Correlation functions of 2D Ising model and integrable differential equations*

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1 Introduction

Ising model appeared first in statistical physics as a toy model for describing phase transitions in magnets. However, during the last 40 years it has been used mainly in quantum field theory as a "theoretical laboratory" for the development of new ideas and concepts. The most bright examples of this kind are renormalization group and conformal field theory. The reason for such interest to the Ising model is simple: it is integrable. I will try to explain here what is meant by integrability. My second goal is to explain how in this model of *quantum* field theory *classical* integrable differential equations arise. To achieve this goal, we have to pass a long way. Therefore, the results will be stated in rather declarative manner. I will simply try to mention mathematical constructions which appear at various stages of the theory.

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2 The Model

The Ising model is defined on the square lattice with $M \times N$ sites (Fig.1). In each site we place a

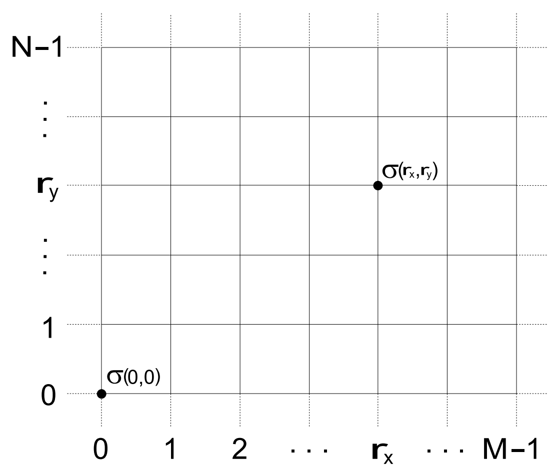

Figure 1: Ising lattice and numeration of spins

spin variable which takes on the values $\sigma(i,j) = \pm 1$. The hamiltonian of the model

$$H[\sigma] = -J \sum_{i=0}^{M-1} \sum_{j=0}^{N-1} \{\sigma(i,j)\sigma(i+1,j) + \sigma(i,j)\sigma(i,j+1)\}$$

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depends on the interactions between nearest neighbors only. One usually deals with the case of periodic boundary conditions

$$\sigma(M,j) = \sigma(0,j) \qquad \sigma(i,N) = \sigma(i,0)$$

i. e. the lattice is considered to be wrapped on two-dimensional torus.

According to the main axiom of statistical physics, the probability of a spin configuration is given by

$$P_{[\sigma]} = \frac{1}{Z}\, e^{-\frac{H[\sigma]}{T}}.$$

The sum of probabilities of all states should be equal to 1. This condition allows us to determine the coefficient Z:

$$Z = \sum_{[\sigma]} e^{-\frac{H[\sigma]}{T}} \tag{1}$$

where the sum is taken over all spin configurations. This coefficient is called *partition function* and plays very important role in the whole theory. Almost all thermodynamical quantities (such as pressure, specific heat and so on) can be determined through the logarythmic derivatives of Z. The second main quantity of interest is the two-point correlation function, which represents the averaged product of two spins in different positions:

$$\langle\sigma(0,0)\sigma(i,j)\rangle = \frac{1}{Z}\sum_{[\sigma]} \sigma(0,0)\sigma(i,j)\, e^{-\frac{H[\sigma]}{T}} \tag{2}$$

This is a good point to outline deep connection between statistical physics and quantum field theory. To this end, let us consider the simplest action for interacting fields – so-called euclidean φ^4 action

$$S[\varphi] = \int d^D x \left\{ \frac{1}{2}\,\partial_\mu\varphi\partial^\mu\varphi + \frac{1}{2}\,m^2\varphi^2 + g\varphi^4 \right\}$$

Thermodynamics of the field system is again described by "partition function"

$$W = \int \mathcal{D}\varphi\; e^{-S[\varphi]}$$

while two-point Green functions are given by

$$\langle\varphi(x_1)\varphi(x_2)\rangle = \frac{1}{W}\int \mathcal{D}\varphi\; e^{-S[\varphi]}\varphi(x_1)\varphi(x_2)$$

After comparison we see that the QFT action corresponds to hamiltonian of statistical physics, and the Green functions (which are directly related to observable quantities like masses of particles, cross-sections etc.) correspond to correlation functions. This analogy allows to transfer the results obtained in statistical physics to QFT and vice versa.

3 Partition function and spontaneous magnetization

It should be noted that in thermodynamics not partition function but its asymptotics plays a role. More presicely, the *free energy* per site

$$f = \lim_{N,M\to\infty} \frac{\ln Z}{MN}$$

must be some finite number. The free energy of the Ising model was calculated by Onsager in 1944 and the result is [1]

$$f = \ln 4s + \frac{1}{8\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} dp\, dq\; \ln\left(s + s^{-1} - \cos p - \cos q\right),$$

where $s = \sinh\frac{2J}{T}$. This result demonstrates the presence of phase transition of the second kind (here it is interpreted as the transition between ferro- and paramagnetic phase). Mathematically it means that the specific heat $c \sim \frac{\partial f}{\partial T}$ has a singularity at some point. In our case it is the point $s = 1$ where the specific heat has logarithmic divergence $c \sim \ln |1 - s|$.

Later the method of Onsager was generalized to several other models. Now it is known under the name of *Yang-Baxter equation*. Its main disadvantage is that it does not allow to find the correlation functions. Though people have found the way to solve this problem, it works only for the Ising model. This is why it is so important for quantum field theory: it is a single model where we know exact answer for correlators.

Let us consider the quantity

$$\mathfrak{M}^2 = \lim_{r_x, r_y \to \infty} \langle \sigma(0,0)\sigma(r_x, r_y)\rangle$$

$\mathfrak{M}$ is called spontaneous magnetization. In the Ising model it has nonvanishing value [2] in the ferromagnetic phase, while in the paramagnetic phase it is equal to zero:

$$\mathfrak{M}^2 = \begin{cases} (1 - s^{-4})^{1/4}, & s > 1 \\ 0, & s < 1 \end{cases}$$

When s tends to its critical value $s \to s_c = 1$, the above expression (in the ferromagnetic phase) has the behaviour $\mathfrak{M}^2 \sim (s - s_c)^{1/4}$. This appears to be a general property of all thermodynamic quantities: when parameters of the system (temperature, pressure, etc.) are close to critical, each quantity Q (specific heat, magnetization and so on) behaves like

$$Q \sim (s - s_c)^{\alpha}$$

The numbers α are called *critical indices*. They are very important for the following reason: physicists believe in the hypothesis of universality, which says that the systems with the same symmetry must have the same behaviour in the vicinity of critical point. In particular, critical indices of such systems must coincide.

4 Correlation function

As we have seen, the Ising model represents the greatest interest near its critical point, since there it describes the whole $\mathbb{Z}_2$–universality class. Following this arguments, let us consider Ising correlator $\langle \sigma(0,0)\sigma(r_x, r_y)\rangle$ in the scaling limit. It implies that

$$|s - 1| \to 0, \qquad r_x, r_y \to \infty$$

whereas scaled coordinates x, y

$$|s - 1| r_x \to \sqrt{2}\, x, \qquad |s - 1| r_y \to \sqrt{2}\, y$$

are kept finite. This limit represents continuum version of our initial lattice model and is of extreme interest to quantum field theory. Scaled correlation function was found by Wu et al in 1976 [3] and is given by the following formulae:

$$\langle \sigma(0,0)\sigma(r_x, r_y)\rangle = \begin{cases} \mathfrak{M}^2 \sum\limits_{n=0}^{\infty} g_{2n}(x,y) = \mathfrak{M}^2 \tau_-(x,y), & s > 1, \\ \mathfrak{M}^2 \sum\limits_{n=0}^{\infty} g_{2n+1}(x,y) = \mathfrak{M}^2 \tau_+(x,y), & s < 1. \end{cases} \tag{3}$$

The functions $g_n(x,y)$ depend on single variable $r = \sqrt{x^2 + y^2}$ and are represented as n-fold integrals:

$$g_n(x,y) = \frac{1}{n!} \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \prod_{k=1}^{n} \frac{dq_k\, e^{-r\sqrt{1+q_k^2}}}{2\pi\sqrt{1+q_k^2}} \prod_{1 \le i < j \le n} \left(\frac{q_i - q_j}{\sqrt{1+q_i^2} + \sqrt{1+q_j^2}} \right)^2 \tag{4}$$

It should be noted that the correlator is given by different expressions in paramagnetic and ferromagnetic phase[1]. We also see that the answer is not very simple. However, it turned out that the functions $\tau_-(r)$ and $\tau_+(r)$ satisfy some ordinary integrable differential equations.

[1]Corresponding quantum field theory has nontrivial particle content only in the paramagnetic phase, where momentum representation of the correlation function has simple pole.

For example, when we consider the function

$$\eta(\theta) = \frac{\tau_-(2\theta) - \tau_+(2\theta)}{\tau_-(2\theta) + \tau_+(2\theta)},$$

it can be shown to satisfy the equation [3]

$$\eta'' = \frac{(\eta')^2}{\eta} - \frac{\eta'}{\theta} + \eta^3 - \eta^{-1} \tag{5}$$

Actually, this is a special case of the so-called Painlevé III equation. Even more striking fact is that, if we denote

$$\zeta(t) = t\frac{d \ln \tau_\pm(t)}{dt},$$

one obtains

$$(t\zeta'')^2 = 4(t\zeta' - \zeta)^2 - 4(\zeta')^2(t\zeta' - \zeta) + (\zeta')^2 \tag{6}$$

the equation for the τ-function of Painlevé V in both cases [4].

Painlevé equations have their mathematical origins in the classification of differential equations by singularities of their solutions. Consider an equation of the second order

$$\frac{d^2 w}{dz^2} = F\left(\frac{dw}{dz}, w, z\right), \tag{7}$$

where the function F is rational in w and w' and locally analytic in z. If solutions of (7) have no movable branching points and essential singularities, the equation is linearizable or can be reduced to one of six special nonlinear differential equations. This result and corresponding equations were found by Painlevé and for this reason they bear his name.

Recall that till now we have been treating the case $M, N \to \infty$, i. e. our torus was degenerated into plane. Natural question arises: what shall we obtain keeping one or two of the dimensions finite? For the partition function the answer was found by Onsager. For the correlator, however, no other results were known until 2001, when the formfactor expansion for the correlation function on the cylinder was found [5]. The result has the form similar to (3)–(4), but we will not write it explicitly. The only thing one should note is that the correlators $\tau_\pm(x, y)$ are no longer rotationally invariant.

The question we want to address now – do there exist some analogs of differential equations for scaled correlators in the case of the cylinder? To answer, let us introduce complex coordinates

$$z = \frac{x + iy}{2} \qquad \bar{z} = \frac{x - iy}{2}$$

If we consider the function φ, similar to the logarythm of our previous function η

$$\varphi = \ln \frac{\tau_- + \tau_+}{\tau_- - \tau_+},$$

instead of Painlevé III we obtain celebrated sinh-Gordon equation [6]

$$\varphi_{z\bar{z}} = \frac{1}{2} \sinh 2\varphi \tag{8}$$

Moreover, both functions $\tau_\pm$ satisfy the same three determinant equations:

$$\begin{vmatrix} \tau & \tau_z & \tau_{zz} \\ \tau_{\bar{z}} & \tau_{z\bar{z}} & \tau_{zz\bar{z}} \\ \tau_{\bar{z}\bar{z}} & \tau_{z\bar{z}\bar{z}} & \tau_{zz\bar{z}\bar{z}} \end{vmatrix} = \begin{vmatrix} \tau_{z\bar{z}} & \tau_z & \tau_{zz} \\ \tau_{\bar{z}} & \tau & \tau_z \\ \tau_{\bar{z}\bar{z}} & \tau_{\bar{z}} & \tau_{z\bar{z}} \end{vmatrix}, \tag{9}$$

$$\begin{vmatrix} \tau & \tau_z & \tau_{\bar{z}} & \tau_{z\bar{z}} \\ \tau_z & \tau_{zz} & \tau_{z\bar{z}} & \tau_{zz\bar{z}} \\ \tau_{\bar{z}} & \tau_{z\bar{z}} & \tau_{\bar{z}\bar{z}} & \tau_{z\bar{z}\bar{z}} \\ \tau_{zz} & \tau_{zzz} & \tau_{zz\bar{z}} & \tau_{zzz\bar{z}} \end{vmatrix} = \begin{vmatrix} \tau & \tau_z & \tau_{\bar{z}} & \tau_{z\bar{z}} \\ \tau_z & \tau_{zz} & \tau_{z\bar{z}} & \tau_{zz\bar{z}} \\ \tau_{\bar{z}} & \tau_{z\bar{z}} & \tau_{\bar{z}\bar{z}} & \tau_{z\bar{z}\bar{z}} \\ \tau_{\bar{z}\bar{z}} & \tau_{z\bar{z}\bar{z}} & \tau_{\bar{z}\bar{z}\bar{z}} & \tau_{z\bar{z}\bar{z}\bar{z}} \end{vmatrix} = 0. \tag{10}$$

5 Discussion

In the last 20 years there has been a great deal of progress in the study of critical phenomena by means of conformal field theory. To pass to conformal limit of the Ising model, one should rescale $x \to mx$, $y \to my$ and formally take $m \to 0$. While the equations (10) remain unchanged after this transformation, the relation (9) reduces to

$$\begin{vmatrix} \tau & \tau_z & \tau_{zz} \\ \tau_{\bar{z}} & \tau_{z\bar{z}} & \tau_{zz\bar{z}} \\ \tau_{\bar{z}\bar{z}} & \tau_{z\bar{z}\bar{z}} & \tau_{zz\bar{z}\bar{z}} \end{vmatrix} = 0. \tag{11}$$

The last equation is of fourth order. Therefore, if we find its solution, depending on four arbitrary functions, it will be general (at least in Cauchy-Kowalevskaya sense). In fact, one can easily find such a solution

$$\tau(z, \bar{z}) = f(z)g(\bar{z}) + h(z)s(\bar{z}).$$

If we substitute this expression into the equations (10), we will obtain

$$f(z) = \text{const} \cdot h(z), \quad g(\bar{z}) = \text{const} \cdot s(\bar{z}).$$

Though there also exist some special solutions like

$$\tau(z, \bar{z}) = e^{\alpha z + \gamma \bar{z}}(f(z) + g(\bar{z})),$$

they do not correspond to the correlation functions of the Ising model. Thus as the final answer one obtains

$$\tau(z, \bar{z}, \beta) = f(z, \beta)g(\bar{z}, \beta).$$

Now let us consider more general situation. In conformal field theory, two-point correlation function of arbitrary primary field ϕ on a cylinder has the form (cf [7])

$$\langle \phi(0,0)\phi(z,\bar{z})\rangle \sim \left(\sinh \frac{2\pi z}{\beta}\right)^{-2h} \left(\sinh \frac{2\pi \bar{z}}{\beta}\right)^{-2\bar{h}} \tag{12}$$

Conformal dimensions of the Ising spin operator are $h = \bar{h} = \frac{1}{4}$. However, the expression (12) is a product of holomorphic and antiholomorphic function for arbitrary h, $\bar{h}$. Hence the equations (10) and (11) are in fact certain manifestations of conformal symmetry and are not Ising-specific. We can try to extend this idea to the case $m \neq 0$ and conclude with the following

Conjecture. Correlators of massive fields, corresponding to conformal primary fields, should satisfy differential equations (9) and (10).

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