

**AUTOMORPHIC MOTIVES AND  
THE ICHINO-IKEDA CONJECTURE**

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Conference in honor of Jean-Marc Fontaine

(thanks to Blasius, Gan, Gross, N. Harris)

March 25, 2010 (uncorrected version)

## MAIN FORMULA

(This lecture): (a real lecture):: (CDO): (Real money)

CDO = Collateralized Debt Obligation = bundle of more or less toxic speculations on future production (just like this lecture).

Goal: to determine compatibility of two conjectures on special values of  $L$ -functions:

**Deligne** (motivic)

(in some indefinite future Bloch-Kato/Fontaine-Perrin-Riou) ;

and **Ichino-Ikeda** (automorphic).

## 0. LINEAR ALGEBRA

Consider a *polarized regular motive*  $M$  over  $\mathcal{K}$ , pure of weight  $w = n - 1$ , of rank  $n$  over its coefficient field  $E$ .

$\mathcal{K}/F$  a CM quadratic extension of a totally real field.

$c \in \text{Gal}(\mathcal{K}/F)$  complex conjugation.

**Absolutely everything** is true in this generality.

But to simplify notation (usually)  $F = \mathbb{Q}$ ,  $\mathcal{K}$  an imaginary quadratic subfield of  $\mathbb{C}$ .

This is a collection of realizations

$$M = (\{M_\lambda\}, M_B, M_{DR}, M_{Hodge}, M_{cris}) :$$

just vector spaces with associated structures.

Here  $\lambda$  runs over finite places of  $E$ ,

$$M_{Hodge} = gr_F^\bullet M_{DR} = \bigoplus M^{p_i, q_i}$$

a  $\mathcal{K}$ -vector space (really: free rank  $n$   $\mathcal{K} \otimes E$ -module)

In practice we need the  $\lambda$ -adic Galois representation  $M_\lambda$  (for  $L$ -function)

and  $M_{Hodge}$  (for periods).

*Regularity hypothesis:*  $\dim M^{p,w-p} \leq 1$  for all  $i$ .

Thus we have a decreasing set of  $p_i$ :

$$p_1 > p_2 > \cdots > p_n \quad \dim H^{p_i, q_i} = 1$$

For each  $i$  choose a rational vector  $0 \neq \omega_i \in M^{p_i, q_i}$ .

The conjugate motive  $M^c$  has Hodge types  $(p_i^c, q_i^c)$ .

Let  $RM = M \oplus M^c$  over  $F$  ( $= \mathbb{Q}$ ).

*Polarization hypothesis:*

$$\langle, \rangle_M: M \otimes M^c \rightarrow E(1 - n).$$

Finally,  $F_\infty$  is an  $E$ -linear isomorphism of Betti realizations

$$F_\infty : M_B \xrightarrow{\sim} M_B^c$$

(the *king of Frobenii*, according to Hida).

Induces a map

$$F_\infty : M_{Hodge} \otimes \mathbb{C} \xrightarrow{\sim} M_{Hodge}^c \otimes \mathbb{C};$$

$$F_\infty(M^{p_i, q_i}) = (M^c)^{p_{n+1-i}^c, q_{n+1-i}^c}.$$

*does not* respect the  $\mathcal{K}$ -rational structure.

$$Q_i = Q_i(M) = \langle \omega_i, F_\infty(\omega_i) \rangle_M \in (\mathbb{C} \otimes \mathcal{K} \otimes E)^\times / (\mathcal{K} \otimes E)^\times$$

(we will work only mod  $\overline{\mathbb{Q}}^\times$  in this talk.)

The polarization induces one relation:

$$\prod_{i=1}^n Q_i(M) = (2\pi i)^{n(n-1)} \delta(M)^{-2}.$$

Here  $\delta(M)$  is the determinant of

$$I_\infty : M_B \otimes \mathbb{C} \xrightarrow{\sim} M_{DR} \otimes \mathbb{C}.$$

This is the same as the comparison map

$$\det(M)_B \otimes \mathbb{C} \xrightarrow{\sim} \det(M)_{DR} \otimes \mathbb{C}.$$

$\det(M)$  is the (abelian) motive, of a CM Hecke character.

Thus

$$\prod_{i=1}^n Q_i(M) = 1 \pmod{A^\times}$$

where  $A^\times \subset \mathbb{C}^\times$  is the subgroup generated by periods of CM motives.

To avoid having to keep track of CM periods we work in  $\mathbb{C}^\times / A^\times$ .

(Anyway, I haven't had time to finish the necessary calculations.)

**Deligne's conjecture.** If  $m$  is a *critical* integer for  $M$ , there is a *Deligne period*

$$c^+(M(m)) \in \mathbb{C}^\times / \overline{\mathbb{Q}}^\times$$

defined using  $M_{Hodge}$  and  $F_\infty$ , such that

$$L(M, m) / c^+(M(m)) \in \overline{\mathbb{Q}}.$$

**Principle:** Many  $c^+(M(m))$  can be expressed in terms of the  $Q_i(M)$ .

## 1. AUTOMORPHIC MOTIVES

$\Pi$  a cuspidal cohomological automorphic representation of  $GL(n, \mathcal{K})$ .

*Polarization condition*

$$\Pi^\vee \xrightarrow{\sim} \Pi^c.$$

$E = E(\Pi)$  the field of definition of  $\Pi_f$ , a CM field.

$M_{\Pi, \lambda}$ .

$\Pi$  gives rise to a compatible system of  $\lambda$ -adic representations

$$\rho_{\Pi, \lambda} \rightarrow GL(n, E_\lambda),$$

where  $\lambda$  runs over places of  $E$ , non-degenerate pairing

$$\rho_{\Pi, \lambda} \otimes \rho_{\Pi, \lambda}^c \rightarrow E_\lambda(1 - n).$$

(cf. Eichler-Shimura)

References :

$n = 2$ : essentially the theory of Galois representations attached to Hilbert modular forms (Carayol, Wiles, Blasius-Rogawski, and Taylor).

$n = 3$ : Blasius-Rogawski.

$n > 3$ , 1988-2005: Clozel, Kottwitz, MH-Taylor, Taylor-Yoshida.

2005+ (After the proof of the Fundamental Lemma by Laumon and Ngô): Book project (mainly Clozel, Labesse, Chenevier, and MH), S.W. Shin.

Complete statement in Chenevier-MH.

$\rho_{\Pi, \lambda}$  are pure of weight  $n - 1$ . (proved at most bad places by Shin except in limit cases; in limit cases results of Clozel, Taylor, etc.)

**Hypothesis:** there is a polarized regular motive  $M = M_\Pi$  over  $\mathcal{K}$  of rank  $n$ , pure of weight  $w = n - 1$ , coefficients in  $E$ , with

$$M_{\Pi,\lambda} = \rho_{\Pi,\lambda}.$$

Relation between  $M$  and  $\Pi$ :

$$L(s, M) = L\left(s + \frac{1-n}{2}, \Pi\right) = L\left(s, \Pi \otimes (|\bullet| \circ \det)^{\frac{1-n}{2}}\right)$$

Actually true! except for some cases (the “limit cases”) treated by  $\lambda$ -adic congruences.

The **Fontaine-Mazur conjecture** in this setting: every such motive (even a single  $M_\lambda$ ) is automorphic.

In most cases (the “motivic cases”)  $\rho_{\Pi,\lambda}$  realized in the cohomology of certain Shimura varieties, defined over  $\mathcal{K}$  with coefficients in certain local systems.

Similarly,  $M_{\Pi}$  can be realized as a Grothendieck motive using Hecke correspondences on generalized Kuga-Sato varieties (mixed Shimura varieties) .

**In the motivic cases,  $M_{\Pi,Hodge} (+ F_{\infty})$  can be related directly to automorphic forms** (see below).

The limit case is important: it includes symmetric powers of modular forms as in proofs of various cases of the Sato-Tate conjecture and arises in all potential automorphy arguments.

$M_{\Pi,B}$ . Purely hypothetical (but see Taylor on descent of Shimura varieties)

Interpret  $F_\infty$  as complex conjugation of automorphic forms,  $Q_i(M_\Pi)$ : Petersson (square) norm of automorphic form  $\omega_i$ .

$M_{\Pi,DR}$ .

Since  $\Pi$  is cohomological,  $M_\Pi$  is *regular*, which means that

$$p_i > p_{i+1}, p_i^c > p_{i+1}^c$$

There is a simple expression for the Hodge types  $p_i, p_i^c$  in terms of the component  $\Pi_\infty$  of  $\Pi$ .

**Problem 1.** *Study  $M_{\Pi,cris}$  in the automorphic setting.*

The comparison isomorphisms of  $p$ -adic Hodge theory are used, e.g. in Clozel-MH-Taylor (to compute Zariski tangent space of crystalline deformation functor).

$M_{\Pi,cris}$  plays no direct role (but see Mokrane-Tilouine as well as earlier work of Coleman, Gross, Buzzard-Taylor on  $p$ -adic cohomology and modular forms).

## 2. PERIODS

Shimura varieties  $Sh(V)$  attached to unitary groups  $U(V)$ ,

$V$  an  $n$ -dimensional hermitian space over  $\mathcal{K}$ .

For each  $V$ , base change map

(cohomological automorphic representations of  $U(V)$ )



(cohomological automorphic reps. of  $GL(n)_{\mathcal{K}}$ )

Studied in detail by Labesse, (S. Morel for  $F = \mathbb{Q}$ ).

Inverse image of  $\Pi$ : the  $L$ -packet  $\Pi_V = \{\pi_\alpha\}$  of  $U(V)$  attached to  $\Pi$ .

Depending on local conditions,  $\Pi_V$  may be empty, but often is rich enough to define variants of the motive  $M$ , as follows.

First case:  $V = V_1$  has signature  $(1, n - 1)$ . Pick  $\pi = \pi_\infty \otimes \pi_f \in \Pi_{V_1}$ , fix  $\pi_f$ .

$$\dim Sh(V_1) = n - 1$$

Local system  $L(\Pi_\infty)/Sh(V_1)$  ( $\ell$ -adic or VHS) defined by  $\Pi_\infty$

Of geometric origin (in direct image of cohomology of a family of abelian varieties over  $Sh(V_1)$ ).

The group  $U(V_1)(\mathbf{A}^f)$  (actually  $GU(V_1)(\mathbf{A}^f)$ ) acts on  $L(\Pi_\infty)$  and  $Sh(V_1)$ .

Define

$$M(V_1, \Pi) = \text{Hom}_{U(V_1)(\mathbf{A}^f)}(\pi_f, H^{n-1}(\text{Sh}(V_1), L(\Pi_\infty))).$$

Usually  $\dim M(V_1, \Pi) = n$ : this is the motive over  $\mathcal{K}$  we want to call  $M_\Pi$ , up to an explicit abelian twist.

$n$ -dimensional because the archimedean (discrete series)  $L$ -packet  $\Pi_{V_1, \infty}$  has  $n$ -members – basis of Hodge realization indexed by  $\Pi_{V_1, \infty}$  (see below).

More generally, if  $U(V) = U(r, s)$  with  $r + s = n$  the signature, then

$$|\Pi_{V, \infty}| = d(V) := \binom{n}{r}$$

so  $M(V, \Pi)$ , defined by the formula analogous to that used with  $V = V_1$ , should be a motive of rank  $d(V)$ . In fact we have (Kottwitz + stable trace formula)

$$M(V, \Pi)_\lambda \xrightarrow{\sim} \wedge^r M_{\Pi, \lambda} \otimes M(\xi)$$

for an explicit CM Hecke character  $\xi$ .

## The point to keep in mind:

Cohomology of Shimura varieties not only gives us a large collection of  $n$ -dimensional polarized regular motives. It also gives us motivic realizations of all exterior powers of the  $\lambda$ -adic realizations.

**Problem 2.** *The Tate conjecture implies the corresponding relations at the level of motives. Prove them independently, at least for the Hodge realization (see below); i.e., prove the corresponding period relations.*

(All motives have coefficients in some Hecke field  $E = E(\Pi)$ , but I ignore this.)

We actually know more than this. The automorphic theory naturally gives us a  $\mathcal{K}$ -rational basis (actually  $\mathcal{K} \otimes E$ -rational)  $(\omega_1, \dots, \omega_n)$  of the Hodge realization

$$M_{Hodge} := \bigoplus_{i=1}^n H^{p_i, q_i}(M_\Pi),$$

which is naturally a  $\mathcal{K}$ -vector space via the identification of the  $H^{p_i, q_i}$  with coherent cohomology.

Thus the pure tensors

$$\omega_I = \omega_I(\Pi) := \omega_{i_1} \wedge \omega_{i_2} \wedge \dots \wedge \omega_{i_r}; \quad I = (i_1 < i_2 < \dots < i_r)$$

form a canonical basis for  $\wedge^r M_{Hodge}$ .

(For general  $F$ , we have to tensor over the  $\sigma \in \text{Hom}(F, \mathbb{R})$ , but the idea is the same, and we get rational bases of pure tensors for each  $M(V, \Pi)_{Hodge}$  in terms of the  $\omega_{i, \sigma}$ .)

The *periods* of the  $\omega_i$  – integrals over rational homology classes – are completely inaccessible by automorphic means.

But  $\omega_i$  is a concrete automorphic form on  $U(V_1)$ , generating a specific  $\pi_i \in \Pi_{V_1}$ , and its Petersson (square) norm  $Q_i = Q_i(\Pi)$  is a well-defined invariant in  $\mathbb{C}^\times / \overline{\mathbb{Q}}^\times$ , up to algebraic factors.

In particular,  $\omega_1$  is a *holomorphic* automorphic form.

Similarly, if  $V$  is fixed, for each  $\pi_I \in \Pi_V$ , there is a concrete automorphic form  $\omega_{I,V}$  on  $U(V)$ , generating an automorphic representation of  $U(V)$  isomorphic to  $\pi_I \otimes \pi_f$  when  $|I| = r$ ; these form a rational basis for the Hodge realization of the motive  $M(V, \Pi)$ , canonical up to (rational) scalar multiples.

(Here we are making the assumption that  $U(V)(\mathbf{A}^f) \xrightarrow{\sim} U(V)$  so their local representations can be identified. This is legitimate when  $n$  is odd but not when  $n$  is even, in which case the construction needs to be modified.)

For example,  $\omega_{I,V}$  is holomorphic, as is the formal wedge product  $\omega_I$ , if and only if  $I = \{1, \dots, r\}$ .

Let  $Q_{I,V}$  be the Petersson square norm of  $\omega_{I,V}$ . Under the hypothetical isomorphism

$$M(V, \Pi) \xrightarrow{\sim} \wedge^r M_{\Pi} \otimes M(\xi)$$

we want to identify  $\omega_{I,V}$  with the wedge product  $\omega_I$ .

**Problem 3.** *Prove that  $Q_{I,V} = \prod_{i \in I} Q_i \pmod{A^\times}$ , more precisely up to the CM period corresponding to the abelian twist), as predicted by the Tate conjecture.*

When  $F = \mathbb{Q}$  and  $\Pi \xrightarrow{\sim} \Pi^c$ , known (MH) for the holomorphic  $Q_I$ , up to an undetermined constant that depends only on  $\Pi_\infty$ .

The proof is based on the theta correspondence.

MH-Li-Sun (2009-2010): the theta correspondence alone cannot solve Problem 3 for general  $I$ !

Can it be solved without the Tate conjecture?

The interest of the  $Q$ -invariants: for certain motives  $\mathcal{M}$  constructed from polarized regular  $M$ , the *Deligne periods*  $c^+(\mathcal{M})$  (of Deligne's conjecture on special values) can be expressed formally in terms of the  $Q_i$  and some abelian (CM) periods.

In the heuristic discussion that follows, assume the period relations hold for the  $Q$ -invariants. In particular, we identify the formal wedge product  $\omega_I$  with the automorphic form  $\omega_{I,V}$  on  $U(V)$ .

### 3. THE GROSS-PRASAD CONJECTURES

Let now  $\Pi$  and  $\Pi'$  be cohomological automorphic representations of  $GL(n)_{\mathcal{K}}$  and  $GL(n-1)_{\mathcal{K}}$ .

Choose hermitian spaces  $V$  and  $V'$ ,  $V = V' \oplus E$  for a non-isotropic line  $E$ ,  $\dim V = n$ .

Let  $\pi \in \Pi_V$ ,  $\pi' \in \Pi'_{V'}$ ,  $G = U(V)$ ,  $G' = U(V')$ .

Consider the abstract space of linear forms

$$L(\pi, \pi') = \text{Hom}_{G'}(\pi \otimes \pi', \mathbb{C})$$

where  $G' \subset G \times G'$  is embedded diagonally.

The following theorem is due to Aizenbud-Gurevich-Rallis-Schiffman (for the non-archimedean factors) and Sun-Zhu (for the archimedean factors):

**Theorem 3.**  $\dim L(\pi, \pi') = \prod_v \dim L(\pi_v, \pi'_v) \leq 1$ ,  
*where the factorization is defined in the obvious way.*

Waldspurger has proved much more (so far only for orthogonal groups).

The full Gross-Prasad conjecture in this setting is roughly

$$\sum_{\pi \in \Pi_V, \pi' \in \Pi_{V'}} \dim L(\pi, \pi') \leq 1$$

with equality if the  $L$ -function considered by Ichino-Ikeda (see below) does not vanish at  $s = \frac{1}{2}$ .

Waldspurger has proved the non-archimedean part of this statement (for tempered  $L$ -packets, which should include all the local terms of  $\Pi_V$  and  $\Pi_{V'}$ ).

The archimedean statement is not yet proved but will be assumed in what follows.

## CONSEQUENCES OF THEOREM 3

**Global consequences.** Suppose  $\pi = \pi_I$  for some subset  $I \subset \{1, \dots, n\}$  as above, with  $|I| = r$ ,  $G_\infty = U(r, s)$ . Assume  $\pi' = \pi_{I'}$  similarly.

### Hypotheses.

(a)  $\dim L(\pi, \pi') = 1$

(b) A generator  $L \in L(\pi, \pi')$  is given by integration

$$L_{can}(f, f') = \int_{G'(F) \backslash G'(\mathbf{A})} f(g') f'(g') dg'$$

where  $dg'$  is Tamagawa measure.

The original global Gross-Prasad conjecture is roughly the following:

**Problem 4.** Assuming (a), prove that (b) is true if and only if  $L(\frac{1}{2}, \Pi \otimes \Pi') \neq 0$ .

The Ichino-Ikeda conjecture is a refined version of Problem 4, expressing the  $L$ -value in terms of the integral  $L_{can}$  and other terms.

**Proposition.** *The spaces  $\pi$  and  $\pi'$  have  $\overline{\mathbb{Q}}$ -rational structures generated by  $\omega_I(\Pi)$  and  $\omega_{I'}(\Pi')$ . In particular,  $L(\pi, \pi')$  has a generator  $L_0$  rational with respect to these structures.*

One can replace  $\overline{\mathbb{Q}}$  by  $\mathcal{K} \otimes E$ . For  $\pi_f$  and  $\pi'_f$ , standard considerations about automorphic cohomology.

For the archimedean part, use algebraic realization of  $\pi_{I, \infty}, \pi_{I', \infty}$  as Zuckerman derived functor modules.

(For this need to choose max. compact, max torus  $K_\infty \supset T_\infty$ ; OK over a CM point attached to  $\mathcal{K}$ .)

**Corollary.** *There is a constant  $P(\pi, \pi') \in \mathbb{C}^\times / \overline{\mathbb{Q}}^\times$  (an **occult period**) such that, for any  $f \in \pi$ ,  $f' \in \pi'$ , rational over  $\overline{\mathbb{Q}}$ ,  $L_0(f, f')$  is a  $\overline{\mathbb{Q}}$ -multiple of  $P(\pi, \pi')$ .*

**Proposition.** *If  $\pi_I$  is holomorphic,  $P(\pi, \pi') = 1$ .*

Indeed,  $\pi_{I, \infty} |_{G'(\mathbb{R})}$  is a discrete (countable) direct sum of holomorphic representations. Thus if  $L(\pi, \pi') \neq 0$ ,  $\pi'$  must be *antiholomorphic*, and the integral  $L_{can}$  is just cup-product in coherent cohomology.

With the rational structures I (haven't) defined, we can thus take  $L_0 = L_{can}$ .

More generally, for certain pairs  $(I, I')$ ,  $L_{can}$  can be interpreted as a cup product in coherent cohomology, and then one again has  $L_0 = L_{can}$ . We then say the pair  $(\pi_{I, \infty}, \pi'_{I', \infty})$  is *coherent*.

One has  $\omega_i \in H^{p_i, q_i} \xrightarrow{\sim} H^{i-1}(Sh(V_1), \mathcal{E}_i)$  for some automorphic vector bundle  $\mathcal{E}_i$  on  $Sh(V_1)$ .

(Don't always have  $q_i = i - 1$ :  $\mathcal{E}_i$  has a built-in twist by (the infinity type of) a CM Hecke character.)

Similarly,  $\omega_I$  belongs to  $H^j(Sh(V), \mathcal{E}_I)$  where  $j$  is the number of elements in  $I$  *not* in the set  $\{1, \dots, r\}$ .

**Problem 5.** *Determine the set of coherent pairs  $(\pi_{I, \infty}, \pi'_{I', \infty})$ ;  $L_0 = L_{can}$  (possibly up to an element of  $A^\times$ , to account for the built-in twist).*

In some cases, for example, when  $\pi_I$  is holomorphic, one can get not only direct cup product pairings but also cup products after application of “nearly holomorphic” differential operators (Maass operators). So a special case of Problem 5 is

**Problem 6.** *Classify nearly holomorphic differential operators on higher coherent cohomology.*

For holomorphic cohomology these have been classified long ago.

When  $L_0 = L_{can}$ , we say the pair  $(\pi_{I,\infty}, \pi'_{I',\infty})$  is *coherent*.

## FOUR PAGES RELEVANT TO THIS CONFERENCE

When  $V_{\mathbb{R}}$  is a definite hermitian space – so  $G(\mathbb{R})$  and  $G'(\mathbb{R})$  are necessarily compact –

$$\dim \pi_{\infty}, \dim \pi'_{\infty} < \infty.$$

The classification of pairs such that  $L(\pi, \pi') \neq 0$  given by classical branching laws for algebraic representations of  $U(n)$  (or  $GL(n)$ ).

At a finite prime  $p$

$$G_p = G(\mathbb{Q}_p) \xrightarrow{\sim} \begin{cases} GL(n, \mathbb{Q}_p) \\ \text{some unitary group} \end{cases}$$

Let  $\pi_p \otimes \pi'_p$  (resp.  $W \otimes W'$ ) be a smooth (resp. algebraic) irreducible representations of  $G_p \times G'_p$ , with coefficients in  $\mathbb{C}_p$ . Suppose

$$L(\pi_p, \pi'_p) \neq 0; L(W, W') := \text{Hom}_{G'_p}(W \otimes W', \mathbb{C}_p) \neq 0.$$

Let  $\sigma, \sigma'$  be irreducible Banach space completions of the locally algebraic representations  $\pi_p \otimes W, \pi'_p \otimes W'$ .

Define

$$L(\sigma, \sigma') = \text{Hom}_{G'_p}^{\text{cont}}(\sigma \otimes \sigma', \mathbb{C}_p).$$

Locally algebraic vectors dense

$$\Rightarrow \dim_{\mathbb{C}_p}(L(\sigma, \sigma')) \leq 1.$$

**Problem 7.** *For which Banach space completions of  $\pi_p \otimes W$  and  $\pi'_p \otimes W'$  do the non-trivial elements of  $L(\pi_p, \pi'_p) \otimes L(W, W')$  extend continuously?*

$GL(2, \mathbb{Q}_p)$  ( $n = 2$ ) is already interesting.

$L(\pi_p, \pi'_p)$  determined by Waldspurger, Tunnell, H. Saito.

Non-vanishing depends on sign of the local  $\varepsilon$  factor of  $\pi_p \times \pi'_p$  (by Waldspurger, this is true more generally).

Specialists expect to classify Banach space completions by admissible filtrations on the local  $(\phi, N)$ -module attached to  $\pi_p \otimes W$ .

Can the  $\varepsilon$  factor be extended to allow for the admissible filtration?

$U(2) \times U(1)$  at a non-split prime  $\leftrightarrow G = GL(2, \mathbb{Q}_p)$ ,  
 $G' = L^\times$ ,  $L$  is a quadratic extension of  $\mathbb{Q}_p$

What can be said about the Banach representations of  $G$  as  $G'$ -modules?

As for modular representations:

**Problem 8.**  $G = GL(2, K)$  for some  $p$ -adic field  $K$ ,  
 $[L : K] = 2$ ,  $\pi$  an irreducible smooth representation of  $G$  over  $k = \overline{\mathbb{F}}_p$ .

*What is the structure of  $\pi$  as  $L^\times$ -module?*

*Breuil and Paskunas have constructed uncountable families of such  $\pi$ ; do they all have the same  $L^\times$ -module structure?*

## 4. THE ICHINO-IKEDA CONJECTURES

Ichino and Ikeda formulated an explicit conjectural expression for the central value  $L(\frac{1}{2}, \pi \otimes \pi')$  when  $\pi$  and  $\pi'$  are representations of  $SO(n)$  and  $SO(n-1)$ , respectively.

Neal Harris (Ph.D. thesis under the direction of W. T. Gan) is working out the analogue for pairs  $(G, G')$  as above:

Let  $f \in \pi$ ,  $f' \in \pi'$ .

$$\frac{|L_{can}(f, f')|^2}{|f, f|^2 |f', f'|^2} = 2^\beta C_0 \Delta_{U(n)}^S \prod_{v \in S} Z_v(f_v, f'_v) P^S\left(\frac{1}{2}, \pi, \pi'\right)$$

where  $L_{can}(\bullet, \bullet')$  is the period integral and

$$P^S(s, \pi, \pi') = \frac{L^S(s, \pi \times \pi')}{L^S(s + \frac{1}{2}, \pi, Ad) L^S(s + \frac{1}{2}, \pi', Ad)}$$

is the main term.

Here

- (1)  $S$  is a finite set of places including archimedean places
- (2) the superscript  $S$  refers to partial  $L$ -functions
- (3)  $C_0$  is an elementary constant (quotient of volumes, can be taken in  $\mathbb{Q}^\times$ )
- (4)  $\Delta_{U(n)}^S = \alpha\pi^m$ ,  $\alpha \in \overline{\mathbb{Q}}$ ,  $m \in \mathbb{Z}$  (a product of special values of abelian  $L$ -functions),
- (5) each  $Z_v(f_v, f'_v) = \frac{I(f_v, f'_v)}{|f_v|^2 |f'_v|^2}$  can be taken algebraic except for  $v \mid \infty$ ;
- (6)  $L(*, *, Ad)$  is the Langlands  $L$ -function attached to the adjoint representation of the  $L$ -group of  $G$  (or  $G'$ ).

Modulo  $A^\times$ , and being hopeful about the term  $Z_\infty$ , we have

$$\frac{|L_{can}(f, f')|^2}{|f, f|^2 |f', f'|^2} \equiv \frac{L(\frac{1}{2}, \pi \times \pi')}{L(1, \pi, Ad)L(1, \pi', Ad)}.$$

**Problem 9.** *Formulate a  $p$ -adic Ichino-Ikeda conjecture (for completed cohomology?)*

(cf. Villegas-Zagier, Stevens, A. Mori, MH-Tilouine.)

All special values are theoretically covered by definite case (Chenevier, Emerton).

But this must be very difficult to prove!

TWO MORE PAGES RELEVANT  
TO THIS CONFERENCE

**Remarks.**

The terms  $Z_v(f_v, f'_v)$  are explicit integrals of matrix coefficients of representations.

**Problem 10.** *Compare these with local terms in Bloch-Kato-Fontaine-Perrin-Riou conjecture.*

For  $p$ -adic families, the  $Q$ -invariants are not sufficiently precise. For  $n = 2$ , to get the correct periods, have to divide by congruence ideal (Hida).

**Problem 11.** (*Very hard.*) *What about higher dimensions?*

Background motivation for construction of  $p$ -adic  $L$ -functions:

Generalization of Skinner-Wiles, irreducibility of automorphic Galois representations.

## 5. RESULTS

Let

$$M = M_{\Pi}, M' = M_{\Pi'}, f \in \pi_I(\overline{\mathbb{Q}}), f' \in \pi'_{I'}(\overline{\mathbb{Q}}).$$

Then the left hand side is

$$\frac{|L_{can}(f, f')|^2}{Q_I(M)Q_{I'}(M')}.$$

Meanwhile, the right-hand side can be rewritten

$$\frac{L(\frac{w+1}{2}, M \otimes M')}{L(1, M, Ad)L(1, M', Ad)}$$

$$w = (n - 1) + (n - 2) = 2n - 3.$$

All values in this quotient are critical in Deligne's sense.

Thus, if  $L(\pi, \pi') \neq 0$ , Deligne's conjecture and the Ichino-Ikeda conjecture express the occult  $P(\pi, \pi')$  in terms of Deligne periods and the  $Q$ -invariants.

In fact, one can express the Deligne periods in terms of  $Q$ -invariants and (explicit) abelian periods, as I explain below.

**Problem 12.** *When the pair  $(\pi_{I, \infty}, \pi'_{I', \infty})$  is coherent, verify that this expression is compatible with the formula  $P(\pi, \pi') = 1$ .*

*Warning:* the Gross-Prasad conjecture implies that the central value is often calculated by an occult period.

So far I have only checked the simplest cases.

**Proposition.** *In the coherent case, the expressions are compatible, up to factors in  $A^\times$ , when  $\pi_I$  is holomorphic or when  $V = V_1$ .*

The proof is based on explicit expressions for the Deligne periods of the motives that appear in the right-hand side.

The formulas for the adjoint motives are uniform:

$$c^+(Ad(M)(1)) \equiv \prod_{i=1}^n Q_i(M)^{-i} \pmod{A^\times}.$$

Since  $\prod_{i=1}^n Q_i(M) \in A^\times$ , a more suggestive expression for this is

$$\prod_{i=1}^n Q_i(M)^{\frac{n+1}{2}-i}.$$

Likewise for  $M'$ .

The formula for  $c^+(M \otimes M'(\frac{w+1}{2}))$  depends on the relative positions of the Hodge types of  $M$  and  $M'$ . So does the pair  $(I, I')$  such that  $L(\pi_I, \pi'_{I'}) \neq 0$ . The main point is that  $c^+(M \otimes M'(\frac{w+1}{2}))$  can be expressed explicitly in terms of the  $Q_i(M)$  and  $Q_j(M')$ , and of CM periods.