

**Examen**  
Durée 3 heures

1. Let  $A, B, T \in \mathcal{B}(H)$  with  $T = T^*$ . We suppose that  $AT = TB$  and  $TA = BT$ .
  - (a) Let  $g \in C(\mathbb{R})$  be an even function. Show that  $g(T)$  commutes with  $A$  and  $B$ .
  - (b) Let  $f \in C(\mathbb{R})$  be an odd function. Show that  $Af(T) = f(T)B$ .

**Answer** — We first show by recursion on  $n$  the property :

$$\mathcal{P}_n : \quad T^{2n}A = AT^{2n}, \quad T^{2n}B = BT^{2n}, \quad T^{2n+1}A = T^{2n+1}B, \quad T^{2n+1}B = AT^{2n+1}.$$

For  $n = 0$ ,  $\mathcal{P}_0$  is a straightforward consequence of the hypotheses  $AT = TB$  and  $TA = BT$ . Now assume that  $\mathcal{P}_n$  has been proved. Then we write :

$$T^{2n+2}A = T^{2n+1}TA = T^{2n+1}BT = AT^{2n+1}T = AT^{2n+2}$$

and we obtain by a similar computation that  $T^{2n+2}B = AT^{2n+2}$ . Similarly we obtain

$$T^{2n+3}A = T^{2n+2}TA = T^{2n+2}BT = BT^{2n+2}T = BT^{2n+3}$$

and also  $T^{2n+3}B = AT^{2n+3}$ . Hence  $\mathcal{P}_{n+1}$  holds. It follows that, for any odd polynomial  $P$ , we have  $AP(T) = P(T)B$  and, for any even polynomial  $Q$ ,  $[A, Q(T)] = [B, Q(T)] = 0$ . Now we use the fact that  $T$  is self-adjoint, so that we can define a continuous functional calculus  $\mathcal{C}^0(\text{Sp}T, \mathbb{C}) \ni f \mapsto f(T) \in \mathcal{B}(H)$ , which satisfies in particular  $\|f(T)\|_{\mathcal{B}(H)} \leq \|f\|_{\mathcal{C}^0(\text{Sp}T)}$ . Moreover by the Stone–Weierstrass theorem the set of restrictions of polynomial functions on  $\text{Sp}T$  is dense in  $\mathcal{C}^0(\text{Sp}T, \mathbb{C})$ . Hence for any, say, even function  $g \in \mathcal{C}^0(\text{Sp}T, \mathbb{C})$ , there exists a sequence of polynomials  $(Q_n)_n$  s.t.  $Q_n|_{\text{Sp}T}$  converges to  $g$ . Then one proves easily that if we set  $\widehat{Q}_n(x) = \frac{1}{2}(Q_n(x) + Q_n(-x))$ , then  $\widehat{Q}_n$  is obviously even and  $\widehat{Q}_n|_{\text{Sp}T}$  converges to  $g$  also in  $\mathcal{C}^0(\text{Sp}T, \mathbb{C})$  and hence  $\widehat{Q}_n(T)$  converges to  $g(T)$  in  $\mathcal{B}(H)$ . Since  $[\widehat{Q}_n(T), A] = [\widehat{Q}_n(T), B] = 0$ , (a) follows. The proof of (b) is similar.

2. Let  $U$  be the bilateral shift on  $\ell^2(\mathbb{Z})$  ie,  $Ue_n = e_{n+1}$  where  $(e_n)_{n \in \mathbb{Z}}$  is the canonical basis. We recall that the spectrum of  $U$  is the unit circle. Let  $k \in \mathbb{Z}$ . Show that  $e_k$  is a cyclic vector for  $U$  and compute the associated spectral measure.

**Answer** — The vector  $e_k$  is cyclic iff the vector subspace spanned by  $(U^n(U^*)^m e_k)_{n,m \in \mathbb{N}}$  is dense in  $\ell^2(\mathbb{Z})$ . Since  $U^* = U^{-1}$  (this can be seen from the very definition of  $U$  or as a consequence of the fact that  $U$  is unitary because its spectrum is a subset of the unit circle), we have  $U^n(U^*)^m = U^{n-m}$ , so that this amounts to show that the vector subspace spanned by  $(U^n e_k)_{n \in \mathbb{Z}}$  is dense in  $\ell^2(\mathbb{Z})$ . But this is clearly the fact, for  $(e_{n+k})_{n \in \mathbb{Z}}$  being

a Hilbert basis of  $\ell^2(\mathbb{Z})$ .

The spectral measure  $d\mu_{e_k}$  on the circle  $S^1 \subset \mathbb{C}$  is defined by the relation

$$\forall f \in \mathcal{C}^0(S^1, \mathbb{C}), \quad \int_{S^1} f(e^{i\theta}) d\mu_{e_k}(\theta) = \langle e_k, f(U)e_k \rangle.$$

By using a Fourier series decomposition  $f(e^{i\theta}) = \sum_{n \in \mathbb{Z}} \widehat{f}_n e^{in\theta}$  and the fact that  $U^* = U^{-1}$ , we obtain  $f(U) = \sum_{n \in \mathbb{Z}} \widehat{f}_n U^n$ . Hence

$$\langle e_k, f(U)e_k \rangle = \sum_{n \in \mathbb{Z}} \langle e_k, \widehat{f}_n U^n e_k \rangle = \sum_{n \in \mathbb{Z}} \widehat{f}_n \langle e_k, e_{n+k} \rangle = \widehat{f}_0 = \frac{1}{2\pi} \int_{S^1} f(e^{i\theta}) d\theta,$$

so that

$$d\mu_{e_k}(\theta) = \frac{1}{2\pi} d\theta.$$

3. Let  $a = (a_n)_{n \in \mathbb{N}}$  be a sequence of complex numbers and  $M_a$  be the following operator on  $\ell^2(\mathbb{N})$

$$\mathcal{D}(M_a) = \{x = (x_n)_n \in \ell^2(\mathbb{N}) : \sum_n |a_n|^2 |x_n|^2 < \infty\}, \quad M_a x = (a_n x_n)_n.$$

- (a) Show that  $M_a$  is densely defined and closed.

**Answer** — The fact that  $\mathcal{D}(M_a)$  is dense in  $\ell^2(\mathbb{N})$  is a consequence of the fact that  $\mathcal{D}(M_a)$  contains all finite sequences and that finite sequences are dense in  $\ell^2(\mathbb{N})$ . We now need to show that  $\text{Gr}M_a$  is a closed subspace of  $\ell^2(\mathbb{N}) \times \ell^2(\mathbb{N})$ . Consider a sequence  $(x^k, M_a x^k)_{k \in \mathbb{N}}$  with values in  $\text{Gr}M_a$  and assume that  $(x^k, M_a x^k)$  converges to some  $(x, y) \in \ell^2(\mathbb{N}) \times \ell^2(\mathbb{N})$ , when  $k \rightarrow +\infty$ . Setting  $x^k = (x_0^k, x_1^k, \dots)$  and  $x = (x_0, x_1, \dots)$ , this implies immediately that, for any fixed  $n \in \mathbb{N}$ ,  $\lim_{k \rightarrow \infty} x_n^k = x_n$  in  $\mathbb{C}$  and hence that  $\lim_{k \rightarrow \infty} a_n x_n^k = a_n x_n$  in  $\mathbb{C}$ . Hence since we also have  $\lim_{k \rightarrow \infty} a_n x_n^k = y_n$ , this implies that  $a_n x_n = y_n$ ,  $\forall n \in \mathbb{N}$ . Hence, since  $y \in \ell^2(\mathbb{N})$ ,  $\sum_{n \in \mathbb{N}} |a_n|^2 |x_n|^2 < +\infty$  and we deduce that  $x \in \mathcal{D}(M_a)$  and  $y = M_a x$ , i.e.  $(x, y) \in \text{Gr}M_a$ . Hence  $\text{Gr}M_a$  is closed.

- (b) Show that  $\text{sp}(M_a) = \overline{\{a_n : n \in \mathbb{N}\}}$ .

**Answer** — We first observe that any value  $a_n$  is an eigenvalue of  $M_a$  for at least the eigenvector  $e_n$ . Hence  $\{a_n : n \in \mathbb{N}\} \subset \text{Sp}_p M_a \subset \text{Sp}M_a$ . Since we know from the course that the spectrum of any operator is closed, this implies that  $\overline{\{a_n : n \in \mathbb{N}\}} \subset \text{Sp}M_a$ . Now let  $b \in \mathbb{C} \setminus \overline{\{a_n : n \in \mathbb{N}\}}$ . Then, since  $\overline{\{a_n : n \in \mathbb{N}\}}$  is closed, there exists some  $\varepsilon > 0$  s.t.  $B(b, \varepsilon) \cap \overline{\{a_n : n \in \mathbb{N}\}} = \emptyset$ . We will then first show that  $M_a - b$  is a bijection between  $\mathcal{D}(M_a)$  and  $\ell^2(\mathbb{N})$ : given some  $y \in \ell^2(\mathbb{N})$  we need to prove that there exists a unique  $x \in \mathcal{D}(M_a)$  s.t.  $(M_a - b)x = y$ . If such an  $x$  would exist, it would be the unique solution of the equation  $(a_n - b)x_n = y_n \iff x_n = y_n / (a_n - b)$ ,  $\forall n \in \mathbb{N}$ . Lastly one needs to prove that  $(b - M_a)^{-1} : \ell^2(\mathbb{N}) \rightarrow \ell^2(\mathbb{N})$  is bounded, a consequence of :

$$\|(b - M_a)^{-1}y\|^2 = \sum_{n \in \mathbb{N}} \frac{|y_n|^2}{|b - a_n|^2} \leq \sum_{n \in \mathbb{N}} \frac{|y_n|^2}{\varepsilon^2} = \frac{\|y\|^2}{\varepsilon^2}.$$

Hence the resolvent set of  $M_a$  contains  $\mathbb{C} \setminus \overline{\{a_n : n \in \mathbb{N}\}}$ , which is equivalent to say that  $\text{Sp}M_a \subset \overline{\{a_n : n \in \mathbb{N}\}}$ .

4. Let  $a = (a_n)_{n \in \mathbb{Z}}$  be a sequence of complex numbers indexed by  $\mathbb{Z}$  and  $T_a$  be the operator defined on  $\ell^2(\mathbb{Z})$  by

$$\mathcal{D}(T_a) = \{x = (x_n)_{n \in \mathbb{Z}} \in \ell^2(\mathbb{Z}) : \sum_{n \in \mathbb{Z}} |a_n|^2 |x_{-n}|^2 < \infty\}, \quad T_a x = (a_n x_{-n})_{n \in \mathbb{Z}}.$$

- (a) Show that  $T_a$  is densely defined and closed.

**Answer** — Use the same method as in question 3 – (a).

- (b) Compute  $T_a^*$ .

**Answer** — We first compute its domain  $\mathcal{D}(T_a^*)$  : this is the set of  $y \in \ell^2(\mathbb{Z})$  s.t. the linear form

$$\begin{aligned} \mathcal{D}(T_a) &\longrightarrow \mathbb{C} \\ x &\longmapsto \langle y, T_a x \rangle \end{aligned}$$

admits a continuous extension on  $\ell^2(\mathbb{Z})$  (which is then unique since  $\mathcal{D}(T_a)$  is dense in  $\ell^2(\mathbb{Z})$ ). By Riesz' theorem this property is equivalent to say that there exists some  $z \in \ell^2(\mathbb{Z})$  s.t.  $\langle y, T_a x \rangle = \langle z, x \rangle, \forall x \in \mathcal{D}(T_a)$ . But

$$\langle y, T_a x \rangle = \sum_{n \in \mathbb{Z}} \overline{y_n} a_n x_{-n} = \sum_{n \in \mathbb{Z}} \overline{y_{-n}} a_{-n} x_n.$$

Hence such a  $z$  exists, its satisfies  $z_n = \overline{a_{-n}} y_{-n}$ . Thus  $y \in \mathcal{D}(T_a^*)$  iff  $\sum_{n \in \mathbb{Z}} |\overline{a_{-n}} y_{-n}|^2 < +\infty$ , i.e.

$$\mathcal{D}(T_a^*) = \{y \in \ell^2(\mathbb{Z}) : \sum_{n \in \mathbb{Z}} |a_n y_n|^2 < +\infty\} \quad \text{and} \quad \forall y \in \mathcal{D}(T_a^*), \quad T_a^* y = (\overline{a_{-n}} y_{-n})_{n \in \mathbb{Z}}.$$

- (c) Find a necessary and sufficient condition on  $a$  for  $T_a$  to be normal.

**Answer** — For any  $x \in \mathcal{D}(T_a T_a^*) := \{x \in \mathcal{D}(T_a^*) : T_a^* x \in \mathcal{D}(T_a)\}$ ,

$$T_a T_a^* x = T_a ((\overline{a_{-n}} x_{-n})_n) = (a_n (\overline{a_n} x_n))_n = (|a_n|^2 x_n)_n,$$

whereas for any  $x \in \mathcal{D}(T_a^* T_a)$ ,

$$T_a^* T_a x = T_a^* ((a_n x_{-n})_n) = (\overline{a_{-n}} (a_{-n} x_n))_n = (|a_{-n}|^2 x_n)_n,$$

Whatever  $\mathcal{D}(T_a T_a^*)$  and  $\mathcal{D}(T_a^* T_a)$  are, they contain the space of finite sequences. Hence a necessary condition for  $T_a$  to be normal is that, for any finite sequence  $x = (x_n)_n$ ,  $|a_n|^2 x_n = |a_{-n}|^2 x_n, \forall n \in \mathbb{Z}$ , which implies that :

$$|a_n| = |a_{-n}|, \quad \forall n \in \mathbb{Z}.$$

Conversely it is clear from the preceding computation that, if the above condition holds, then  $\mathcal{D}(T_a T_a^*) = \mathcal{D}(T_a^* T_a)$  and  $T_a T_a^* = T_a^* T_a$ , i.e.  $T_a$  is normal.

- (d) Find a necessary and sufficient condition on  $a$  for  $T_a$  to be bounded.

**Answer** — The operator  $T_a$  is bounded iff  $\mathcal{D}(T_a) = \ell^2(\mathbb{Z})$  and there exists a constant  $C \in [0, +\infty)$  s.t.  $\forall x \in \ell^2(\mathbb{Z}), \|T_a x\| \leq C \|x\|$ . Testing this condition with  $x = e_n$ , for any  $n \in \mathbb{Z}$  implies that  $|a_n| \leq C$ . Conversely, it is easy to check that the latter condition implies that  $T_a$  is bounded. Hence the necessary and sufficient condition is : the sequence  $(a_n)_{n \in \mathbb{Z}}$  is bounded.

(e) Compute  $\text{sp}(T_a^*T_a)$  and  $\text{sp}(T_aT_a^*)$ .

**Answer** — We have seen in question c) that  $T_a^*T_a$  has the domain  $\{x \in \ell^2(\mathbb{Z}) : \sum_{n \in \mathbb{Z}} |a_{-n}|^4 |x_n|^2 < +\infty\}$  and is defined by  $T_a^*T_a(x) = (|a_{-n}|^2 x_n)_n$ . Hence by a reasoning similar to the question 3 – (b), we deduce that the spectrum of  $T_a^*T_a$  is  $\overline{\{|a_{-n}|^2 : n \in \mathbb{Z}\}} = \overline{\{|a_n|^2 : n \in \mathbb{Z}\}}$ . Similarly  $T_aT_a^*$  has the domain  $\{x \in \ell^2(\mathbb{Z}) : \sum_{n \in \mathbb{Z}} |a_n|^4 |x_n|^2 < +\infty\}$  and is defined by  $T_aT_a^*(x) = (|a_n|^2 x_n)_n$ . Hence the spectrum of  $T_aT_a^*$  is also  $\overline{\{|a_n|^2 : n \in \mathbb{Z}\}}$ .

(f) Find a necessary and sufficient condition on  $a$  for  $T_a^*T_a$  (resp.  $T_aT_a^*$ ) to be compact.

**Answer** — Assume that  $T_a^*T_a$  is compact. Note also that this operator is self-adjoint. Then it follows from the course that the spectrum of  $T_a^*T_a$  is equal to  $\{0\} \cup \Lambda$ , where  $\Lambda$  is a subset of  $\mathbb{C} \setminus \{0\}$  which is at most countable and has no accumulation point, excepted may be 0. In particular, for any  $r > 0$ ,  $\text{Sp}T_a^*T_a \cap (\mathbb{C} \setminus B(0, r^2))$  is finite. Moreover  $\Lambda$  is composed of eigenvalues  $\lambda$  associated with *finite* dimensional vector eigenspaces. However we have seen in the preceding question that  $\text{Sp}T_a^*T_a = \overline{\{|a_n|^2 : n \in \mathbb{Z}\}}$  and the dimension of the eigenspace corresponding to any value  $\lambda \in \Lambda$  is the cardinal of  $\{n \in \mathbb{Z} : |a_n|^2 = \lambda\}$ . We hence deduce that the number of values  $n \in \mathbb{Z}$  s.t.  $|a_n|^2 \geq r^2$  is finite. In particular, if we set  $N(r) := \sup\{|n| \in \mathbb{Z} : |a_n|^2 \geq r^2\}$ , we have  $\forall n \in \mathbb{Z}$  s.t.  $|n| > N(r)$ ,  $|a_n| < r$ . Hence  $\lim_{|n| \rightarrow \infty} a_n = 0$ . Conversely if  $\lim_{|n| \rightarrow \infty} a_n = 0$ , then we define for any  $r > 0$  the operator  $K_r$  on  $\ell^2(\mathbb{Z})$  by

$$K_r x = \sum_{n \in \mathbb{Z}; |a_n|^2 \geq r^2} |a_{-n}|^2 x_n e_n.$$

Then  $\|T_a^*T_a - K_r\| \leq r^2$ , so that  $\lim_{r \rightarrow 0} \|T_a^*T_a - K_r\| = 0$  and each  $K_r$  is a finite rank operator. Hence  $T_a^*T_a$  is compact.

A similar reasoning shows that  $T_aT_a^*$  is compact iff the same condition holds, i.e.

$$\lim_{|n| \rightarrow \infty} a_n = 0.$$

5. Let  $\mathcal{H}$  be a complex separable Hilbert space. Let  $A$  be a non bounded self-adjoint operator such that its domain  $\mathcal{D}(A)$  is dense in  $\mathcal{H}$ . We assume that  $A$  is positive, i.e. that  $\forall \varphi \in \mathcal{D}(A)$ ,  $\langle \varphi, A\varphi \rangle \geq 0$ .

(a) recall the spectral decomposition theorem for  $A$ ;

**Answer** — There exists a topological space  $X$  endowed with a Borelian measure  $\mu$ , an unitary operator  $U : \mathcal{H} \rightarrow L^2(X, \mu; \mathbb{C})$  and a real-valued Borelian<sup>1</sup> function  $f : X \rightarrow \mathbb{C}$  s.t., for any function  $\varphi \in U(\mathcal{D}(A))$ ,

$$(UAU^{-1}\varphi)(x) = f(x)\varphi(x), \quad \text{for } \mu - \text{a.e. } x \in X.$$

In the following we will write  $L^2(X) = L^2(X, \mu; \mathbb{C})$  for shortness. Note that the positivity condition translates here as  $\forall \varphi \in U(\mathcal{D}(A))$ ,  $\int_X f(x)|\varphi(x)|^2 d\mu(x) \geq 0$ . Moreover since  $\mathcal{D}(A)$  is dense in  $\mathcal{H}$ ,  $U(\mathcal{D}(A))$  is dense in  $L^2(X)$  and hence the latter condition implies that  $f$  is real and  $f \geq 0$ ,  $\mu$ -a.e.

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1. not necessarily bounded

- (b) Prove that, for any  $t > 0$ , one can define the operator  $e^{-tA}$  and that this operator is bounded on  $\mathcal{H}$ ;

**Answer** — Since the function  $x \mapsto e^{-tf(x)}$  is Borelian and bounded, it suffices to define  $e^{-tA}$  by  $:\forall \varphi \in L^2(X)$ ,

$$(Ue^{-tA}U^{-1}\varphi)(x) = e^{-tf(x)}\varphi(x), \quad \text{for } \mu - \text{a.e. } x \in X.$$

This operator is bounded and its norm is equal to the supremum of  $x \mapsto e^{-tf(x)}$  over  $X$ .

**Beware** : it is not possible to use the series  $\sum_{n \geq 0} (-tA)^n/n!$ , which does not converge since  $A$  is not bounded.

- (c) Prove that  $\forall t, s > 0$ ,  $e^{-tA}e^{-sA} = e^{-(t+s)A}$ ;

**Answer** — This follows from the definition of  $e^{-tA}$  and the fact that  $e^{-tf(x)}e^{-sf(x)} = e^{-(t+s)f(x)}$ .

- (d) Prove that  $\forall t_0 > 0$ ,  $\lim_{t \rightarrow 0, t > 0} e^{-tA} = e^{-t_0A}$ ;

**Answer** — We need to evaluate the limit of  $\|e^{-tA} - e^{-t_0A}\|_{\mathcal{H}}$ , when  $t \rightarrow t_0$ . Without loss of generality we can assume that  $t_0/2 < t < 3t_0/2$ . For any  $u \in \mathcal{H}$ , let  $\varphi := Uu \in L^2(X)$ , so that  $u = U^{-1}\varphi$ . Then

$$\|(e^{-tA} - e^{-t_0A})u\|_{\mathcal{H}}^2 = \|(e^{-tf} - e^{-t_0f})\varphi\|_{L^2}^2 = \int_X |e^{-tf(x)} - e^{-t_0f(x)}|^2 |\varphi(x)|^2 d\mu(x).$$

By the mean value theorem, for any  $x \in X$ , there exists some  $\tau$  between  $t_0$  and  $t$  s.t.  $|e^{-tf(x)} - e^{-t_0f(x)}| = |-f(x)e^{-\tau f(x)}(t - t_0)| = |f(x)|e^{-t_0f(x)/2}|t - t_0| \leq C|t - t_0|$ , where  $C := \sup_{z > 0} ze^{-t_0z/2}$  (actually  $C = 2/et_0$ ). Hence

$$\|(e^{-tA} - e^{-t_0A})u\|_{\mathcal{H}}^2 \leq C^2(t - t_0)^2 \int_X |\varphi(x)|^2 d\mu(x) = C^2(t - t_0)^2 \|\varphi\|_{L^2}^2 = C^2(t - t_0)^2 \|u\|_{\mathcal{H}}^2$$

and  $\|e^{-tA} - e^{-t_0A}\|_{\mathcal{H}} \leq C|t - t_0|$ , which implies the result.

- (e) Prove that  $\forall u \in D(A)$ ,

$$\lim_{t > 0, t \rightarrow 0} \frac{e^{-tA}u - u}{t} = -Au$$

**Answer** — For any  $u \in D(A)$  and  $\varphi := Uu \in L^2(X)$ , so that  $u = U^{-1}\varphi$ , we have, for  $t > 0$  :

$$\begin{aligned} \left\| \left( \frac{e^{-tA} - 1}{t} + A \right) u \right\|_{\mathcal{H}}^2 &= \left\| \left( \frac{e^{-tf} - 1}{t} + f \right) \varphi \right\|_{L^2}^2 \\ &= \int_X \left| \frac{e^{-tf(x)} - 1}{t} + f(x) \right|^2 |\varphi(x)|^2 d\mu(x) \end{aligned}$$

Using the mean value theorem, we know that, for any  $x \in X$ , there exists some  $\tau \in (0, t)$  s.t.

$$\frac{e^{-tf(x)} - 1}{t} + f(x) = -f(x)e^{-\tau f(x)} + f(x) = f(x)(1 - e^{-\tau f(x)}),$$

and hence

$$\left\| \left( \frac{e^{-tA} - 1}{t} + A \right) u \right\|_{\mathcal{H}}^2 = \int_X |1 - e^{-\tau f(x)}|^2 f(x) |\varphi(x)|^2 d\mu(x).$$

The function  $|1 - e^{-\tau f(x)}|^2 f(x) |\varphi(x)|^2$  is dominated by  $f(x) |\varphi(x)|^2$ , which is  $\mu$ -integrable and it converges pointwise to 0 when  $t$  (and hence  $\tau$ ) converges to 0. Hence we can apply Lebesgue's dominated convergence theorem to conclude that  $\frac{e^{-tA}u - u}{t} - Au$  converges to 0.

- (f) We define  $D(A^n)$  for all  $n \in \mathbb{N}$  recursively by :  $D(A^0) := \mathcal{H}$  and  $\forall n \in \mathbb{N}$ ,  $D(A^{n+1}) := \{u \in D(A^n) \mid A^n u \in D(A)\}$ . Prove that  $\forall t > 0$ ,  $\forall n \in \mathbb{N}$ ,  $\forall u \in \mathcal{H}$ ,  $e^{-tA}u \in D(A^n)$ .

**Answer** — This is just a consequence of the fact that, for any  $n \in \mathbb{N}$ ,  $x \mapsto f(x)^n e^{-tf(x)}$  is bounded on  $X$ , since  $f$  is a nonnegative function.

### Supplementary questions (an application) <sup>2</sup>

- (g) Let  $\Omega \subset \mathbb{R}^n$  be a bounded domain with smooth boundary. Let  $A$  be the non bounded operator on  $L^2(\Omega, \mathbb{C})$  with domain  $D(A) = H^2(\Omega, \mathbb{C}) \cap H_0^1(\Omega, \mathbb{C})$  and such that  $\forall u \in D(A)$ ,  $Au = -\Delta u$ . Show that  $A$  is symmetric.

**Answer** — We must show that  $D(A) \subset D(A^*)$  and that  $A$  and  $A^*$  coincides on  $D(A)$ . In other words it suffices to check that, for any  $u \in D(A)$ , the linear form

$$\begin{aligned} D(A) &\longrightarrow \mathbb{C} \\ v &\longmapsto \langle u, Av \rangle \end{aligned}$$

admits a continuous extension on  $L^2(\Omega)$ , which coincides with  $v \mapsto \langle Au, v \rangle$ . But, since  $u$  and  $v$  belongs to  $H^2(\Omega, \mathbb{C}) \cap H_0^1(\Omega, \mathbb{C})$ , we have

$$\langle u, Av \rangle = - \int_{\Omega} \bar{u} \Delta v = - \int_{\Omega} v \Delta \bar{u} + \int_{\partial\Omega} v \frac{\partial \bar{u}}{\partial n} - \bar{u} \frac{\partial v}{\partial n} = - \int_{\Omega} v \Delta \bar{u} = \langle Au, v \rangle,$$

where we have used the fact that the boundary integral cancels because  $u$  and  $v$  vanish on  $\partial\Omega$  ( $u, v \in H_0^1(\Omega, \mathbb{C})$ ).

In the following we will admit that  $A$  is self-adjoint ;

- (h) Prove that, for any  $f \in L^2(\Omega, \mathbb{C})$ , there exists <sup>3</sup> a  $\mathcal{C}^1$  map

$$\begin{aligned} u : (0, +\infty) &\longrightarrow D(A) \\ t &\longmapsto u(t, \cdot) \end{aligned}$$

such that :

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2. One could use the following formula, valid for  $f, g \in H^2(\Omega, \mathbb{C})$

$$\int_{\partial\Omega} f \frac{\partial g}{\partial n} - g \frac{\partial f}{\partial n} = \int_{\Omega} f \Delta g - g \Delta f,$$

where  $\frac{\partial}{\partial n}$  denotes the derivatives with respect to the exterior normal vector to the boundary  $\partial\Omega$ .

3. Sorry, there were a slight mistake in the subject : the equation was  $\frac{\partial u}{\partial t}(t, x) = \Delta u(t, x)$  (and not =  $-\Delta u(t, x)$ ).

- $u(0, \cdot) = f$ ;
- $\forall t > 0, u(t, \cdot) \in \mathcal{C}^\infty(\Omega)$  and  $u(t, \cdot)$  vanishes on  $\partial\Omega$ ;
- $\forall t > 0, \forall x \in \Omega,$

$$\frac{\partial u}{\partial t}(t, x) = \Delta u(t, x).$$

**Answer** — We just take  $u(t, x) = (e^{t\Delta} f)(x) = (e^{-tA} f)(x), \forall t \geq 0, \forall x \in X,$  with the convention that, for  $t = 0, e^{-0 \cdot A} = 1$  is the identity map.

We first note that,  $\forall t > 0, e^{t\Delta}$  maps  $L^2(\Omega)$  to  $\cap_{n \in \mathbb{N}} H_0^n(\Omega, \mathbb{C}),$  because of the result of question (f) (where  $H_0^n(\Omega, \mathbb{C})$  is the space of functions  $f$  in  $L^2(\Omega, \mathbb{C})$  such that all derivatives of  $f$  of order less than or equal to  $n$  are in  $L^2(\Omega, \mathbb{C})$  and such that  $f$  vanishes on  $\partial\Omega$ ). But this space is contained in the space of  $\mathcal{C}^\infty$  functions on  $\Omega$  which vanish on  $\partial\Omega$ .

The map  $u$  is clearly continuous on  $(0, +\infty),$  as a consequence of question (d). It is moreover continuous at  $t = 0$  : this can be proved by starting the same computation as in question (d), which gives

$$\|e^{-tA}u - u\|_{\mathcal{H}}^2 = \int_X |e^{-tf(x)} - 1|^2 |\varphi(x)|^2 d\mu(x)$$

and by using Lebesgue's dominated convergence theorem as in question (e). However  $u$  is not  $\mathcal{C}^1$  at 0 : we can only prove that, for any  $f \in D(A),$  the map  $t \mapsto e^{t\Delta} f$  is derivable at 0, with a derivative equal to  $\Delta f,$  as a straightforward consequence of question (e). Lastly  $u$  is continuously derivable on  $(0, +\infty),$  since, we can write, for any  $f \in L^2(\Omega, \mathbb{C})$  and for any  $t > 0$  :

$$\frac{u(t+s, \cdot) - u(t, \cdot)}{s} = \frac{e^{(t+s)\Delta} f - e^{t\Delta} f}{s} = \frac{e^{s\Delta} (e^{t\Delta} f) - e^{t\Delta} f}{s}$$

and we remark that  $e^{t\Delta} f \in D(A)$  because of question (f). Hence we can apply question (e) with the function  $e^{t\Delta} f$  to prove that this quotient has a limit when  $s$  goes to 0, which is equal to  $\Delta (e^{t\Delta} f) = \Delta u(t, \cdot).$  Hence  $\frac{\partial u}{\partial t}(t, x) = \Delta u(t, x), \forall t > 0, \forall x \in X.$  The fact that  $t \mapsto \frac{\partial u}{\partial t}(t, \cdot) = \Delta u(t, \cdot)$  is continuous can be seen by writing

$$\Delta u(t, \cdot) = \Delta (e^{t\Delta} f) = e^{(t-\tau)\Delta} (\Delta e^{\tau\Delta} f),$$

for some fixed  $\tau$  such that  $0 < \tau < t$  : question (f) says us again that, for any  $f \in L^2(\Omega, \mathbb{C}), \Delta e^{\tau\Delta} f$  belongs to  $L^2(\Omega, \mathbb{C})$  and question (d) says us that its image by  $e^{(t-\tau)\Delta}$  depends continuously on  $t.$