

## Motivations and setting

What do we mean by ‘first integrals’  
A non linear Klein–Gordon equation

## The general case

A classical analogue of Fock spaces  
A generating function  
How to do it ?

## Perspectives

The multisymplectic formalism  
References  
Some projects

# Covariant Hamiltonian description of relativistic fields and the construction of observable quantities

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Conformal Geometry: invariant theory and the variational method

Roscoff, June 30– July 4, 2008

# The question

- ▶ Find **first integrals** for solutions  $u : \mathbb{R}^n \longrightarrow \mathbb{R}$  of

$$\frac{\partial^2 u}{\partial t^2} - \Delta u + m^2 u + W(u, du) = 0.$$

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- ▶ Motivation: *the quantization of fields*. The space

$$\mathcal{E}_W := \{u : \mathbb{R}^n \longrightarrow \mathbb{R} \mid \square u + m^2 u + W(u, du) = 0\}$$

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- ▶ Using each of these first integrals one defines *covariantly* an *observable* functional on  $\mathcal{E}_W$

# The linear Klein–Gordon equation

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- ▶ Fix  $\varphi \in \mathcal{E}_0$  and  $t \in \mathbb{R}$ . We define a functional on the set of maps  $u : \mathbb{R}^n \longrightarrow \mathbb{R}$  by

$$I_t^\varphi[u]_t := \int_{\mathbb{R}^{n-1}} \left( u \frac{\partial \varphi}{\partial t} - \frac{\partial u}{\partial t} \varphi \right) \Big|_{x^0=t} d\vec{x} = \int_t \overleftrightarrow{\frac{\partial}{\partial t}} \varphi.$$

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- ▶ Claim: if  $u \in \mathcal{E}_0$  then  $I_t^\varphi[u]_t$  is time independant (this can be interpreted using Noether's theorem)
- ▶ However if  $\varphi \in \mathcal{E}_0$  and  $u \in \mathcal{E}_W$  with  $W \neq 0$ , then  $I_t^\varphi[u]_t \neq I_s^\varphi[u]_s$  if  $t \neq s$  in general.

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We assume  $\varphi \in \mathcal{E}_0$  and  $u \in \mathcal{E}_W$  with  $W(u) = u^2$ .

► Add a correction to  $I_t^\varphi[u]_t$ :  $\mathcal{F}_{t,2}^\varphi[u]_t \otimes [u]_t :=$

$$- \int_t \int_t u(y_1) u(y_2) \overleftrightarrow{\partial}_{y_1} \overleftrightarrow{\partial}_{y_2} \int_0^t G(y_1 - z) G(y_2 - z) \int_0 G(z - x) \overleftrightarrow{\partial}_x \varphi(x),$$

where  $G$  is the solution of:

$$\square G + m^2 G = 0, \quad G|_{x^0=0} = 0 \quad \text{and} \quad \frac{\partial G}{\partial x^0} |_{x^0=0} = \delta_0.$$

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- ▶ For shortness we write:

$$\mathcal{F}_{t,2}^\varphi[u]_t \otimes [u]_t := - \int_t \int_t u_1 u_2 \overleftrightarrow{\partial}_1 \overleftrightarrow{\partial}_2 \int_0^t G_{1z} G_{2z} \int_0 G_{zx} \overleftrightarrow{\partial}_x \varphi$$

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- ▶ Then if we denote  $\mathcal{F}_{t,1}^\varphi[u]_t := I_t^\varphi[u]_t$ , the functional

$$\mathcal{F}_{t,1}^\varphi[u]_t + \mathcal{F}_{t,2}^\varphi[u]_t \otimes [u]_t$$

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- ▶ Idea: build a series

$$\mathcal{F}_t^\varphi[u]_t = \mathcal{F}_{t,1}^\varphi[u]_t^{\otimes 1} + \mathcal{F}_{t,2}^\varphi[u]_t^{\otimes 2} + \mathcal{F}_{t,3}^\varphi[u]_t^{\otimes 3} + \dots$$

to obtain a conserved quantity.

## The result by D. HARRIVEL (Annales IHP, 2006)

- ▶ For any  $\varphi \in \mathcal{E}_0$ , there exists a family (depending on  $t$ ) of formal series  $\mathcal{F}_t^\varphi[u]_t$  such that  $\mathcal{F}_0^\varphi[u]_0 = \mathcal{F}_{0,1}^\varphi[u]_0$  (coincidence at time 0) and  $\mathcal{F}_t^\varphi[u]_t = \mathcal{F}_0^\varphi[u]_0$  (conservation), i.e.

$$\forall t, \quad \sum_{p=1}^{\infty} \mathcal{F}_{t,p}^\varphi[u]_t^{\otimes p} = \mathcal{F}_{0,1}^\varphi[u]_0 = I_0^\varphi[u]_0,$$

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- ▶ If  $s > (n-1)/2$ ,

$$\begin{aligned} \varphi &\in \mathcal{C}^0([0, T], H^{-s}(\mathbb{R}^{n-1})) \cap \mathcal{C}^1([0, T], H^{-s-1}(\mathbb{R}^{n-1})), \\ u &\in \mathcal{C}^0([0, T], H^{s+2}(\mathbb{R}^{n-1})) \cap \mathcal{C}^1([0, T], H^{s+1}(\mathbb{R}^{n-1})) \text{ and} \\ &T, \|u\| \text{ are small enough,} \end{aligned}$$

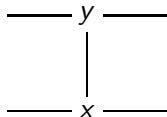
then the series  $\mathcal{F}_t^\varphi[u]_t$  converges for  $t \in [0, T]$ .

## Construction of the series

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- ▶ and the second term is:

$$\mathcal{F}_{t,2}^\varphi[u]_t^{\otimes 2} = - \begin{array}{c} \text{--- } y_1 \text{ ---} \quad \text{--- } y_2 \text{ ---} \\ \diagdown \quad \diagup \\ \quad z \\ | \\ \text{--- } x \text{ ---} \end{array}$$

## The general case

# A classical analogue of Fock spaces

We consider a formal algebra  $\mathcal{A}$  spanned by the symbols  $(\phi_{\triangleright}(x), \phi_{\triangleleft}(y))_{x,y \in \mathbb{R}^n}$  and acting on two vector spaces  $\mathbb{F}$  and  $\mathbb{F}^*$  in duality, such that

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# A generating function...

- ▶ For any  $|f\rangle \in \mathbb{F}$ ,  $t \in \mathbb{R}$ , set

$$\mathcal{F}_t[u]_t := \langle [u]_t | T \exp \left( - \int_0^t W(\phi_{\triangleright}(z), d\phi_{\triangleright}(z)) \phi_{\triangleleft}(z) \right) | f \rangle,$$

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- ▶ and an example for  $|f\rangle$  is:

$$|f\rangle = |f_{\varphi}\rangle := \int_{x^0=0} \phi_{\triangleright}(x) \overleftrightarrow{\partial}_x \varphi(x) d\vec{x} |0\rangle,$$

for some function  $\varphi$ . Then  $\mathcal{F}_0[u]_0 = I_0^{\varphi}[u]_0$ .

... which produces a first integral

**Theorem** (D. HARRIVEL, F.H., arXiv:0704.2674)

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then the series  $\mathcal{F}_t^\varphi[u]_t$  converges for  $t \in [0, T]$  (and is really conserved).

## An example

- ▶ If  $\varphi|_{t=0} = 0$  and  $\frac{\partial\varphi}{\partial t}|_{t=0} = \delta_{\vec{x}}$ , then  $I_0^\varphi[u] = u(0, \vec{x})$ ;

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$$u(x) = \langle [u]_t | T \exp \left( - \int_0^t W(\phi_{\triangleright}(z), d\phi_{\triangleright}(z)) \phi_{\triangleleft}(z) \right) \phi_{\triangleright}(x) | 0 \rangle,$$

for  $x = (0, \vec{x})$ .

## How to do it ?

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 $[x \longmapsto G(y - x)] \in \mathcal{E}_0$
- ▶ Remark: then actually  $\langle [u]_t |$  is the Dirac mass at the only map  $\varphi_{[u]_t} \in \mathcal{E}_0$  which has the same Cauchy data as  $u$  at time  $t$ .

## A first difficulty

- ▶ The 'creation' operators  $\phi_{\triangleright}(x)$  and the 'annihilation' operators  $\phi_{\triangleleft}(y)$  cannot be defined simultaneously;
  - ▶ building  $\phi_{\triangleright}(x)$  requires at least that the fields in  $\mathcal{E}_0$  should be continuous on  $\mathbb{R}^n$
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- ▶ then the  $\phi_{\triangleright}(x)$ 's are well defined and the  $\phi_{\triangleleft}(y)$ 's are not defined but

$$\int_t u \overleftrightarrow{\partial} \phi_{\triangleleft}, \quad - \int_t (\square u + m^2 u) \phi_{\triangleleft}, \quad \int_t W(\phi_{\triangleright}, d\phi_{\triangleright}) \phi_{\triangleleft}$$

are well defined.

## A second difficulty

- ▶ The ‘annihilation operators’  $\int_t u \overleftrightarrow{\partial} \phi_{\triangleleft}$ ,  $\int_t W(\phi_{\triangleright}, d\phi_{\triangleright})\phi_{\triangleleft}$ , etc. are not bounded from  $\mathbb{F}$  to itself;

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$$\left( \mathbb{F}_r^{(k)} \right)_{k \in \mathbb{N}; r \in ]0, \infty[}$$

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- ▶ Then the annihilation operators  $\mathbb{D}$  are bounded from  $\mathbb{F}_r^{(1)}$  to  $\mathbb{F}_r$ ;
- ▶ As a consequence the (time ordered) exponential of an annihilation operator is bounded from  $\mathbb{F}_r$  to some  $\mathbb{F}_{e^{-tX}(r)}$ , where  $X(r) := \|\mathbb{D}\|_{\mathbb{F}_r^{(1)} \rightarrow \mathbb{F}_r} \frac{dr}{dr}$ .

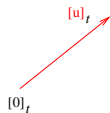
# Structure of the proof I

$\langle 0|$

$|0\rangle_t$

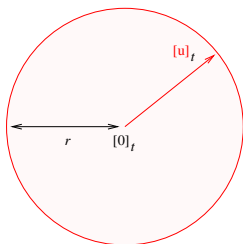
## Structure of the proof II

$$\langle 0 | \exp \left( \int_t u(y) \overleftrightarrow{\partial}_y \phi_{\triangleleft}(y) \right)$$



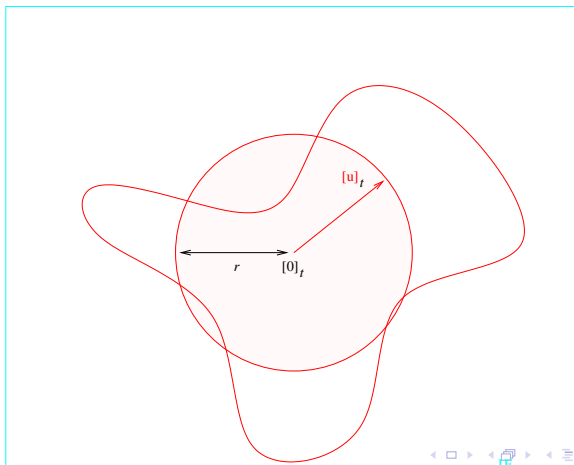
## Structure of the proof III

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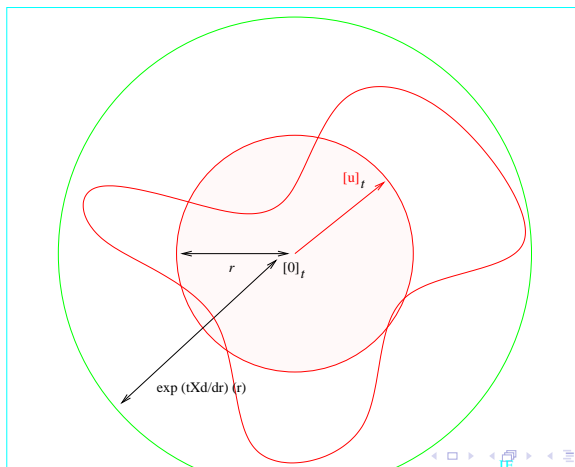
## Structure of the proof IV

$$\langle 0 | \exp \left( \int_t u(y) \overleftrightarrow{\partial}_y \phi_{\triangleleft}(y) \right) T \exp \left( - \int_0^t W(\phi_{\triangleright}(z), d\phi_{\triangleright}(z)) \phi_{\triangleleft}(z) \right)$$



## Structure of the proof V

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## Where does it come from ?

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- ▶ This can be done by using the **multisymplectic formalism**...

# Multisymplectic manifolds

An  $n$ -multisymplectic manifold is a manifold  $\mathcal{M}$  endowed with an  $(n + 1)$ -form  $\omega$  which satisfies:

- ▶  $\omega$  is *closed*, i.e.  $d\omega = 0$

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- ▶  $\omega$  is *non degenerate*, i.e. for any tangent vector field  $\xi$  on  $\mathcal{M}$ ,  $\xi \lrcorner \omega = 0$  if and only if  $\xi = 0$

(Remark: in general the dimension of  $\mathcal{M}$  can be anything.)

An example is:

$\mathcal{M} = \Lambda^n T^*(\mathbb{R}^n \times \mathbb{R})$  with coordinates:

▶  $x = (x^0, x^1, \dots, x^{n-1})$  on  $\mathbb{R}^n$ ,

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- ▶ let  $\beta := dx^0 \wedge \dots \wedge dx^{n-1}$  and  $\beta_\mu := \frac{\partial}{\partial x^\mu} \lrcorner \beta$ , then a basis of  $\Lambda^n T_{(x,y)}(\mathbb{R}^n \times \mathbb{R})$  is  $(\beta, dy \wedge \beta_0, \dots, dy \wedge \beta_{n-1})$  and we denote the coordinates on  $\Lambda^n T_{(x,y)}(\mathbb{R}^n \times \mathbb{R})$  in this basis by  $(e, p^0, \dots, p^{n-1})$

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- ▶ then the multisymplectic form reads

$$\omega = de \wedge \beta + dp^0 \wedge dy \wedge \beta_0 + \dots + dp^{n-1} \wedge dy \wedge \beta_{n-1}.$$

# The dynamics

A classical solution of the fields equation is pictured by an  $n$ -dimensional submanifold  $\Gamma \subset \mathcal{M}$  which is a solution of the *Hamilton equations*:



$$\Lambda^n T\Gamma \lrcorner \omega = 0 \text{ mod } d\mathcal{H},$$

where  $\mathcal{H} : \mathcal{M} \longrightarrow \mathbb{R}$  is a given function ('Hamiltonian'). We call a **Hamiltonian  $n$ -curve** any solution of this equation and we denote by  $\mathcal{E}$  the set of all Hamiltonian  $n$ -curves.

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- ▶ For instance in  $\mathcal{M} = \Lambda^n T^*(\mathbb{R}^n \times \mathbb{R})$  the Hamilton equations with

$$\mathcal{H}(x, y, e, p) = e + \frac{1}{2} \eta_{\mu\nu} p^\mu p^\nu + \frac{1}{2} m^2 y^2 + V(u)$$

correspond to  $\square u + m^2 u + V'(u) = 0$ .

# Observable functionals I

An observable functional is built from

- ▶ an *observable*  $(n - 1)$ -form which is a  $(n - 1)$ -form  $F$  on  $\mathcal{M}$  such that there exists a tangent vector field  $\xi_F$  such that

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- ▶ Then one set

$$\int_{\Sigma} F : \begin{array}{l} \mathcal{E} \longrightarrow \mathbb{R} \\ \Gamma \longmapsto \int_{\Gamma \cap \Sigma} F \end{array}$$

# Observable functionals II

Furthermore we can define a ‘Poisson’ bracket between two  $(n - 1)$ -forms  $F$  and  $G$  by:

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- ▶ is a Poisson bracket, which coincides with the standard Poisson bracket of fields theory.

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- ▶ (this relation expresses that the equation is invariant by  $\xi_F$  and the independence on  $\Sigma$  is then a version of Noether's theorem)

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If  $\mathcal{M} = \Lambda^n T^*(\mathbb{R}^n \times \mathbb{R})$  and  $\mathcal{H} = e + \frac{1}{2}\eta_{\mu\nu} p^\mu p^\nu + \frac{1}{2}m^2 y^2 + V(u)$ ,  
 there are **very few solutions** to the symmetry condition  
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- ▶ Then

$$\text{(for } \Sigma_0 := \{t = 0\}), \quad \int_{\Sigma_0} F_\varphi = \int_0 \overset{\leftarrow}{u} \frac{\partial}{\partial t} \varphi$$

## Connection with our main result

(in progress)

If  $V \neq 0$ , we found the invariant functional:

$$\langle 0 | \exp \left( \int_{\Sigma \cap \Gamma} F_{\triangleleft} \right) T \exp \left( - \int_{\Sigma_0}^{\Sigma} W_{\triangleright} \phi_{\triangleleft} \right) \int_{\Sigma_0 \cap \Gamma_{\triangleright}} F_{\varphi} | 0 \rangle,$$

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- ▶ The corresponding 'symmetry' can be derived.

## References on the use of series

Series expansions of solution to nonlinear ODE's:

- ▶ K.T. Chen (1957), Chen–Fliess (M. Fliess, 1981), W. Magnus (1954) (many applications to control theory)
- ▶ the Butcher series which explain the structure of Runge–Kutta methods: J.C. Butcher, E. Hairer and G. Wanner (1974).  
 Relation with the Hopf algebra of D. Kreimer (1998) on the renormalization theory discovered by C. Brouder (2000).

Concerning nonlinear PDE's:

- ▶ known to physicists since J. Schwinger and R. Feynman
- ▶ few precise references: M. Dütsch, K. Fredenhagen (2003)
- ▶ first rigorous result (i.e. with a proof of convergence of the series) by D. Harrivel.

## References on the multisymplectic formalism

Series expansions of solution to nonlinear ODE's:

- ▶ C. Carathéodory (1929), T. De Donder, H. Weyl (1935), T. Lepage (1936)
- ▶ P. Dedecker (1953)
- ▶ J. Kijowski, H. Goldschmidt–S. Sternberg, K. Gawędski (1972–73), J. Kijowski–W. Szczyrba (1976), J. Kijowski–W.M. Tulczyjew (1979), E. Binz–J. Śniatycki–H. Fisher (1989)
- ▶ M.J. Gotay–J. Isenberg–J.E. Marsden–R. Montgomery–J. Śniatycki–P.B. Yasskin (1998)
- ▶ I. V. Kanatchikov (1997-98), S. Hrabak (1999)
- ▶ F. Hélein–J. Kouneiher (2004)

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- ▶ Quantize...