

# A NOTE ON WEAK AMENABILITY FOR REDUCED FREE PRODUCTS OF DISCRETE QUANTUM GROUPS

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ABSTRACT. We prove that the Cowling-Haagerup constant of a reduced free product of weakly amenable discrete quantum groups with Cowling-Haagerup constant equal to 1 is again equal to 1. We also review the basic theory of weak amenability for discrete quantum groups.

## 1. INTRODUCTION

In geometric group theory, weak amenability is an approximation property which is satisfied by a large class of groups (for example free or even hyperbolic groups [Oza08]) but is strong enough to give interesting properties for the von Neumann algebras associated to these groups (for example related to deformation/rigidity techniques [OP10a], [OP10b]). The stability of this property under free products is still an open question. However, using a very general version of the Khintchine inequality, E. Ricard and Q. Xu were able to prove in [RX06] that if  $(G_i)$  is a family of weakly amenable discrete groups *with Cowling-Haagerup constant equal to 1*, then their free product is again weakly amenable, and its Cowling-Haagerup constant is also equal to 1. The proof uses a classical characterization of those bounded functions on a group giving rise to completely bounded multipliers.

This characterization has recently been generalized to arbitrary locally compact quantum groups by M. Daws in [Daw11b]. Using it, we will prove an analogue of Ricard and Xu's result in the setting of discrete quantum groups and give some new examples of non-commutative and non-cocommutative discrete quantum groups having Cowling-Haagerup constant equal to 1. Before that, we give a brief survey on the notion of weak amenability for discrete quantum groups. Along the way, we prove Theorem 3.11 which is a slight generalization of [KR99, Thm 5.14].

## 2. PRELIMINARIES

2.1. **Notations.** All scalar products will be taken to be *left-linear*. For two Hilbert spaces  $H$  and  $K$ ,  $\mathcal{B}(H, K)$  will denote the set of bounded linear maps from  $H$  to  $K$  and  $\mathcal{B}(H) := \mathcal{B}(H, H)$ . In the same way we use the notations  $\mathcal{K}(H, K)$  and  $\mathcal{K}(H)$  for

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compact linear maps. We will denote by  $\mathcal{B}(H)_*$  the predual of  $\mathcal{B}(H)$ , i.e. the Banach space of all normal linear forms on  $\mathcal{B}(H)$ . On any tensor product  $A \otimes B$ , we define the flip operator

$$\Sigma : \begin{cases} A \otimes B & \rightarrow B \otimes A \\ x \otimes y & \mapsto y \otimes x \end{cases}$$

We will use the usual leg-numbering notations : for an operator  $X$  acting on a tensor product we set  $X_{12} := X \otimes 1$ ,  $X_{23} := 1 \otimes X$  and  $X_{13} := (\Sigma \otimes 1)(1 \otimes X)(\Sigma \otimes 1)$ . The identity map of a  $C^*$ -algebra  $A$  will be denoted  $\iota_A$  or simply  $\iota$  if there is no possible confusion. For a subset  $B$  of a topological vector space  $C$ ,  $\overline{\text{span}}B$  will denote the *closed linear span* of  $B$  in  $C$ . The symbol  $\otimes$  will denote the *minimal* (or spatial) tensor product of  $C^*$ -algebras or the topological tensor product of Hilbert spaces.

**2.2. Compact and discrete quantum groups.** Discrete quantum groups will be seen as duals of compact quantum groups in the sense of Woronowicz. We briefly present the basic theory of compact quantum groups as introduced in [Wor98]. Another survey, encompassing the non-separable case, can be found in [MVD98].

**Definition 2.1.** A *compact quantum group*  $\mathbb{G}$  is a pair  $(C(\mathbb{G}), \Delta)$  where  $C(\mathbb{G})$  is a unital  $C^*$ -algebra and  $\Delta : C(\mathbb{G}) \rightarrow C(\mathbb{G}) \otimes C(\mathbb{G})$  is a unital  $*$ -homomorphism such that

$$(\Delta \otimes \iota) \circ \Delta = (\iota \otimes \Delta) \circ \Delta \text{ and } \overline{\Delta(C(\mathbb{G}))(1 \otimes C(\mathbb{G}))} = C(\mathbb{G}) \otimes C(\mathbb{G}) = \overline{\Delta(C(\mathbb{G}))(C(\mathbb{G}) \otimes 1)}.$$

The main feature of compact quantum groups is the existence of a Haar measure, which happens to be both left and right invariant (see [Wor98, Thm 1.3] or [MVD98, Thm 4.4]).

**Proposition 2.2.** *Let  $\mathbb{G}$  be a compact quantum group, there is a unique Haar state on  $\mathbb{G}$ , that is to say a state  $h$  on  $C(\mathbb{G})$  such that for all  $a \in C(\mathbb{G})$ ,*

$$(\iota \otimes h) \circ \Delta(a) = h(a).1$$

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Let  $(L^2(\mathbb{G}), \xi_h)$  be the associated GNS construction and let  $C_{\text{red}}(\mathbb{G})$  be the image of  $C(\mathbb{G})$  under the GNS map, called the *reduced form* of  $\mathbb{G}$ . Let  $W$  be the unique unitary operator on  $L^2(\mathbb{G}) \otimes L^2(\mathbb{G})$  such that

$$W^*(\xi \otimes a\xi_h) = \Delta(a)(\xi \otimes \xi_h)$$

for  $\xi \in L^2(\mathbb{G})$  and  $a \in C(\mathbb{G})$  and let  $\widehat{W} := \Sigma W^* \Sigma$ . Then  $W$  is a *multiplicative unitary* in the sense of [BS93], i.e.  $W_{12}W_{13}W_{23} = W_{23}W_{12}$  and we have the following equalities :

$$C_{\text{red}}(\mathbb{G}) = \overline{\text{span}}(\iota \otimes \mathcal{B}(H)_*)(W)$$

$$\Delta(x) = W^*(1 \otimes x)W$$

Moreover, we can define the *dual discrete quantum group*  $\widehat{\mathbb{G}} = (C_0(\widehat{\mathbb{G}}), \widehat{\Delta})$  by

$$C_0(\widehat{\mathbb{G}}) = \overline{\text{span}}(\mathcal{B}(H)_* \otimes \iota)(W)$$

$$\widehat{\Delta}(x) = \Sigma W(x \otimes 1)W^* \Sigma$$

This data defines a *discrete quantum group*. One can prove that  $W$  is in fact a multiplier of  $C(\mathbb{G}) \otimes C_0(\widehat{\mathbb{G}})$ . We define two von Neumann algebras associated to these quantum groups as  $L^\infty(\mathbb{G}) = C(\mathbb{G})''$  and  $\ell^\infty(\widehat{\mathbb{G}}) = C_0(\widehat{\mathbb{G}})''$  in  $\mathcal{B}(L^2(\mathbb{G}))$ . We will need a few facts about the representation theory of compact quantum groups.

**Definition 2.3.** A *finite-dimensional representation* of a compact quantum group  $\mathbb{G}$  is an element  $u = (u_{i,j}) \in M_n(C(\mathbb{G}))$  such that

$$\Delta(u_{i,j}) = \sum_k u_{i,k} \otimes u_{k,j}.$$

The elements  $u_{i,j}$  are called the *coefficients* of the representation  $u$ . Such a representation will often be seen as an element of  $M_n(\mathbb{C}) \otimes C(\mathbb{G})$ .

The following generalization of the classical Peter-Weyl theorem holds (see [Wor98, Section 6]).

**Theorem 2.4** (Woronowicz). *Every irreducible representation of a compact quantum group is finite dimensional and every unitary representation is unitarily equivalent to a sum of irreducible ones. Moreover, the linear span of the coefficients of all irreducible representations is a dense Hopf \*-algebra denoted  $\mathcal{C}(\mathbb{G})$ .*

Let  $\text{Irr}(\mathbb{G})$  be the set of isomorphism classes of irreducible representations of  $\mathbb{G}$ . If  $\alpha \in \text{Irr}(\mathbb{G})$ , we will denote by  $u^\alpha$  a representative and  $H_\alpha$  the finite dimensional Hilbert space on which it acts. There are isomorphisms

$$C_0(\widehat{\mathbb{G}}) = \oplus_{\alpha \in \text{Irr}(\mathbb{G})} \mathcal{B}(H_\alpha) \text{ and } \ell^\infty(\widehat{\mathbb{G}}) = \prod_{\alpha \in \text{Irr}(\mathbb{G})} \mathcal{B}(H_\alpha).$$

The minimal central projection in  $\ell^\infty(\widehat{\mathbb{G}})$  corresponding to  $\alpha$  will be denoted  $p_\alpha$ .

### 3. WEAK AMENABILITY

We now study the notion of weak amenability for discrete quantum groups. In the classical case, this notion is defined by the existence of some bounded functions on the group giving rise to completely bounded multipliers. The notion of completely bounded multipliers for locally compact quantum groups has now been largely studied, see for instance [HNR11, JNR09, KR99, Daw11a, Daw11b]. For the sake of completeness and because it is much simpler in the context of discrete quantum groups, we give a brief description of the main result.

If  $G$  is a discrete group, one way to construct useful multipliers for its reduced  $C^*$ -algebra is to start with a bounded function  $\varphi : G \rightarrow \mathbb{C}$ . Its associated multiplier is the operator  $m_\varphi$  defined on the linear span of  $\{\lambda(g)\}$  ( $\lambda$  being the left regular representation) by  $m_\varphi(\lambda(g)) = \varphi(g)\lambda(g)$ . One then looks for some criterion on  $\varphi$  ensuring that  $m_\varphi$  extends to a (completely) bounded map on  $C_r^*(G)$ .

**Definition 3.1.** Let  $\widehat{\mathbb{G}}$  be a discrete quantum group and  $a \in \ell^\infty(\widehat{\mathbb{G}})$ . The *left multiplier* associated to  $a$  is the map  $m_a : \mathcal{C}(\mathbb{G}) \rightarrow \mathcal{C}(\mathbb{G})$  defined by

$$(m_a \otimes \iota)(u^\alpha) = (1 \otimes ap_\alpha)u^\alpha,$$

for any irreducible representation  $\alpha$  of  $\mathbb{G}$ .

*Remark 3.2.* This definition can be rephrased simply as

$$(m_a \otimes \iota)(W) = (1 \otimes a)W.$$

This means that for any  $\omega \in \mathcal{B}(L^2(\mathbb{G}))_*$ , one has

$$m_a((\iota \otimes \omega)(W)) = (\iota \otimes \omega)((1 \otimes a)W) = (\iota \otimes \omega a)(W),$$

which is the usual definition of multipliers for locally compact quantum groups.

*Remark 3.3.* Let us assume that there exists  $\omega_a \in \mathcal{B}(L^2(\mathbb{G}))_*$  such that  $(\omega_a \otimes \iota)(W) = a$ , then  $m_a = (\omega_a \otimes \iota) \circ \Delta$ . Indeed,

$$\begin{aligned} (m_a \otimes \iota)(W) &= (1 \otimes a)W \\ &= (\omega_a \otimes \iota \otimes \iota)(W_{13})W \\ &= (\omega_a \otimes \iota \otimes \iota)(W_{13}W_{23}) \\ &= (\omega_a \otimes \iota \otimes \iota) \circ (\Delta \otimes \iota)(W) \\ &= ((\omega_a \otimes \iota) \circ \Delta) \otimes \iota(W). \end{aligned}$$

This links definition 3.1 with the "convolution operators" used by M. Brannan in [Bra11] to study the Haagerup property for some particular discrete quantum groups.

**Definition 3.4.** A net  $(a_i)$  of elements of  $\ell^\infty(\widehat{\mathbb{G}})$  is said to *converge pointwise* to  $a \in \ell^\infty(\widehat{\mathbb{G}})$  if

$$a_i p_\alpha \rightarrow a p_\alpha$$

for any irreducible representation  $\alpha$  of  $\mathbb{G}$ . An element  $a \in \ell^\infty(\widehat{\mathbb{G}})$  is said to have *finite support* if  $a p_\alpha$  is non-zero only for a finite number of irreducible representations  $\alpha$ .

*Remark 3.5.* If  $a_\lambda$  is a sequence in  $\ell^\infty(\widehat{\mathbb{G}})$  converging pointwise to  $a$ , then  $\widehat{\varepsilon}(a_\lambda) \rightarrow \widehat{\varepsilon}(1)$  since  $x p_\varepsilon = \widehat{\varepsilon}(x) p_\varepsilon$ .

Before defining weak amenability for discrete quantum groups, we need an intrinsic characterization of those bounded functions giving rise to completely bounded multipliers. For a discrete group  $G$ , it is known that a bounded function  $\varphi : G \rightarrow \mathbb{C}$  gives rise to a completely bounded multiplier if and only if there exists a Hilbert space  $K$  and two families  $(\xi_s)_{s \in G}$  and  $(\eta_t)_{t \in G}$  of vectors in  $K$  such that for all  $s, t \in G$ ,  $\varphi(s) = \langle \eta_t, \xi_{st} \rangle$  (which is usually written  $\varphi(st^{-1}) = \langle \eta_t, \xi_s \rangle$ ). Such a characterization has been generalized to the setting of locally compact quantum groups by M. Daws [Daw11b, Prop 4.1 and Thm 4.2]. Due to its generality, Daws' proof is quite complicated and subtle, but restricting to the discrete case enables us to drop many technicalities and keep only the heart of the proof, which we give here for the sake of completeness.

**Theorem 3.6** (Daws). *Let  $\widehat{\mathbb{G}}$  be a discrete quantum group and  $a \in \ell^\infty(\widehat{\mathbb{G}})$ . Then  $m_a$  extends to a completely bounded multiplier on  $\mathcal{B}(L^2(\mathbb{G}))$  if and only if there exists a*

Hilbert space  $K$  and two maps  $\alpha, \beta \in \mathcal{B}(L^2(\mathbb{G}), L^2(\mathbb{G}) \otimes K)$  such that  $\|\alpha\|\|\beta\| = \|m_a\|_{cb}$  and

$$(1) \quad (1 \otimes \beta)^* \widehat{W}_{12}^* (1 \otimes \alpha) \widehat{W} = a \otimes 1.$$

Moreover, we then have  $m_a(x) = \beta^*(x \otimes 1)\alpha$ .

*Proof.* We only consider the case when  $m_a$  extends to a completely contractive map on  $\mathcal{B}(L^2(\mathbb{G}))$  still denoted  $m_a$ . By Wittstock's factorization theorem (see for example [BO08, Thm B.7]), there is a representation  $\pi : \mathcal{B}(L^2(\mathbb{G})) \rightarrow \mathcal{B}(K)$  and two isometries  $P, Q \in \mathcal{B}(L^2(\mathbb{G}), K)$  such that for all  $x \in \mathcal{B}(L^2(\mathbb{G}))$ ,  $m_a(x) = Q^* \pi(x) P$ . Set  $U = (\iota \otimes \pi)(\widehat{W})$  and define two maps  $\alpha$  and  $\beta$  by

$$\begin{cases} \alpha &= U^*(1 \otimes P)\widehat{W}(1 \otimes \xi_h) \\ \beta &= U^*(1 \otimes Q)\widehat{W}(1 \otimes \xi_h) \end{cases}$$

These are contractive linear maps from  $L^2(\mathbb{G})$  to  $L^2(\mathbb{G}) \otimes K$ . If we set

$$X = (1 \otimes \beta)^* \widehat{W}_{12}^* (1 \otimes \alpha) \widehat{W},$$

then the pentagon equation gives

$$\begin{aligned} X &= (1 \otimes 1 \otimes \xi_h)^* \widehat{W}_{23}^* (1 \otimes 1 \otimes Q^*) U_{23} \widehat{W}_{12}^* U_{23}^* \\ &\quad (1 \otimes 1 \otimes P) \widehat{W}_{23} (1 \otimes 1 \otimes \xi_h) \widehat{W} \\ &= (1 \otimes 1 \otimes \xi_h)^* \widehat{W}_{23}^* (1 \otimes 1 \otimes Q^*) U_{23} \widehat{W}_{12}^* U_{23}^* \\ &\quad (1 \otimes 1 \otimes P) \widehat{W}_{23} \widehat{W}_{12} (1 \otimes 1 \otimes \xi_h) \\ &= (1 \otimes 1 \otimes \xi_h)^* \widehat{W}_{23}^* (1 \otimes 1 \otimes Q^*) U_{23} \widehat{W}_{12}^* U_{23}^* \\ &\quad (1 \otimes 1 \otimes P) \widehat{W}_{12} \widehat{W}_{13} \widehat{W}_{23} (1 \otimes 1 \otimes \xi_h) \\ &= (1 \otimes 1 \otimes \xi_h)^* \widehat{W}_{23}^* (1 \otimes 1 \otimes Q^*) U_{23} \widehat{W}_{12}^* U_{23}^* \widehat{W}_{12} \\ &\quad (1 \otimes 1 \otimes P) \widehat{W}_{13} \widehat{W}_{23} (1 \otimes 1 \otimes \xi_h). \end{aligned}$$

Using the fact that  $\widehat{W}_{12}^* U_{23}^* \widehat{W}_{12} = U_{23}^* U_{13}^*$ , we get

$$\begin{aligned} X &= (1 \otimes 1 \otimes \xi_h)^* \widehat{W}_{23}^* (1 \otimes 1 \otimes Q^*) U_{13}^* \\ &\quad (1 \otimes 1 \otimes P) \widehat{W}_{13} \widehat{W}_{23} (1 \otimes 1 \otimes \xi_h) \\ &= (1 \otimes 1 \otimes \xi_h)^* \widehat{W}_{23}^* (1 \otimes 1 \otimes Q^*) U_{13}^* (1 \otimes 1 \otimes P) \widehat{W}_{13} \widehat{W}_{23} (1 \otimes 1 \otimes \xi_h) \\ &= (1 \otimes 1 \otimes \xi_h)^* \widehat{W}_{23}^* (\iota \otimes m_a)(\widehat{W}^*)_{13} \widehat{W}_{13} \widehat{W}_{23} (1 \otimes 1 \otimes \xi_h) \end{aligned}$$

Now we observe that

$$\begin{aligned} (\iota \otimes m_a)(\widehat{W}^*)_{13} &= (\iota \otimes m_a)(\Sigma W \Sigma)_{13} \\ &= \Sigma_{13} (m_a \otimes \iota)(W)_{13} \Sigma_{13} \\ &= \Sigma_{13} ((1 \otimes a)W)_{13} \Sigma_{13} \\ &= (a \otimes 1 \otimes 1) \Sigma_{13} W_{13} \Sigma_{13} \\ &= (a \otimes 1 \otimes 1) \widehat{W}_{13}^*. \end{aligned}$$

Thus,  $X = (1 \otimes 1 \otimes \xi_h)^* \widehat{W}_{23}^* (a \otimes 1 \otimes 1) \widehat{W}_{13}^* \widehat{W}_{13} \widehat{W}_{23} (1 \otimes 1 \otimes \xi_h) = a \otimes 1$ .

Assume now the existence of  $\alpha$  and  $\beta$  satisfying equation (1), then

$$\begin{aligned}
(m_a \otimes \iota)(W) &= (1 \otimes a)W \\
&= \Sigma(a \otimes 1) \widehat{W}^* \Sigma \\
&= \Sigma(1 \otimes \beta^*) \widehat{W}_{12}^* (1 \otimes \alpha) \widehat{W} \widehat{W}^* \Sigma \\
&= \Sigma(1 \otimes \beta^*) \widehat{W}_{12}^* (1 \otimes \alpha) \Sigma \\
&= \Sigma(1 \otimes \beta^*) \Sigma_{12} W_{12} \Sigma_{12} (1 \otimes \alpha) \Sigma \\
&= (\beta^* \otimes 1) \Sigma_{23} W_{12} \Sigma_{23} (\alpha \otimes 1) \\
&= (\beta^* \otimes 1) W_{13} (\alpha \otimes 1).
\end{aligned}$$

Thus,  $x \mapsto \beta^*(x \otimes 1)\alpha$  is a completely bounded extension of  $m_a$  to  $\mathcal{B}(L^2(\mathbb{G}))$ .  $\square$

From this we may deduce a result which is well-known in the classical case.

**Corollary 3.7.** *Let  $\widehat{\mathbb{G}}$  be a discrete quantum group and let  $a \in \ell^\infty(\widehat{\mathbb{G}})$ . The following are equivalent*

- (1)  $m_a$  extends to a completely bounded map on  $\mathcal{B}(L^2(\mathbb{G}))$ .
- (2)  $m_a$  extends to a completely bounded map on  $C_{\text{red}}(\mathbb{G})$ .
- (3)  $m_a$  extends to a completely bounded map on  $L^\infty(\mathbb{G})$ .

Moreover, the completely bounded norms of these maps are all equal.

*Proof.* We only prove the equivalence of (1) and (2), the same argument applies for the equivalence between (1) and (3).

Assume (1), then as  $m_a$  maps  $\mathcal{C}(\mathbb{G})$  into itself, its completely bounded extension restricts to a completely bounded map on  $C_{\text{red}}(\mathbb{G})$  with smaller norm.

Assume (2), then  $\alpha$  and  $\beta$  can still be defined if  $m_a$  is a completely bounded map on  $C_{\text{red}}(\mathbb{G})$  ( $\pi$  being then a representation of  $C_{\text{red}}(\mathbb{G})$ ) and the formula

$$x \mapsto \beta^*(x \otimes 1)\alpha$$

defines a completely bounded map on all of  $\mathcal{B}(L^2(\mathbb{G}))$  with norm less than  $\|\alpha\|\|\beta\|$ .  $\square$

We are now able to give a definition of weak amenability for discrete quantum groups.

**Definition 3.8.** A discrete quantum group  $\widehat{\mathbb{G}}$  is said to be *weakly amenable* if there exists a net  $(a_\lambda)$  of elements of  $\ell^\infty(\widehat{\mathbb{G}})$  such that

- $a_\lambda$  has finite support for all  $\lambda$ .
- $(a_\lambda)$  converges pointwise to 1.
- $K := \limsup_\lambda \|m_{a_\lambda}\|_{cb}$  is finite.

The lower bound of the constants  $K$  for all nets satisfying these properties is denoted  $\Lambda_{cb}(\widehat{\mathbb{G}})$  and called the *Cowling-Haagerup constant* of  $\widehat{\mathbb{G}}$ . By convention,  $\Lambda_{cb}(\widehat{\mathbb{G}}) = \infty$  if  $\widehat{\mathbb{G}}$  is not weakly amenable.

It is clear on the definition that a discrete group  $G$  is weakly amenable in the classical sense if and only if the commutative discrete quantum group  $(C_0(G), \Delta_G)$  is weakly amenable (and the constants are the same). We recall the following notions of weak amenability for operator algebras.

**Definition 3.9.** A  $C^*$ -algebra  $A$  is said to be *weakly amenable* if there exists a net  $(T_\lambda)$  of linear maps from  $A$  to itself such that

- $T_\lambda$  has finite rank for all  $\lambda$ .
- $\|T_\lambda(x) - x\| \rightarrow 0$  for all  $x \in A$ .
- $K := \limsup_\lambda \|T_\lambda\|_{cb}$  is finite.

The lower bound of the constants  $K$  for all nets satisfying these properties is denoted  $\Lambda_{cb}(A)$  and called the *Cowling-Haagerup constant* of  $A$ . By convention,  $\Lambda_{cb}(A) = \infty$  if the  $C^*$ -algebra  $A$  is not weakly amenable.

A von Neumann algebra  $N$  is said to be *weakly amenable* if there exists a net  $(T_\lambda)$  of normal linear maps from  $N$  to itself such that

- $T_\lambda$  has finite rank for all  $\lambda$ .
- $T_\lambda(x) - x \rightarrow 0$  ultraweakly for all  $x \in N$ .
- $K := \limsup_i \|T_\lambda\|_{cb}$  is finite.

The lower bound of the constants  $K$  for all nets satisfying these properties is denoted  $\Lambda_{cb}(N)$  and called the *Cowling-Haagerup constant* of  $N$ . By convention,  $\Lambda_{cb}(N) = \infty$  if the von Neumann algebra  $N$  is not weakly amenable.

*Remark 3.10.* Note that given a von Neumann algebra  $N$ , its Cowling-Haagerup constant as a  $C^*$ -algebra need not be equal to its Cowling-Haagerup constant as a von Neumann algebra. For instance,  $\mathcal{B}(\ell^2(\mathbb{N}))$  is weakly amenable as a von Neumann algebra (it is even amenable) but not as a  $C^*$ -algebra (it is not even exact). Except otherwise stated,  $\Lambda_{cb}(N)$  will always denote the Cowling-Haagerup constant of  $N$  as a von Neumann algebra.

The following theorem was first proved by J. Kraus and Z.-J. Ruan in [KR99] for discrete quantum groups of *Kac type* (i.e. with the Haar state on  $\mathbb{G}$  being tracial). We give here a proof which works for any discrete quantum group.

**Theorem 3.11.** *Let  $\widehat{\mathbb{G}}$  be a discrete quantum group, then*

$$\Lambda_{cb}(\widehat{\mathbb{G}}) = \Lambda_{cb}(C_{red}(\mathbb{G})) = \Lambda_{cb}(L^\infty(\mathbb{G})).$$

*Proof.* Let  $(a_\lambda)$  be a net satisfying the hypothesis of definition 3.8, then the maps  $m_{a_\lambda}$  are unital normal finite rank uniformly completely bounded maps and converge pointwise to the identity, giving

$$\Lambda_{cb}(L^\infty(\mathbb{G})) \leq \Lambda_{cb}(\widehat{\mathbb{G}}) \text{ and } \Lambda_{cb}(C_{red}(\mathbb{G})) \leq \Lambda_{cb}(\widehat{\mathbb{G}}).$$

Let now  $(T_\lambda)$  be a net of unital finite rank uniformly completely bounded maps on  $L^\infty(\mathbb{G})$  converging pointwise to the identity and set

$$a_\lambda := (h \otimes \iota)((T_\lambda \otimes \iota)(W)W^*) \in \ell^\infty(\widehat{\mathbb{G}}).$$

These are elements converging to 1 pointwise. Let us prove that we can perturb the maps  $T_\lambda$  so that the  $a_\lambda$ 's have finite support. Let  $(\xi_1, \dots, \xi_n)$  be an orthonormal basis of the image of  $T_\lambda$  and set  $\omega_i(x) := \langle \xi_i, T_\lambda(x) \rangle$ . Then  $\omega_i$  is a normal linear form on  $L^\infty(\mathbb{G})$  and for all  $x \in L^\infty(\mathbb{G})$ ,  $T_\lambda(x) = \sum_{i=1}^n \omega_i(x) \xi_i$ . Let  $\eta > 0$  and chose elements  $\zeta_i \in \mathcal{C}(\mathbb{G})$  such that

$$\|\zeta_i - \xi_i\| \leq \frac{\eta}{\sup \|\omega_i\|}.$$

Then the equation  $\tilde{T}_\lambda(x) = \sum \omega_i(x) \zeta_i$  defines a completely bounded finite rank map on  $L^\infty(\mathbb{G})$  such that  $\|T_\lambda - \tilde{T}_\lambda\|_{cb} \leq \eta$  and giving rise to an element of  $\ell^\infty(\widehat{\mathbb{G}})$  with finite support. To study the completely bounded norm of  $m_{a_\lambda}$ , first remark that the co-product  $\Delta$  is an isometry from  $\mathcal{C}(\mathbb{G})$  onto its image with respect to the scalar product  $\langle a, b \rangle = h(ab^*)$ . Hence, we can extend it to an isometry from  $L^2(\mathbb{G})$  to  $L^2(\mathbb{G}) \otimes L^2(\mathbb{G})$  and then consider its adjoint operator  $\Delta^*$ . With these considerations, the following formula holds :

$$(2) \quad m_{a_\lambda} = \Delta^* \circ (T_\lambda \otimes \iota) \circ \Delta.$$

To prove this equality, let  $\alpha$  and  $\beta$  be two irreducible representations of  $\mathbb{G}$  and recall (see e.g. [Tim08, Prop 5.3.8]) that

$$h(u_{i,j}^\alpha (u_{l,m}^\beta)^*) = \delta_{\alpha,\beta} \delta_{i,l} \frac{F_{j,m}^\alpha}{\dim_q(\alpha)}$$

for some coefficients  $F_{j,m}^\alpha$  which are equal to  $\delta_{j,m}$  if and only if the group is of Kac type. We compute on the one hand

$$\begin{aligned} \langle \Delta^* \circ (T_\lambda \otimes \iota) \circ \Delta(u_{i,j}^\alpha), u_{l,m}^\beta \rangle &= \sum_k \langle (T_\lambda(u_{i,k}^\alpha) \otimes u_{k,j}^\alpha), \Delta(u_{l,m}^\beta) \rangle \\ &= \sum_{k,t} h(T_\lambda(u_{i,k}^\alpha) (u_{l,t}^\beta)^*) h(u_{k,j}^\alpha (u_{t,m}^\beta)^*) \\ &= \sum_k h(T_\lambda(u_{i,k}^\alpha) (u_{l,k}^\alpha)^*) \delta_{\alpha,\beta} \frac{F_{j,m}^\alpha}{\dim_q(\alpha)} \\ &= \delta_{\alpha,\beta} \frac{F_{j,m}^\alpha}{\dim_q(\alpha)} \sum_k h((1 \otimes e_i^*)(T_\lambda \otimes \iota)(u^\alpha)(1 \otimes e_k) \\ &\quad (1 \otimes e_k^*)(u^\alpha)^*(1 \otimes e_l)) \\ &= \delta_{\alpha,\beta} \frac{F_{j,m}^\alpha}{\dim_q(\alpha)} h((1 \otimes e_i^*)(T_\lambda \otimes \iota)(u^\alpha)(u^\alpha)^*(1 \otimes e_l)) \\ &= \delta_{\alpha,\beta} \frac{F_{j,m}^\alpha}{\dim_q(\alpha)} h((1 \otimes e_i^*)(1 \otimes a_\lambda p_\alpha)(1 \otimes e_l)) \\ &= \delta_{\alpha,\beta} \frac{F_{j,m}^\alpha}{\dim_q(\alpha)} \langle e_i, a_\lambda p_\alpha e_l \rangle \end{aligned}$$

and on the other hand

$$\begin{aligned}
 \langle m_{a_\lambda}(u_{i,j}^\alpha), u_{l,m}^\beta \rangle &= h(m_{a_\lambda}(u_{i,j}^\alpha)(u_{l,m}^\beta)^*) \\
 &= h((1 \otimes e_i^*)(1 \otimes a_\lambda p_\alpha) u^\alpha (1 \otimes e_j)(u_{l,m}^\beta)^*) \\
 &= \sum_k h((1 \otimes e_i^*)(1 \otimes a_\lambda p_\alpha)(1 \otimes e_k)(1 \otimes e_k^*) u^\alpha (1 \otimes e_j)(u_{l,m}^\beta)^*) \\
 &= \sum_k \langle a_\lambda^* p_\alpha e_i, e_k \rangle h(u_{k,j}^\alpha (u_{l,m}^\beta)^*) \\
 &= \langle e_i, a_\lambda p_\alpha e_l \rangle \delta_{\alpha,\beta} \frac{F_{j,m}^\alpha}{\dim_q(\alpha)}.
 \end{aligned}$$

Equation 2 implies that  $\|m_{a_\lambda}\|_{cb} \leq \|T_\lambda\|_{cb}$ , yielding

$$\Lambda_{cb}(\widehat{\mathbb{G}}) \leq \Lambda_{cb}(L^\infty(\mathbb{G})) \text{ and } \Lambda_{cb}(\widehat{\mathbb{G}}) \leq \Lambda_{cb}(C_{\text{red}}(\mathbb{G})).$$

□

**Corollary 3.12.** *Let  $\widehat{\mathbb{G}}$  be a weakly amenable discrete quantum group, then  $\widehat{\mathbb{G}}$  is exact.*

*Proof.* Any weakly amenable C\*-algebra is exact according to [KW99]. Combining this with the fact from [Bla01, Prop. 4.1] that a discrete quantum group  $\widehat{\mathbb{G}}$  is exact if and only if  $C_{\text{red}}(\mathbb{G})$  is an exact C\*-algebra yields the result □

As a consequence of Theorem 3.11, we can deduce a few permanence properties. First of all, we can consider discrete quantum subgroups in the following sense : let  $\mathbb{G}$  be a compact quantum group and let  $C(\mathbb{H})$  be a unital C\*-subalgebra of  $C(\mathbb{G})$  which becomes a compact quantum group when endowed with the restriction of the co-product of  $\mathbb{G}$  (this means in particular that  $\Delta(C(\mathbb{H})) \subseteq C(\mathbb{H}) \otimes C(\mathbb{H})$ ), then  $\widehat{\mathbb{H}}$  will be called a *discrete quantum subgroup* of  $\widehat{\mathbb{G}}$ . According to [Ver04, Lemma 2.2], there is a unique conditional expectation from  $C(\mathbb{G})$  to  $C(\mathbb{H})$ . Thus we have  $\Lambda_{cb}(\mathbb{H}) \leq \Lambda_{cb}(\mathbb{G})$ .

For any two reduced compact quantum groups  $\mathbb{G}$  and  $\mathbb{H}$ , there is a unique compact quantum group structure on the spatial tensor product  $C(\mathbb{G}) \otimes C(\mathbb{H})$  turning  $C(\mathbb{G})$  and  $C(\mathbb{H})$  into sub-Hopf-C\*-algebras under the canonical inclusions, which is defined in [Wan95b]. By analogy with the classical case, the dual of this compact quantum group will be called the *direct product* of  $\widehat{\mathbb{G}}$  and  $\widehat{\mathbb{H}}$  and denoted  $\widehat{\mathbb{G}} \times \widehat{\mathbb{H}}$ .

**Corollary 3.13.** *Let  $\widehat{\mathbb{G}}$  and  $\widehat{\mathbb{H}}$  be two discrete quantum groups, then*

$$\Lambda_{cb}(\widehat{\mathbb{H}} \times \widehat{\mathbb{G}}) = \Lambda_{cb}(\widehat{\mathbb{H}}) \Lambda_{cb}(\widehat{\mathbb{G}}).$$

*Proof.* It is true for any two C\*-algebras A and B that  $\Lambda_{cb}(A \otimes B) = \Lambda_{cb}(A) \Lambda_{cb}(B)$  (see [BO08, Thm 12.3.13]). This, combined with Theorem 3.11, gives the result. □

Let  $(\widehat{\mathbb{G}}_i)$  be a family of discrete quantum groups together with \*-homomorphisms

$$\pi_{i,j} : C(\mathbb{G}_i) \rightarrow C(\mathbb{G}_j)$$

intertwining the comultiplications and satisfying  $\pi_{j,k} \circ \pi_{i,j} = \pi_{i,k}$ . We will call the data  $(\widehat{\mathbb{G}}_i, \pi_{i,j})$  an *inductive system of discrete quantum groups*. The inductive limit  $C^*$ -algebra  $C(\widehat{\mathbb{G}})$  of this system can be endowed with a natural compact quantum group structure, see for example [BGS11, Lemma 1.1]. We will say that  $\widehat{\mathbb{G}}$  is the *inductive limit* of the system  $(\widehat{\mathbb{G}}_i, \pi_{i,j})$ .

Assume the maps  $\pi_{i,j}$  to be injective, then we can identify the  $C(\mathbb{G}_i)$ 's with sub-Hopf- $C^*$ -algebras of  $C(\widehat{\mathbb{G}})$  satisfying

$$\overline{\bigcup C(\mathbb{G}_i)} = C(\widehat{\mathbb{G}}).$$

This implies that any irreducible representation of some  $\mathbb{G}_i$  yields an irreducible representation of  $\widehat{\mathbb{G}}$ . Moreover,

$$\mathcal{A} := \bigcup_i \mathcal{C}(\mathbb{G}_i)$$

is a dense Hopf- $*$ -subalgebra of  $C(\widehat{\mathbb{G}})$  spanned by coefficients of irreducible representations. Because of Shur's orthogonality relations, this implies that the coefficients of all irreducible representations of  $\widehat{\mathbb{G}}$  are in  $\mathcal{A}$ , i.e.  $\mathcal{A} = \mathcal{C}(\widehat{\mathbb{G}})$ . This means that any irreducible representation of  $\widehat{\mathbb{G}}$  comes from an irreducible representation of some  $\mathbb{G}_i$ .

**Corollary 3.14.** *Let  $(\widehat{\mathbb{G}}_i, \pi_{i,j})$  be an inductive system of discrete quantum groups with inductive limit  $\widehat{\mathbb{G}}$  and limit maps  $\pi_i : C(\mathbb{G}_i) \rightarrow C(\widehat{\mathbb{G}})$ , then*

$$\sup_i \Lambda_{cb}(\pi_i(C_{red}(\mathbb{G}_i))) = \Lambda_{cb}(\widehat{\mathbb{G}}).$$

*In particular, if all the connecting maps are injective, the inductive limit is weakly amenable if and only if the quantum groups are all weakly amenable with uniformly bounded Cowling-Haagerup constant.*

*Proof.* The inequality  $\sup_i \Lambda_{cb}(\pi_i(C_{red}(\mathbb{G}_i))) \leq \Lambda_{cb}(\widehat{\mathbb{G}})$  is straightforward from the fact that  $(\pi_i(C_{red}(\mathbb{G}_i)), \Delta)$  is the dual of a discrete quantum subgroup of  $\widehat{\mathbb{G}}$ . The second one can be seen as a consequence of the following more general result.  $\square$

**Proposition 3.15.** *Let  $(A_i, \pi_{i,j})$  be a direct system of  $C^*$ -algebras such that for each  $i$ , there is a conditional expectation  $\mathbb{E}_i$  from the inductive limit  $A$  onto the subalgebra  $\pi_i(A_i)$ . Then  $\Lambda_{cb}(A) \leq \sup_i \Lambda_{cb}(\pi_i(A_i))$ .*

*Proof.* The proof is certainly well-known, but we give it for completeness. Let  $\varepsilon > 0$  and  $\mathcal{F} \subseteq A$  be a finite subset, set  $\Lambda = \sup_i (\Lambda_{cb}(\pi_i(A_i)))$  and

$$\eta = \frac{\sqrt{(2 + \Lambda)^2 + 4\varepsilon} - (2 + \Lambda)}{2}.$$

We can see  $A$  as the closure of the union of the  $\pi_i(A_i)$  (see for example [RLL00, Prop 6.2.4]), thus there is an index  $i_0$  and a finite subset  $\mathcal{G}$  of  $A_{i_0}$  such that  $d(\mathcal{F}, \mathcal{G}) \leq \eta$ . Let  $T$  be a finite rank linear map from  $\pi_{i_0}(A_{i_0})$  to itself approximating the identity up to  $\eta$  on  $\mathcal{G}$  and with

$$\|T\|_{cb} \leq \Lambda + \varepsilon.$$

Set  $T_{\mathcal{F},\varepsilon} = T \circ E_i$ . This is a finite rank linear maps from  $A$  to itself with  $\|T_{\mathcal{F},\varepsilon}\|_{cb} \leq \Lambda + \eta$ . Moreover, for any  $x \in \mathcal{F}$ , if  $y$  is an element of  $\mathcal{G}$  such that  $\|x - y\| \leq \eta$ , one has

$$\begin{aligned} \|T_{\mathcal{F},\varepsilon}(x) - x\| &= \|T \circ E_{i_0}(x) - x\| \\ &\leq \|T \circ E_{i_0}(y) - y\| + \|T \circ E_{i_0}(x - y) - (x - y)\| \\ &\leq \eta + \|T \circ E_{i_0}\| \|x - y\| + \|x - y\| \\ &\leq \eta + (\Lambda + \eta)\eta + \eta \\ &= \eta(2 + \Lambda + \eta) \\ &\leq \varepsilon. \end{aligned}$$

As  $\eta$  tends to 0 when  $\varepsilon$  tends to 0, we get the desired result.  $\square$

#### 4. FREE PRODUCTS

Recall that given two discrete quantum groups  $\widehat{\mathbb{G}}$  and  $\widehat{\mathbb{H}}$ , there is a unique compact quantum group structure on the reduced free product  $C(\mathbb{G}) * C(\mathbb{H})$  with respect to the Haar states turning  $C(\mathbb{G})$  and  $C(\mathbb{H})$  into sub-Hopf- $C^*$ -algebras under the canonical inclusions, which is defined in [Wan95a]. By analogy with the classical case, the dual of this compact quantum group will be called the *reduced free product* of  $\widehat{\mathbb{G}}$  and  $\widehat{\mathbb{H}}$  and denoted  $\widehat{\mathbb{G}} * \widehat{\mathbb{H}}$ .

The fact that a free product of amenable groups has Cowling-Haagerup constant equal to 1 (though it may not be amenable) was first proved by M. Bożejko and M.A. Picardello in [BP93] (even allowing amalgamation over a finite subgroup). But it can also be recovered as an easy consequence of the following theorem [RX06, Thm 4.3].

**Theorem 4.1** (Ricard, Xu). *Let  $(A_i, \varphi_i)_{i \in I}$  be  $C^*$ -algebras with distinguished states  $(\varphi_i)$  having faithful GNS construction. Assume that for each  $i$ , there is a net of finite rank unital completely positive maps  $(V_{i,j})$  on  $A_i$  converging to the identity pointwise and preserving the state (i.e.  $\varphi_i$  is cp-approximable in the sense of [Eck10, Def 1.1]). Then, the reduced free product of the family  $(A_i, \varphi_i)$  has Cowling-Haagerup constant equal to 1.*

Thus we only need to find such a net of unital completely positive maps which leaves the Haar states invariant when the discrete quantum groups are amenable (i.e. the reduced form of their duals admit a bounded counit). This is given by the following characterization of amenability [Tom06, Thm 3.8].

**Theorem 4.2** (Tomatsu). *A discrete quantum group  $\widehat{\mathbb{G}}$  is amenable if and only if there is a net  $(\omega_j)$  of states on  $C_{red}(\mathbb{G})$  such that the nets of completely positive maps  $((\omega_j \otimes \iota) \circ \Delta)$  and  $((\iota \otimes \omega_j) \circ \Delta)$  converge pointwise to the identity.*

The  $h$ -invariance of these maps is given by the left and right invariance of the Haar state on compact quantum groups. Thus we have the following :

**Corollary 4.3.** *Let  $(\widehat{\mathbb{G}}_i)_{i \in I}$  be a family of amenable discrete quantum groups, then*

$$\Lambda_{cb}(*_{i \in I} \widehat{\mathbb{G}}_i) = 1.$$

**Example 4.4.** Let  $(G_i)_{i \in I}$  be any family of compact groups, then their duals in the sense of quantum groups are amenable. Thus  $*_{i \in I}(C(G_i))$  is the dual of a non classical (non-commutative and non-cocommutative) discrete quantum group with Cowling-Haagerup constant equal to 1.

We will now prove that a free product of weakly amenable discrete quantum groups with Cowling-Haagerup constant equal to 1 has Cowling-Haagerup constant equal to 1. This result has been proved in the classical case by E. Ricard and Q. Xu [RX06, Thm 4.3] using the following key result [RX06, Prop 4.11].

**Theorem 4.5** (Ricard, Xu). *Let  $(B_i, \psi_i)_{i \in I}$  be unital  $C^*$ -algebras with distinguished states  $(\psi_i)$  having faithful GNS constructions. Let  $A_i \subseteq B_i$  be unital  $C^*$ -subalgebras such that the states  $\varphi_i = \psi_i|_{A_i}$  also have faithful GNS construction. Assume that for each  $i$ , there is a net of finite rank maps  $(V_{i,j})$  on  $A_i$  converging to the identity pointwise, preserving the state and such that  $\limsup_j \|V_{i,j}\|_{cb} = 1$ . Assume moreover that for each pair  $(i, j)$ , there is a completely positive unital map  $U_{i,j} : A_i \rightarrow B_i$  preserving the state and such that*

$$\|V_{i,j} - U_{i,j}\|_{cb} + \|V_{i,j} - U_{i,j}\|_{\mathcal{B}(L^2(A_i, \varphi_i), L^2(B_i, \psi_i))} + \|V_{i,j} - U_{i,j}\|_{\mathcal{B}(L^2(A_i, \varphi_i)^{op}, L^2(B_i, \psi_i)^{op})} \rightarrow 0.$$

*Then, the reduced free product of the family  $(A_i, \varphi_i)$  has Cowling-Haagerup constant equal to 1.*

Having a characterization of weak amenability in terms of approximation of the identity on the group is the main ingredient to apply this theorem. Thanks to Theorem 3.6, we can prove a quantum version.

**Theorem 4.6.** *Let  $(\widehat{G}_i)_{i \in I}$  be a family of discrete quantum groups with Cowling-Haagerup constant equal to 1, then  $\Lambda_{cb}(*_{i \in I} \widehat{G}_i) = 1$*

*Proof.* Let  $\widehat{G}$  be a discrete quantum group, let  $0 < \eta < 1$  and let  $a \in \ell^\infty(\widehat{G})$  be such that  $\|m_a\|_{cb} \leq 1 + \eta$ . Note that the  $\alpha$  and  $\beta$  given by Theorem 3.6 can be both chosen to have norm less than  $\sqrt{1 + \eta}$ . We set  $\gamma = (\alpha + \beta)/2$  and  $\delta = (\alpha - \beta)/2$ .

Observing that  $m_a(1) = (m_a \otimes \iota)(u^\varepsilon) = (1 \otimes a p_\varepsilon)u^\varepsilon = \widehat{\varepsilon}(a).1$  and assuming  $\widehat{\varepsilon}(a)$  to be non-zero, we can divide  $a$  by it so that  $m_a$  becomes unital (this ensures that  $m_a$  preserves the vector state but we will prove it later on). We also know from [KR99, Prop 2.6] that for any  $x \in C_{\text{red}}(\widehat{G})$ ,

$$\alpha^*(x \otimes 1)\beta = m_a(x^*)^* = m_{\widehat{S}(a)^*}(x).$$

Thus we can, up to replacing  $a$  by  $\frac{1}{2}(a + \widehat{S}(a)^*)$  and using the fact that  $\widehat{S} \circ * \circ \widehat{S} \circ * = \iota$ , assume that

$$m_a(x) = \frac{1}{2}(m_a(x) + m_{\widehat{S}(a)^*}(x)) = \frac{1}{2}((\beta^*(x \otimes 1)\alpha + \alpha^*(x \otimes 1)\beta)) = M_\gamma(x) - M_\delta(x),$$

where  $M_\gamma(x) = \gamma^*(x \otimes 1)\gamma$  and  $M_\delta(x) = \delta^*(x \otimes 1)\delta$ . The maps  $M_\gamma$  and  $M_\delta$  are completely positive thus  $\|M_\gamma\|_{cb} = \|\gamma\|^2 \leq 1 + \eta$  and evaluating at 1 gives  $\|1 + \delta^* \delta\| \leq 1 + \eta$ , i.e.  $\|M_\delta\|_{cb} = \|\delta^* \delta\| \leq \eta$ .

We now want to perturb  $M_\gamma$  into a *unital* completely positive map. To do this, first note that

$$\|1 - \gamma^* \gamma\| = \|\delta^* \delta\| \leq \eta < 1,$$

which implies that  $\gamma^* \gamma$  is invertible, and set  $\tilde{\gamma} = \gamma |\gamma|^{-1}$  where  $|\gamma| = (\gamma^* \gamma)^{1/2}$ . Note that  $\|\tilde{\gamma} - \gamma\| \leq \eta$ . Thus,  $M_{\tilde{\gamma}}$  is a unital completely positive map and

$$\begin{aligned} \|M_{\tilde{\gamma}} - M_\gamma\|_{cb} &= \|M_{\gamma+(\tilde{\gamma}-\gamma)} - M_\gamma\|_{cb} \\ &\leq \|\tilde{\gamma} - \gamma\| \|\gamma\| + \|\tilde{\gamma} - \gamma\| \|\gamma\| + \|\tilde{\gamma} - \gamma\| \|\tilde{\gamma} - \gamma\| \\ &\leq \eta(2 + 3\eta) \leq 5\eta. \end{aligned}$$

This proves that  $M_{\tilde{\gamma}}$  is a unital completely positive map approximating  $m_a$  on  $C(\mathbb{G})$  up to  $6\eta$  in completely bounded norm.

We now have to prove that  $M_{\tilde{\gamma}}$  also approximates  $m_a$  on  $\mathcal{B}(L^2(\mathbb{G}))$ . Let us denote by  $\tau$  the vector state on  $\mathcal{B}(L^2(\mathbb{G}))$  associated to  $\xi_h$ :

$$\tau(x) = \langle x(\xi_h), \xi_h \rangle.$$

**Lemma 4.7.** *Let  $T$  be any bounded linear operator on  $K$  and set*

$$A(T) = (\iota \otimes \pi)(\widehat{W})^*(1 \otimes T)\widehat{W}(\iota \otimes \xi_h) \in \mathcal{B}(L^2(\mathbb{G})).$$

*Then  $\|A(T)\| \leq \|T\|$ ,  $M_{A(T)}$  is a bounded operator on  $(\mathcal{B}(L^2(\mathbb{G}), \tau)$  of norm less than  $\|T\|^2$  and  $\tau(M_{A(T)}(x^*x)) \leq \|T\|^2 \tau(x^*x)$ . If moreover  $A(T)^*A(T)$  is invertible, then  $M_{A(T)|A(T)|^{-1}}$  is  $\tau$ -invariant.*

*Proof.* The inequality  $\|A(T)\| \leq \|T\|$  is obvious since  $\widehat{W}$  and  $(\iota \otimes \pi)(\widehat{W})$  are unitary operators. Note that since by definition  $W(\xi \otimes \xi_h) = \xi \otimes \xi_h$  for any  $\xi \in H$ , we also have  $\widehat{W}(\xi_h \otimes \xi) = \xi_h \otimes \xi$ . Let us prove that  $(\iota \otimes \pi)(\widehat{W})(\xi_h \otimes \zeta) = \xi_h \otimes \zeta$  for any  $\zeta \in K$ . First, for any  $\theta_1, \theta_2, \xi \in L^2(\mathbb{G})$ ,

$$\begin{aligned} \langle (\iota \otimes \omega_{\theta_1, \theta_2})(\widehat{W})\xi_h, \xi \rangle &= \langle \widehat{W}(\xi_h \otimes \theta_1), \xi \otimes \theta_2 \rangle \\ &= \langle \xi_h, \xi \rangle \langle \theta_1, \theta_2 \rangle \\ &= \omega_{\theta_1, \theta_2}(1) \langle \xi_h, \xi \rangle. \end{aligned}$$

Thus by density, we have  $\langle (\iota \otimes \omega)(\widehat{W})\xi_h, \xi \rangle = \omega(1) \langle \xi_h, \xi \rangle$  for any  $\omega \in \mathcal{B}(L^2(\mathbb{G}))_*$ . Secondly, let  $\zeta_1, \zeta_2 \in K$  and  $\xi \in L^2(\mathbb{G})$ , then

$$\begin{aligned} \langle (\iota \otimes \pi)(\widehat{W})(\xi_h \otimes \zeta_1), \xi \otimes \zeta_2 \rangle &= \langle (\iota \otimes \omega_{\zeta_1, \zeta_2} \circ \pi)(\widehat{W})\xi_h, \xi \rangle \\ &= \omega_{\zeta_1, \zeta_2}(\pi(1)) \langle \xi_h, \xi \rangle = \langle \xi_h \otimes \zeta_1, \xi \otimes \zeta_2 \rangle. \end{aligned}$$

Now we can compute

$$\begin{aligned} A(T)\xi_h &= (\iota \otimes \pi)(\widehat{W})^*(1 \otimes T)\widehat{W}(\xi_h \otimes \xi_h) \\ &= (\iota \otimes \pi)(\widehat{W})^*(1 \otimes T)(\xi_h \otimes \xi_h) \\ &= (\iota \otimes \pi)(\widehat{W})^*(\xi_h \otimes T(\xi_h)) \\ &= \xi_h \otimes T(\xi_h). \end{aligned}$$

Thus,  $\langle A(T)^*(x \otimes 1)A(T)\xi_h, \xi_h \rangle = \langle (x \otimes 1)A(T)(\xi_h), A(T)\xi_h \rangle = \langle x(\xi_h), \xi_h \rangle \|T(\xi_h)\|^2$  and using Kadison's inequality we get

$$\begin{aligned} \tau(M_{A(T)}(x)^* M_{A(T)}(x)) &\leq \|A(T)\|^2 \tau(M_{A(T)}(x^* x)) \\ &\leq \|T\|^2 \|A(T)\|^2 \tau(x^* x) \\ &\leq \|T\|^4 \tau(x^* x). \end{aligned}$$

Let us now turn to  $A(T)^* A(T)$ . First,

$$\begin{aligned} A(T)^* A(T)\xi_h &= (\iota \otimes \xi_h^*) \widehat{W}^* (1 \otimes T^*) (\iota \otimes \pi)(\widehat{W})(\xi_h \otimes T(\xi_h)) \\ &= (\iota \otimes \xi_h^*) \widehat{W}^* (1 \otimes T^*) (\xi_h \otimes T(\xi_h)) \\ &= (\iota \otimes \xi_h^*) \widehat{W}^* (\xi_h \otimes T^* T(\xi_h)) \\ &= (\iota \otimes \xi_h^*) (\xi_h \otimes T^* T(\xi_h)) \\ &= \langle \xi_h, T^* T(\xi_h) \rangle \xi_h \\ &= \|T(\xi_h)\|^2 \xi_h \end{aligned}$$

and  $\xi_h$  is an eigenvector for  $A(T)^* A(T)$ . If  $A(T)^* A(T)$  is invertible, then

$$(A(T)^* A(T))^{-1/2} \xi_h = \|T(\xi_h)\|^{-1} \xi_h.$$

Thus  $A(T)|A(T)|^{-1} \xi_h = \xi_h \otimes \|T(\xi_h)\|^{-1} T(\xi_h)$  and

$$\begin{aligned} \tau(M_{A(T)|A(T)|^{-1}}(x)) &= \langle (x \otimes 1)A(T)|A(T)|^{-1} \xi_h, A(T)|A(T)|^{-1} \xi_h \rangle \\ &= \langle x(\xi_h), \xi_h \rangle \\ &= \tau(x). \end{aligned}$$

□

For  $x \in \mathcal{B}(L^2(\mathbb{G}))$ , we set  $\|x\|_2 = \tau(x^* x)^{1/2}$ . Using Lemma 4.7 with  $\delta = A((P - Q)/2)$ , we obtain

$$\|(m_a - M_\gamma)(x)\|_2^2 = \|M_\delta(x)\|_2^2 = \tau(M_\delta(x)^* M_\delta(x)) \leq \|\delta\|^4 \|x\|_2^2 \leq \eta^4 \|x\|_2^2$$

i.e.  $\|(m_a - M_\gamma)(x)\|_2 \leq \eta^2 \|x\|_2$ . We also have  $\gamma = A((P + Q)/2)$ , thus, setting  $T = ((P + Q)/2)$  and observing that

$$\left\| \left( T\xi_h - \frac{1}{\|T\xi_h\|} T\xi_h \right) \right\| = \left\| \xi_h \otimes \left( T\xi_h - \frac{1}{\|T\xi_h\|} T\xi_h \right) \right\| = \|(\gamma - \tilde{\gamma})\xi_h\| \leq \|\gamma - \tilde{\gamma}\| \leq \eta,$$

we can compute again with Lemma 4.7

$$\begin{aligned}
 \tau(M_{\gamma-\tilde{\gamma}}(x^*x)) &\leq \langle (x^*x \otimes 1)(\gamma - \tilde{\gamma})\xi_h, (\gamma - \tilde{\gamma})\xi_h \rangle \\
 &= \left\langle (x^*x \otimes 1) \left( \xi_h \otimes \left( T\xi_h - \frac{1}{\|T\xi_h\|} T\xi_h \right) \right), \xi_h \otimes \left( T\xi_h - \frac{1}{\|T\xi_h\|} T\xi_h \right) \right\rangle \\
 &= \langle (x^*x)\xi_h, \xi_h \rangle \left\| T\xi_h - \frac{1}{\|T\xi_h\|} T\xi_h \right\|^2 \\
 &\leq \eta^2 \tau(x^*x) \\
 &\leq \eta^2 \|x\|_2^2,
 \end{aligned}$$

thus  $\|M_{\gamma-\tilde{\gamma}}(x)\|_2^2 = \tau(M_{\gamma-\tilde{\gamma}}(x)^*M_{\gamma-\tilde{\gamma}}(x)) \leq \|\gamma - \tilde{\gamma}\|^2 \tau(M_{\gamma-\tilde{\gamma}}(x^*x)) \leq \eta^4 \|x\|_2^2$ . Now, for all  $x \in \mathcal{B}(L^2(\mathbb{G}))$ , we have

$$\begin{aligned}
 \|M_\gamma(x) - M_{\tilde{\gamma}}(x)\|_2 &= \|M_\gamma(x) - M_{\gamma+(\tilde{\gamma}-\gamma)}(x)\|_2 \\
 &\leq \|M_{\tilde{\gamma}-\gamma}(x)\|_2 + \tau((\tilde{\gamma}-\gamma)^*(x^* \otimes 1)\gamma\gamma^*(x \otimes 1)(\tilde{\gamma}-\gamma))^{1/2} \\
 &\quad + \tau(\gamma^*(x^* \otimes 1)(\tilde{\gamma}-\gamma)(\tilde{\gamma}-\gamma)^*(x \otimes 1)\gamma)^{1/2} \\
 &\leq \eta^2 \|x\|_2 + (1+\eta)\tau((\tilde{\gamma}-\gamma)^*(x^* \otimes 1)(x \otimes 1)(\tilde{\gamma}-\gamma))^{1/2} \\
 &\quad + \eta\tau(\gamma^*(x^* \otimes 1)(x \otimes 1)\gamma)^{1/2} \\
 &\leq \eta^2 \|x\|_2 + (1+\eta)\tau(M_{\tilde{\gamma}-\gamma}(x^*x))^{1/2} + \eta\tau(M_\gamma(x^*x))^{1/2} \\
 &\leq \eta^2 \|x\|_2 + (1+\eta)\eta\|x\|_2 + \eta(1+\eta)\|x\|_2 \\
 &\leq 5\eta\|x\|_2.
 \end{aligned}$$

Finally,  $\|(m_a - M_{\tilde{\gamma}})(x)\|_2 \leq 6\eta\|x\|_2$ . Moreover, by the last part of Lemma 4.7,  $M_{\tilde{\gamma}}$  preserves  $\tau$ . We also get that  $m_a$  is state preserving since

$$\begin{aligned}
 \tau(m_a(x)) &= \langle \beta^*(x \otimes 1)\alpha(\xi_h), \xi_h \rangle \\
 &= \langle (x \otimes 1)\alpha(\xi_h), \beta(\xi_h) \rangle \\
 &= \langle (x \otimes 1)T(P)(\xi_h), T(Q)(\xi_h) \rangle \\
 &= \langle (x \otimes 1)(\xi_h \otimes P(\xi_h), \xi_h \otimes Q(\xi_h)) \rangle \\
 &= \langle x(\xi_h), \xi_h \rangle \langle P(\xi_h), Q(\xi_h) \rangle \\
 &= \tau(x)\tau(m_a(1))
 \end{aligned}$$

and we have assumed that  $m_a(1) = 1$ .

We can now prove the theorem. For each  $i$ , set  $A_i = C_{\text{red}}(\mathbb{G}_i)$  and  $B_i = \mathcal{B}(L^2(\mathbb{G}_i))$ . Consider a net  $(a_{i,j})_j$  of finite rank elements in  $\ell^\infty(\widehat{\mathbb{G}}_i)$  converging pointwise to the identity and such that  $\limsup_j \|m_{a_{i,j}}\|_{cb} \leq 1$  and note that since  $\widehat{e}(a_{i,j}) \rightarrow 1$  (because of the pointwise convergence assumption), we can, up to extracting a suitable subsequence, assume it to be non-zero. For any  $0 < \eta < 1$ , there is a  $j(\eta)$  such that  $\|m_{a_{i,j(\eta)}}\|_{cb} \leq 1 + \eta$  (the same being automatically true for  $m_{\widehat{S}(a_{i,j(\eta)})}$ ). The procedure above then yields a unital completely positive map approximating  $m_{a_{i,j(\eta)}}$  up to  $6\eta$  in

completely bounded norm and in  $L^2$ -operator norm. Applying Theorem 4.5 proves that  $\Lambda_{cb}(*_i A_i) = 1$  and Theorem 3.11 gives the desired conclusion.  $\square$

**Example 4.8.** The free orthogonal quantum groups  $\widehat{A_o(F)}$  are amenable for any  $F$  in  $GL(2, \mathbb{C})$  such that  $\overline{FF} \in \mathbb{R}.Id$ , thus their Cowling-Haagerup constant is equal to 1. Moreover,  $\widehat{A_u(F)}$  can be seen as a subgroup of  $\mathbb{Z} * \widehat{A_o(F)}$ . Thus  $\Lambda_{cb}(\widehat{A_u(F)}) = 1$  and any free product of some 2-dimensional free quantum groups with duals of compact groups has Cowling-Haagerup constant equal to 1.

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