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COMPLEMENT TO: HAMILTONIAN STATIONARY TORI IN THE COMPLEX PROJECTIVE PLANE

FRÉDÉRIC HÉLEIN and PASCAL ROMON

We present here an extended version of paragraphs 2.1. and 2.2 of [1].

0.1. *The space* $\mathbb{C}P^n$

The complex projective space $\mathbb{C}P^n$ can be identified with the quotient manifold $\left(\mathbb{C}^{n+1} \setminus \{0\}\right)/\mathbb{C}^\star$. We denote by $\pi : \mathbb{C}^{n+1} \setminus \{0\} \longrightarrow \mathbb{C}P^n$ the canonical projection (a.k.a. Hopf fibration): if $z = (z^1, \cdots, z^{n+1}) \in \mathbb{C}^{n+1} \setminus \{0\}$, $\pi(z) = [z]$ is the equivalence class of z modulo $\mathbb{C}^\star$ (the complex punctured line spanned by z) and is also written in homogeneous coordinates $[z^1 : \cdots : z^{n+1}]$.

The tangent bundle — If $[z] \in \mathbb{C}P^n$ a tangent vector $\xi \in T_{[z]}\mathbb{C}P^n$ can be represented by a $\mathbb{C}$-linear map $\ell : \mathbb{C}z \longrightarrow \mathbb{C}^{n+1}$, $\lambda z \longmapsto \ell(\lambda z)$ such that

$$\forall \lambda \in \mathbb{C}, \quad d\pi_{\lambda z}\left(\ell(\lambda z)\right) = \xi. \tag{0.1}$$

Note that ℓ is not unique and given some $\xi \in T_{[z]}\mathbb{C}P^n$ if ℓ and $\tilde{\ell}$ satisfy condition (0.1) with ξ, then there exists $k \in \mathbb{C}$ s.t. $\tilde{\ell}(\lambda z) = \ell(\lambda z) + k\lambda z$. However by using the standard Hermitian product $\langle \cdot, \cdot \rangle_{\mathbb{C}^{n+1}}$ on $\mathbb{C}^{n+1}$ we can select an unique ℓ^0 which satisfies (0.1) and such that

$$\langle \ell^0(z), z \rangle_{\mathbb{C}^{n+1}} = 0. \tag{0.2}$$

Indeed it suffices to start from any ℓ satisfying (0.1) and to set $\ell^0(z) = \ell(z) + kz$. Then condition (0.2) holds if and only if $k = -\frac{\langle \ell(z), z \rangle_{\mathbb{C}^{n+1}}}{|z|^2_{\mathbb{C}^{n+1}}}$.

The Hermitian metric — We can define a Hermitian product on $\mathbb{C}P^n$ as follows. If $\xi_1, \xi_2 \in T_{[z]}\mathbb{C}P^n$ we consider the linear maps ℓ^0_1 and ℓ^0_2 satisfying (0.2) and such that $d\pi_z(\ell^0_1(z)) = \xi_1$ and $d\pi_z(\ell^0_2(z)) = \xi_2$. Then the Hermitian product of ξ_1 and ξ_2 is

$$\langle \xi_1, \xi_2 \rangle_{\mathbb{C}P^n} := \frac{\langle \ell^0_1(z), \ell^0_2(z) \rangle_{\mathbb{C}^{n+1}}}{|z|^2_{\mathbb{C}^{n+1}}},$$

a definition which is obviously invariant by transformations $z \longmapsto \lambda z$. If we had started with linear mappings ℓ_1, ℓ_2 which lift respectively ξ_1 and ξ_2 according to (0.1) but without the condition (0.2) we could recover $\langle \xi_1, \xi_2 \rangle_{\mathbb{C}P^n}$ by substitution of $\ell^0_a(z) = \ell_a(z) - \frac{\langle \ell_a(z), z \rangle_{\mathbb{C}^{n+1}}}{|z|^2_{\mathbb{C}^{n+1}}} z$ (for $a = 1, 2$) in the above definition. It gives

$$\langle \xi_1, \xi_2 \rangle_{\mathbb{C}P^n} = \frac{\langle \ell_1(z), \ell_2(z) \rangle_{\mathbb{C}^{n+1}}}{|z|^2_{\mathbb{C}^{n+1}}} - \frac{\langle \ell_1(z), z \rangle_{\mathbb{C}^{n+1}} \langle z, \ell_2(z) \rangle_{\mathbb{C}^{n+1}}}{|z|^4_{\mathbb{C}^{n+1}}}. \tag{0.3}$$

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Note that the Hermitian metric $\langle\cdot,\cdot\rangle_{\mathbb{C}P^n}$ on $\mathbb{C}P^n$ provides us with an Euclidean metric $\langle\cdot,\cdot\rangle_E$ and a symplectic form ω through† $\langle\cdot,\cdot\rangle_{\mathbb{C}P^n} = \langle\cdot,\cdot\rangle_E - i\omega(\cdot,\cdot)$.

The horizontal distribution and the connection — For each $z \in \mathbb{C}^{n+1}\setminus\{0\}$ we let $\mathcal{H}_z$ to be the complex n-subspace in $\mathbb{C}^{n+1}$ which is Hermitian orthogonal to z (and hence to the fiber of π). Note that the definition of the Hermitian metric on $\mathbb{C}P^n$ is such that $\frac{1}{|z|_{\mathbb{C}^{n+1}}} d\pi_z : \mathcal{H}_z \longrightarrow T_{[z]}\mathbb{C}P^n$ is an isometry between complex Hermitian spaces (and $d\pi_z$ allows us also to orient $\mathcal{H}_z$). We call the subbundle $\mathcal{H} := \cup_{z\in\mathbb{C}^{n+1}\setminus\{0\}}\mathcal{H}_z$ of $T\mathbb{C}^{n+1}\setminus\{0\} \longrightarrow \mathbb{C}^{n+1}\setminus\{0\}$ the *horizontal distribution*. It defines in a natural way a connection $\nabla^{Hopf} \simeq \nabla^H$ on the Hopf bundle $\pi : \mathbb{C}^{n+1}\setminus\{0\} \longrightarrow \mathbb{C}P^n$: if $f : \mathbb{C}P^n \longrightarrow \mathbb{C}^{n+1}\setminus\{0\}$ is a section of this bundle its covariant derivative is

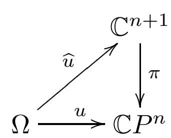
$$\left(\nabla^H_\xi f\right)_{[z]} := \frac{\langle df_{[z]}(\xi), f(z)\rangle_{\mathbb{C}^{n+1}}}{|f(z)|^2_{\mathbb{C}^{n+1}}} f(z).$$

In order to compute the curvature of this connection, let us choose two vector fields ξ_1 and ξ_2 on $\mathbb{C}P^n$ such that $[\xi_1,\xi_2]=0$. Then using (0.3) one computes that

$$\nabla^H_{\xi_1}\left(\nabla^H_{\xi_2} f\right) - \nabla^H_{\xi_2}\left(\nabla^H_{\xi_1} f\right) = \left(\langle\xi_2,\xi_1\rangle_{\mathbb{C}P^n} - \langle\xi_1,\xi_2\rangle_{\mathbb{C}P^n}\right) f = 2i\omega(\xi_1,\xi_2) f.$$

Hence the curvature of this connection is proportional to the symplectic form ω. It has the following consequence:

Proposition 0.1. *Let Ω be a* simply connected *open subset of $\mathbb{R}^n$ and $u : \Omega \longrightarrow \mathbb{C}P^n$ be a smooth Lagrangian immersion, i.e. such that $u^*\omega = 0$. Then there exists a lift*



such that $\left(u^*\nabla^H\right)\widehat{u} = 0$ (where $u^*\nabla^H$ is the pull-back by u of the connection ∇^H). Moreover this lift is unique modulo a multiplicative constant in $\mathbb{C}^\star$, is isotropic (i.e. the pull-back by $\widehat{u}$ of the symplectic form on $\mathbb{C}^{n+1}$ vanishes) and we can choose it in such a way that $|\widehat{u}|_{\mathbb{C}^{n+1}} = 1$ everywhere (i.e. such that $\widehat{u} : \Omega \longrightarrow S^{2n+1}$).

Proof. The curvature of $u^*\nabla^H$ is the pull-back by u of the curvature of ∇^H, i.e. $2iu^*\omega$. But this 2-form vanishes because u is Lagrangian and thus $u^*\nabla^H$ is flat. Hence we can find a parallel section $\widehat{u}$ (unique up to multiplication by a non-zero constant). The condition $\left(u^*\nabla^H\right)\widehat{u} = 0$ writes $\langle d\widehat{u}, \widehat{u}\rangle_{\mathbb{C}^{n+1}} = 0$, which implies $d|\widehat{u}|^2_{\mathbb{C}^{n+1}} = \langle d\widehat{u}, \widehat{u}\rangle_{\mathbb{C}^{n+1}} + \langle \widehat{u}, d\widehat{u}\rangle_{\mathbb{C}^{n+1}} = 0$. Hence $|\widehat{u}|_{\mathbb{R}^{2n+2}}$ is constant and in particular we can choose $\widehat{u}$ to take values in S^{2n+1}. □

Observe that in Proposition 0.1 the tangent n-subspace along $\widehat{u}$ at (x,y) is a sub-

†note the the sign convention may vary in the literature, e.g. some Authors use $\langle\cdot,\cdot\rangle_{\mathbb{C}P^n} = \langle\cdot,\cdot\rangle_E + i\omega(\cdot,\cdot)$.

space of $\mathcal{H}_{\widehat{u}(x,y)}$: we then say that $\widehat{u}$ is a *Legendrian* immersion.

The Levi–Civita connection on $\mathbb{C}P^n$ — It can be expressed in terms of the trivial connection D of $(\mathbb{C}^{n+1}, \langle\cdot,\cdot\rangle_{\mathbb{R}^{2n+2}})$ as follows. Let $U \subset \mathbb{C}P^n$ be an open subset such that we can define a smooth local section $\sigma : U \longrightarrow \mathbb{C}^{n+1}\setminus\{0\}$ of the Hopf bundle. For any $m \in U$ we denote by $\widehat{m} := \sigma(m)$. Let X be a smooth tangent vector field on $\mathbb{C}P^n$ defined on U and, choosing some point $m \in U$, let $\xi \in T_m\mathbb{C}P^n$. In order to define the covariant derivative of X at m along ξ we let $\widehat{\xi} := d\sigma_m(\xi) \in T_{\widehat{m}}\mathbb{C}P^n$ and $\widehat{X} := \sigma_* X$. Then we set

$$\left(\nabla_\xi X\right)(m) := d\pi_{\widehat{m}}\left(\left(D_\xi \widehat{X}\right)(m)\right) - \left(\frac{\langle \widehat{\xi}, \widehat{m}\rangle_{\mathbb{C}^{n+1}}}{|\widehat{m}|^2_{\mathbb{C}^{n+1}}} X(m) + \frac{\langle \widehat{X}(m), \widehat{m}\rangle_{\mathbb{C}^{n+1}}}{|\widehat{m}|^2_{\mathbb{C}^{n+1}}}\xi\right) \tag{0.4}$$

(Of course this definition does not depend on the choice of the local section σ.) One can then check easily that this connection respects the metric and is torsion-free and so it is the Levi–Civita connection on $\mathbb{C}P^n$.

0.2. *The mean curvature vector of a Lagrangian submanifold in* $\mathbb{C}P^n$

We consider here a simply connected Lagrangian submanifold Σ of $\mathbb{C}P^n$ and denote by $u : \Sigma \longrightarrow \mathbb{C}P^n$ its immersion map. According to Proposition 0.1, we can construct a Legendrian lift $\widehat{u} : \Sigma \longrightarrow \mathbb{C}^{n+1}\setminus\{0\}$. We also consider a smooth orthonormal moving frame along u, i.e. sections $E_1, \cdots, E_n$ of $u^*T\mathbb{C}P^n$ such that for all $x \in \Sigma$, $(E_1(x), \cdots, E_n(x))$ is a direct orthonormal basis over $\mathbb{R}$ of $T_{u(x)}\Sigma$ and, for all a, we let $e_a := \widehat{u}_* E_a$. Then for all $x \in \Sigma$, $(e_1(x), \cdots, e_n(x))$ is a direct orthonormal basis over $\mathbb{R}$ of $T_{\widehat{u}(x)}\widehat{u}(\Sigma)$. We next define the *mean curvature vector* $\vec{H}$ of the immersion u which is the normal component of the trace of the second fundamental form. The main point here is that, since $T_{u(x)}\Sigma$ is Lagrangian, it is mapped by the complex structure[†] J of $T_{u(x)}\mathbb{C}P^n$ to the normal space $\left(T_{u(x)}\mathbb{C}P^n\right)^\perp$. And so $(JE_1(x), \cdots, JE_n(x))$ is an orthonormal basis of the normal subspace to $T_{u(x)}\Sigma$ in $T_{u(x)}\mathbb{C}P^n$. Hence

$$\vec{H} := \frac{1}{n}\sum_{a,b=1}^{n} \langle \nabla_{E_a} E_a, JE_b\rangle_E \, JE_b. \tag{0.5}$$

Note that then, since $\langle e_a, \widehat{u}\rangle_{\mathbb{C}^{n+1}} = \langle e_b, \widehat{u}\rangle_{\mathbb{C}^{n+1}} = 0$, we have according to (0.4)

$$\nabla_{E_a} E_b = d\pi_{\widehat{u}(x)}\left(D_{E_a} e_b\right) = d\pi_{\widehat{u}(x)}\left(D_{e_a} e_b\right) .$$

Hence

$$\vec{H} = \frac{1}{n}\sum_{a,b=1}^{n} \langle D_{e_a} e_a, ie_b\rangle_{\mathbb{R}^{2n+2}} \, JE_b. \tag{0.6}$$

The complex volume n*-form along* $\widehat{u}$ — Let us consider on $\mathbb{C}^{n+1}$ the complex volume $(n+1)$-form $\Theta := dz^1 \wedge \cdots \wedge dz^{n+1}$. Then we construct the complex volume n-form along $\widehat{u}$ to be the section $\widehat{\theta}$ of $\widehat{u}^*\left(\Lambda^n\mathcal{H}^* \otimes \mathbb{C}\right)$ by

$$\widehat{\theta}_x := \left(\widehat{u}(x) \,\lrcorner\, \Theta\right)_{|\mathcal{H}_{\widehat{u}(x)}}, \quad \forall x \in \Sigma,$$

[†] the complex structure J is the image of the canonical complex structure i on $\mathcal{H}_{\widehat{u}(x)}$ by $d\pi_{\widehat{u}(x)}$.

where $\lrcorner$ is the interior product. Lastly we define the complex volume n-form θ along u to be the section of $u^*(\Lambda^n T^*\mathbb{C}P^n \otimes \mathbb{C})$ defined by $\theta := \widehat{u}^*\widehat{\theta}$. Using $\widehat{\theta}$ we define the *Lagrangian angle function* along $\widehat{u}$ to be the function $\beta_{\widehat{u}} : \Sigma \longrightarrow \mathbb{R}/2\pi\mathbb{Z}$ such that

$$e^{i\beta_{\widehat{u}}(x)} = \theta_x(E_1(x), \cdots, E_n(x)) = \Theta(\widehat{u}(x), e_1(x), \cdots, e_n(x)), \quad \forall x \in \Sigma.$$

It is not difficult to check that this definition is independent of the choice of the framing $(E_1, \cdots, E_n)$. However all this construction depends on the choice of the Legendrian lift $\widehat{u}$: another choice $\widehat{v}$ would lead to another complex volume n-form along $\hat{v}$, and another Lagrangian angle function $\beta_{\widehat{v}}$, however $\beta_{\widehat{u}} - \beta_{\widehat{v}}$ is a constant in $\mathbb{R}/2\pi\mathbb{Z}$.

LEMMA 0.2. *The mean curvature vector of the Lagrangian surface Σ can be computed using the Lagrangian angle function $\beta_{\widehat{u}}$ through the relation*

$$\vec{H} = \frac{1}{n} J\nabla\beta_{\widehat{u}}. \tag{0.7}$$

Proof. We set here $\beta = \beta_{\widehat{u}}$. Relation (0.7) is equivalent to proving that

$$\forall V \in T_x\mathbb{C}P^n, \quad n\langle \vec{H}, JV\rangle_E = \langle \nabla\beta, V\rangle_E. \tag{0.8}$$

The left hand side of (0.8) is, by using (0.6) and the fact that J is an isometry, $\sum_{a,b=1}^n \langle D_{e_a}e_a, ie_b\rangle_{\mathbb{R}^{2n+2}} \langle E_b, V\rangle_E$. So we are led to the following computation, which uses first the fact that $\langle e_a, ie_b\rangle_{\mathbb{R}^{2n+2}} = 0$, second the fact that i is a complex structure, and third the fact that the Lie bracket $[e_a, e_b]$ is tangent to Σ:

$$\begin{aligned}
\langle D_{e_a}e_a, ie_b\rangle_{\mathbb{R}^{2n+2}} &= -\langle e_a, iD_{e_a}e_b\rangle_{\mathbb{R}^{2n+2}} \\
&= \langle ie_a, D_{e_a}e_b\rangle_{\mathbb{R}^{2n+2}} \\
&= \langle ie_a, D_{e_b}e_a + [e_a, e_b]\rangle_{\mathbb{R}^{2n+2}} \\
&= \langle ie_a, D_{e_b}e_a\rangle_{\mathbb{R}^{2n+2}}.
\end{aligned}$$

So we have $n\langle \vec{H}, JV\rangle_E = \sum_{a,b=1}^n \langle ie_a, D_{e_b}e_a\rangle_{\mathbb{R}^{2n+2}} \langle E_b, V\rangle_E$. Now the right hand side of (0.8) can be computed as follows.. We derivate both sides of the relation $e^{i\beta} = \Theta_{\widehat{u}}(\widehat{u}, e_1, \cdots, e_n)$ with respect to e_b and use the decomposition of $D_{e_b}e_a$ in the $\mathbb{R}$-orthonormal basis $(\widehat{u}, e_1, \cdots, e_n, i\widehat{u}, ie_1, \cdots, ie_n)$:

$$\begin{aligned}
ie^{i\beta}D_{e_b}\beta &= D_{e_b}e^{i\beta} \\
&= D_{e_b}\left(\Theta(\widehat{u}, e_1, \cdots, e_n)\right) \\
&= \Theta(e_b, e_1, \cdots, e_n) + \sum_{a=1}^n \Theta(\widehat{u}, e_1, \cdots, D_{e_b}e_a, \cdots, e_n) \\
&= \sum_{a=1}^n \Theta(\widehat{u}, e_1, \cdots, e_a, \cdots, e_n)\left(\langle e_a, D_{e_b}e_a\rangle_{\mathbb{R}^{2n+2}} + i\langle ie_a, D_{e_b}e_a\rangle_{\mathbb{R}^{2n+2}}\right) \\
&= ie^{i\beta}\sum_{a=1}^n \langle ie_a, D_{e_b}e_a\rangle_{\mathbb{R}^{2n+2}}.
\end{aligned}$$

Here we used in the last line the fact that $|e_a|^2_{\mathbb{R}^{2n+2}} = 1 \Longrightarrow \langle e_a, D_{e_b}e_a\rangle_{\mathbb{R}^{2n+2}} = 0$. Hence we are left with $D_{e_b}\beta = \sum_{a=1}^n \langle ie_a, D_{e_b}e_a\rangle_{\mathbb{R}^{2n+2}}$, which implies that

$$\langle \nabla\beta, V\rangle_E = \sum_{b=1}^n D_{e_b}\beta\,\langle E_b, V\rangle_E = \sum_{a,b=1}^n \langle ie_a, D_{e_b}e_a\rangle_{\mathbb{R}^{2n+2}} \langle E_b, V\rangle_E.$$

So we have proved (0.8). ☐

A *Hamiltonian stationary* Lagrangian submanifold Σ in $\mathbb{C}P^n$ is a Lagrangian submanifold which is a critical point of the n-volume functional $\mathcal{A}$ under first variations which are *Hamiltonian vector fields* with compact support. This means that for any smooth function with compact support $h \in \mathcal{C}_c^\infty(\mathbb{C}P^n, \mathbb{R})$, we have

$$\delta\mathcal{A}_{\xi_h}(\Sigma) := \int_\Sigma \left\langle \vec{H}, \xi_h \right\rangle_E d\text{vol} = 0,$$

where ξ_h is the Hamiltonian vector field of h, i.e. satisfies $\omega(\xi_h, \cdot) + dh = 0$ or $\xi_h = J\nabla h$. We also remark that if $f \in \mathcal{C}_c^\infty(\Sigma, \mathbb{R})$, then there exist smooth extensions with compact support h of f, i.e. functions $h \in \mathcal{C}_c^\infty(\mathbb{C}P^n, \mathbb{R})$ such that $h_{|\Sigma} = f$, and moreover the normal component of $(\xi_h)_{|\Sigma}$ does not depend on the choice of the extension h (it coincides actually with $J\nabla f$, where ∇ is here the gradient with respect to the induced metric on Σ). So we deduce from Lemma 0.2 that actually $\delta\mathcal{A}_{\xi_h}(\Sigma) = \frac{1}{n}\int_\Sigma \langle \nabla\beta, \nabla f\rangle_E \, d\text{vol}$. This implies the following.

COROLLARY 0.3. *Any Lagrangian submanifold Σ in $\mathbb{C}P^n$ is Hamiltonian stationary if and only if β is a harmonic function on Σ, i.e.*

$$\Delta_\Sigma \beta = 0.$$

The theory above extends to non simply connected surfaces Σ with the following restrictions. Let γ be a homotopically non trivial loop. The Legendrian lift of γ needs not close, so that in general its endpoints $p_1, p_2 \in S^{2n+1}$ are multiples of each other by a factor $e^{i\theta}$. The same holds for the Lagrangian angle: $\beta(p_2) \equiv \beta(p_1) + (n+1)\theta \bmod 2\pi$ (since the tangent plane is also shifted by the Decktransformation $z \longmapsto e^{i\theta}z$). In particular β is not always globally defined on surfaces in $\mathbb{C}P^n$ with non trivial topology, unless the Legendrian lift is globally defined in $S^{2n+1}/\mathbb{Z}_{n+1}$ (here $\mathbb{Z}_{n+1}$ stands for the cubic roots of unity in $SU(n+1)$).

References

[1] F. Hélein, P. Romon, *Hamiltonian stationary tori in the complex projective plane*, Proc. London Roy. Soc. (3) 90 (2005), 472–496.