15, $I^3 = 0$ and $TG^3K(X) = \overline{I}^3$ is a cyclical group. It remains to prove that this group is annihilated by multiplication by two. For this let us consider an arbitrary quadratic extension L/F. Composition $TCH^3X \xrightarrow{res} TCH^3X_L \xrightarrow{N} TCH^3X$ coincides with multiplication by two and the proof of the theorem is concluded as soon as we verify that the mean group is trivial for a suitable choice of L/F. Suppose that L/F splits φ: $\varphi_L \simeq \mathbb{H} \perp \psi_L$, where ψ is a subform of φ, and let Y be a quadric corresponding to ψ. Then by 1.1, $CH^3X_L \simeq CH^2Y_L$. Since $\dim\psi = \dim\varphi - 2 \geqslant 8$, according to Theorem 1.2, group TCH^2Y_L can be nontrivial only if ψ_L is proportional to a Pfister 3-form. Assume that it is so. Then according to Lemma 3.5 there exists a quadratic extension L'/F for which $i(\psi_{L'}) \geqslant 3$. Since ψ is a subform of φ so is $i(\varphi_{L'}) \geqslant 3$. Hence, by 1.1, $TCH^3X_{L'} = 0$. The theorem is proved.

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