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Inventiones mathematicae

Homology of the Infinite Orthogonal and Symplectic Groups over Algebraically Closed Fields

An Appendix to the Paper of A. Suslin

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As a consequence of Suslin's result and the fundamental theorem of Hermitian K-theory [3], we shall prove the following statement:

Theorem. *Let k be an algebraically closed field and k_0 the algebraic closure of the prime field. Then, the obvious homomorphisms*

$$O_{2n}(k_0) \to O_{2n}(k) \quad \textit{and} \quad Sp_{2n}(k_0) \to Sp_{2n}(k)$$

induce isomorphisms

$$\varinjlim H_*(O_{2n}(k_0); \mathbf{Z}/q) \xrightarrow{\approx} \varinjlim H_*(O_{2n}(k); \mathbf{Z}/q)$$
$$\varinjlim H_*(Sp_{2n}(k_0); \mathbf{Z}/q) \xrightarrow{\approx} \varinjlim H_*(Sp_{2n}(k); \mathbf{Z}/q)$$

where q is arbitrary if $\mathrm{char}(k) \neq 2$ *and q is odd if* $\mathrm{char}(k) = 2$.

Proof. Let F denote either k or k_0. Here $Sp_{2n}(F)$ is the usual symplectic group and $O_{2n}(F)$ the orthogonal group of F^{2n} provided with the "hyperbolic" quadratic form

$$(x_1, \ldots, x_n, y_1, \ldots, y_n) \mapsto \sum_{i=1}^{n} x_i y_i.$$

Therefore $O(F) = \varinjlim O_{2n}(F)$ and $Sp(F) = \varinjlim Sp_{2n}(F)$ are just the groups denoted by ${}_1O(F)$ and ${}_{-1}O(F)$ in [2]. If $\mathrm{char}(F) \neq 2$, since $\sqrt{-1} \in F$, $O(F)$ may be considered as the direct limit of the orthogonal groups of F^n provided with the quadratic form defined by the sum of the squares of the coordinates. Since homology commutes with direct limits, we have to prove isomorphisms

$$H_*(O(k); \mathbf{Z}/q) \approx H_*(O(k_0); \mathbf{Z}/q) \quad \text{and} \quad H_*(Sp(k); \mathbf{Z}/q) \approx H_*(Sp(k_0); \mathbf{Z}/q).$$

Let us recall now some basic definitions and results of [2, 3] adapted to this situation. The Hermitian K-theory spectrum ${}_\varepsilon\mathscr{L}(F)$, with $\varepsilon = \pm 1$, is defined as follows ([2], p. 212-218):

$$\begin{aligned} {}_\varepsilon\mathscr{L}_n(F) &= \Omega^n({}_\varepsilon L_0(F) \times B_\varepsilon O(F)^+) && \text{for } n \geqq 0 \\ {}_\varepsilon\mathscr{L}_n(F) &= {}_\varepsilon L_0(S^{-n}F) \times B_\varepsilon O(S^{-n}F)^+ && \text{for } n < 0. \end{aligned}$$

The Hermitian "L-groups" are given by the formulas

$$_\varepsilon L_n(F)=\pi_0({}_\varepsilon\mathcal{L}_n(F)), \qquad {}_\varepsilon L_n(F;\mathbf{Z}/q)=\pi_0({}_\varepsilon\mathcal{L}_n(F);\mathbf{Z}/q)=\pi_i({}_\varepsilon\mathcal{L}_{n-i}(F);\mathbf{Z}/q)$$

(the last equality may be taken as a definition for $i\geqq 2$). For example ${}_{-1}L_0(F)\approx {}_1L_0(F)\approx\mathbf{Z}$ (this is just the Grothendieck group of the category of F-vector spaces provided with a non degenerate skew symmetric or quadratic form). We have also ${}_{-1}L_1(F)\approx H_1(Sp(F);\mathbf{Z})=0$ and

$${}_1L_1(F)\approx H_1(O(F);\mathbf{Z})\approx\mathbf{Z}/2.$$

The Quillen K-theory spectrum $\mathcal{K}(F)$ is described in an analogous way ([2], p. 212–218):

$$\mathcal{K}_n(F)=\Omega^n(K_0(F)\times BGL(F)^+) \qquad \text{for } n\geqq 0,$$
$$\mathcal{K}_n(F)=K_0(S^{-n}F)\times BGL(S^{-n}F)^+ \qquad \text{for } n<0.$$

Note that $K_n(F)=\pi_0(\mathcal{K}_n(F))$ and $K_n(F;\mathbf{Z}/q)\approx\pi_i(\mathcal{K}_{n-i}(F);\mathbf{Z}/q)$ for $i\geqq 2$. Note that $K_n(F)$ (resp. ${}_\varepsilon L_n(F)$) are for $n>0$ the homotopy groups of an infinite loop space which homology is precisely that of $GL(F)$ (resp. ${}_\varepsilon O(F)$). Note finally that for $n<0$, $K_n(F)=0$ ([4], p. 73), but ${}_\varepsilon L_n(F)\neq 0$ in general (compare with [3], p. 278).

Finally, there are maps of spectra

$$\mathcal{K}_n(F)\to{}_\varepsilon\mathcal{L}_n(F) \quad\text{and}\quad {}_\varepsilon\mathcal{L}_n(F)\to\mathcal{K}_n(F)$$

induced by the hyperbolic and the forgetful functor respectively. If we denote by ${}_\varepsilon\mathcal{U}_n(F)$ and ${}_\varepsilon\mathcal{V}_n(F)$ the homotopy fibers of these two maps, then the fundamental theorem of [3] implies the existence of an homotopy equivalence

$${}_\varepsilon\mathcal{V}_n(F)\approx\Omega\,{}_{-\varepsilon}\mathcal{U}_n(F)$$

if char$(F)\neq 2$ and

$${}_\varepsilon\mathcal{V}_n(F)_{(2)}\sim\Omega\,{}_{-\varepsilon}\mathcal{U}_n(F)_{(2)}$$

if char $(F)=2$ ([3], p. 260; cf. [2], p. 253 for the definition of $X_{(2)}$ when X is an H-space). Therefore, if we define ${}_\varepsilon V_n(F)$ (resp. ${}_\varepsilon U_n(F)$) as $\pi_0({}_\varepsilon\mathcal{V}_n(F))$ (resp. $\pi_0({}_\varepsilon\mathcal{U}_n(F))$, then we have an isomorphism ${}_\varepsilon V_n(F)\approx{}_{-\varepsilon}U_{n+1}(F)$ if char$(F)\neq 2$ and ${}_\varepsilon V_n(F)\otimes\mathbf{Z}[\frac{1}{2}]\approx{}_{-\varepsilon}U_{n+1}(F)\otimes\mathbf{Z}[\frac{1}{2}]$ if char$(F)=2$. In the same way as above we may introduce U and V groups with coefficients in $\mathbf{Z}/q$ and then we have four types of exact sequences:

$$K_{n+1}(F)\to{}_\varepsilon L_{n+1}(F)\to{}_\varepsilon U_n(F)\to K_n(F)\to{}_\varepsilon L_n(F)$$
$${}_\varepsilon L_{n+1}(F)\to K_{n+1}(F)\to{}_\varepsilon V_n(F)\to{}_\varepsilon L_n(F)\to K_n(F)$$
$$\bar K_{n+1}(F)\to{}_\varepsilon\bar L_{n+1}(F)\to{}_\varepsilon\bar U_n(F)\to\bar K_n(F)\to{}_\varepsilon\bar L_n(F)$$
$${}_\varepsilon\bar L_{n+1}(F)\to\bar K_{n+1}(F)\to{}_\varepsilon\bar V_n(F)\to{}_\varepsilon\bar L_n(F)\to\bar K_n(F)$$

where, for short, we denote by $\bar K$, $\bar L$, $\bar U$ and $\bar V$ the corresponding K, L, U and V groups with coefficients in $\mathbf{Z}/q$. Note that ${}_\varepsilon\bar V_n(F)\approx{}_{-\varepsilon}\bar U_{n+1}(F)$ (q is assumed odd if char$(F)=2$). We are going to prove in three steps that ${}_\varepsilon\bar L_n(k)\approx{}_\varepsilon\bar L_n(k_0)$, i.e. $\pi_n({}_\varepsilon\mathcal{L}(k);\mathbf{Z}/q)\approx\pi_n({}_\varepsilon\mathcal{L}(k_0);\mathbf{Z}/q)$ for all values of n. The Hurewicz theorem mod. q will then imply the theorem of this appendix.

Step 1. *The groups* ${}_\varepsilon\bar{L}_n(k)$ *and* ${}_\varepsilon\bar{L}_n(k_0)$ *are isomorphic for* $n=0$ *and* 1. The exact sequence (E_n):

$${}_\varepsilon L_{n+1}(F) \xrightarrow{\times q} {}_\varepsilon L_{n+1}(F) \to {}_\varepsilon \bar{L}_{n+1}(F) \to {}_\varepsilon L_n(F) \xrightarrow{\times q} {}_\varepsilon L_n(F)$$

written for $n=0$, shows that ${}_\varepsilon\bar{L}_1(F) \approx \operatorname{Coker} {}_\varepsilon L_1(F) \xrightarrow{\times q} {}_\varepsilon L_1(F)$, therefore is 0 if q is odd or $\varepsilon=-1$ and is $\mathbf{Z}/2$ if q is even and $\varepsilon=1$. For the group ${}_\varepsilon\bar{L}_0(F)$ let us distinguish two cases according to the characteristic of F.

If $\operatorname{char}(F) \neq 2$, we write the two exact sequences

$$\begin{array}{ccccccccc}
{}_\varepsilon L_0(F) & \longrightarrow & K_0(F) & \longrightarrow & {}_\varepsilon V_{-1}(F) & \longrightarrow & {}_\varepsilon L_{-1}(F) & \longrightarrow & K_{-1}(F) \\
 & & & & \wr & & & & \\
K_1(F) & \longrightarrow & {}_{-\varepsilon}L_1(F) & \longrightarrow & {}_{-\varepsilon}U_0(F) & \longrightarrow & K_0(F) & \longrightarrow & {}_{-\varepsilon}L_0(F)
\end{array}$$

Let ${}_\varepsilon\alpha$ be the map ${}_\varepsilon L_0(F) \to K_0(F)$. Since ${}_{-1}L_1(F)=0$, the second line in the diagram above shows that $0 = {}_{-1}U_0(F) \approx {}_{+1}V_{-1}(F) \approx \operatorname{Coker} {}_1\alpha$. Since $K_{-1}(F)=0$, we also get ${}_1L_{-1}(F)=0$. In the same way, since ${}_{-1}V_{-1}(F) \approx {}_1U_0(F) \approx \mathbf{Z}/2$ and $\operatorname{Coker} {}_{-1}\alpha = \mathbf{Z}/2$, we have also ${}_{-1}L_{-1}(F)=0$. Therefore, the exact sequence (E_{-1}) above will imply ${}_\varepsilon\bar{L}_0(F) \approx \operatorname{Coker} {}_\varepsilon L_0(F) \xrightarrow{\times q} {}_\varepsilon L_0(F) \approx \mathbf{Z}/q$.

If $\operatorname{char}(F)=2$, let us denote $\tilde{K}, \tilde{L}, \tilde{U}, \tilde{V}$ the groups K, L, U and V tensorized by $\mathbf{Z}[\frac{1}{2}]$. Then the same arguments as above show that ${}_\varepsilon\tilde{L}_{-1}(F)=0$ since ${}_\varepsilon\tilde{V}_{-1}(F) = {}_{-\varepsilon}\tilde{U}_0(F) = \operatorname{Coker} {}_\varepsilon\tilde{\alpha} = 0$. Therefore ${}_\varepsilon\bar{L}_0(F) = \mathbf{Z}/q$ for q odd.

Step 2. *If* $\operatorname{char}(k) \neq 2$, ${}_\varepsilon L_n(k) \approx {}_\varepsilon L_n(k_0)$ *and* ${}_\varepsilon\bar{L}_n(k) \approx {}_\varepsilon\bar{L}_n(k_0)$ *for* $n<0$.
If $\operatorname{char}(k)=2$, ${}_\varepsilon\bar{L}_n(k) \approx {}_\varepsilon\bar{L}_n(k_0)$ *for* $n<0$.

This step is not really necessary for the proof of our main theorem. It is just added for the sake of completeness. If $\operatorname{char}(F) \neq 2$ we have already proved in Step 1 that ${}_\varepsilon L_{-1}(F)=0$. Now if $p>2$,

$$\begin{aligned}
&{}_\varepsilon L_{-p}(F) \approx {}_\varepsilon V_{-p}(F) \approx {}_{-\varepsilon}U_{-p+1}(F) \approx {}_{-\varepsilon}L_{-p+2}(F),\\
&{}_\varepsilon L_{-2}(F) \approx {}_\varepsilon V_{-2}(F) \approx {}_{-\varepsilon}U_{-1}(F) \approx \operatorname{Coker}(K_0(F) \to {}_{-\varepsilon}L_0(F)).
\end{aligned}$$

Therefore ${}_\varepsilon L_{-p}(k) \approx {}_\varepsilon L_{-p}(k_0)$ by induction on p. The exact sequence (E_n), written for negative values of n, will then imply ${}_\varepsilon\bar{L}_{-p}(k) \approx {}_\varepsilon\bar{L}_{-p}(k_0)$. If $\operatorname{char}(k)=2$, the arguments are completely analogous when replacing the groups K, L, U and V by $\tilde{K}, \tilde{L}, \tilde{U}$ and $\tilde{V}$.

Step 3. *The isomorphism* ${}_\varepsilon\bar{L}_{n-1}(k) \approx {}_\varepsilon\bar{L}_{n-1}(k_0)$ *implies the isomorphism*

$${}_{-\varepsilon}\bar{L}_{n+1}(k) \approx {}_{-\varepsilon}\bar{L}_{n+1}(k_0).$$

This step, together with Step 1 will of course finish the proof of the theorem. Let us write the two exact sequences

$$\begin{array}{ccccccccc}
{}_\varepsilon\bar{L}_n(k) & \longrightarrow & \bar{K}_n(k) & \longrightarrow & {}_\varepsilon\bar{V}_{n-1}(k) & \longrightarrow & {}_\varepsilon\bar{L}_{n-1}(k) & \longrightarrow & \bar{K}_{n-1}(k) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow\approx & & \uparrow \\
{}_\varepsilon\bar{L}_n(k_0) & \longrightarrow & \bar{K}_n(k_0) & \longrightarrow & {}_\varepsilon\bar{V}_{n-1}(k_0) & \longrightarrow & {}_\varepsilon\bar{L}_{n-1}(k_0) & \longrightarrow & \bar{K}_{n-1}(k_0)
\end{array}$$

The main theorem of Suslin implies that the maps $K_n(k_0)\to K_n(k)$ and $K_{n-1}(k_0)\to K_{n-1}(k)$ are isomorphisms. If, at the beginning of his proof of the main theorem, we take for R the functor ${}_{\varepsilon}\bar{V}_{n-1}$, we see that the map ${}_{\varepsilon}\bar{V}_{n-1}(k_0)\to {}_{\varepsilon}\bar{V}_{n-1}(k)$ is injective. The surjectivity of this map is obvious by diagram chasing. Using the fundamental theorem of Hermitian K-theory, we deduce that ${}_{-\varepsilon}\bar{U}_n(k_0)\approx {}_{-\varepsilon}\bar{U}_n(k)$. We have now two other exact sequences

$$\begin{array}{ccccccccc}
{}_{-\varepsilon}\bar{U}_{n+1}(k) & \longrightarrow & \bar{K}_{n+1}(k) & \longrightarrow & {}_{-\varepsilon}\bar{L}_{n+1}(k) & \longrightarrow & {}_{-\varepsilon}\bar{U}_n(k) & \longrightarrow & \bar{K}_n(k) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow\approx & & \uparrow \\
{}_{-\varepsilon}\bar{U}_{n+1}(k_0) & \longrightarrow & K_{n+1}(k_0) & \longrightarrow & {}_{-\varepsilon}\bar{L}_{n+1}(k_0) & \longrightarrow & {}_{-\varepsilon}\bar{U}_n(k_0) & \longrightarrow & \bar{K}_n(k_0)
\end{array}$$

and again ${}_{-\varepsilon}\bar{L}_{n+1}(k_0)\approx {}_{-\varepsilon}\bar{L}_{n+1}(k)$ by the same argument as above (take for R the functor ${}_{-\varepsilon}\bar{L}_{n+1}$).

Corollary. *Let us assume* $\operatorname{char}(k)>0$. *Then:*

$$H_*(O(k);\ \mathbf{Z}/q)\approx H_*(Sp(k);\ \mathbf{Z}/q)\approx \mathbf{Z}/q[p_1, p_2, \ldots]$$

for q odd and $\operatorname{char}(k)\nmid q$, *where p_i are homology classes in dimension* $4i$.

$$H_*(O(k);\ \mathbf{Z}/q)=H_*(Sp(k);\ \mathbf{Z}/q)=\mathbb{Z}/q$$

if q is odd and is a power of the characteristic.

$$H_*(O(k);\ \mathbf{Z}/2)\approx \mathbf{Z}/2[w_1, w_2, \ldots]$$

where $w_i\in H_i(O(k);\ \mathbf{Z}/2)$ *if* $\operatorname{char}(k)\neq 2$.

$$H_*(Sp(k);\ \mathbf{Z}/2)\approx \mathbf{Z}/2)[p_1, p_2, \ldots]$$

where $p_i\in H_{4i}(Sp(k);\ \mathbf{Z}/2)$ *if* $\operatorname{char}(k)\neq 2$.

The corollary follows from the main theorem of this appendix and the computation of $H_*(O(k_0))$ and $H_*(Sp(k_0))$ made by E. Friedländer [1]. The work of K. Vogtman [5] on the stability of the homology of the orthogonal group enables us to write explicitly some homology groups $H_i(O_{2n}(k);\ \mathbf{Z}/q)$. More precisely, $H_i(O_{2n}(k);\ \mathbf{Z}/q)\approx H_i(O(k);\ \mathbf{Z}/q)$ if $n\geqq 3i$.

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