

HERMITIAN SYMMETRIC SPACES OF TUBE TYPE
AND MULTIVARIATE MEIXNER-POLLACZEK POLYNOMIALS

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Abstract Harmonic analysis on Hermitian symmetric spaces of tube type is a natural framework for introducing multivariate Meixner-Pollaczek polynomials. Their main properties are established in this setting: orthogonality, generating and determinantal formulae, difference equations. Furthermore, as a by-product, we derive the radial part of the differential equation for the multivariate Laguerre functions and obtain the differential equation for multivariate Laguerre polynomials previously obtained by Baker and Forrester.

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The one variable Meixner-Pollaczek polynomials $P_m^\alpha(\lambda; \phi)$ can be defined by the Gaussian hypergeometric representation as

$$P_m^{(\frac{\nu}{2})}(\lambda; \phi) = \frac{(\nu)_m}{m!} e^{im\theta} {}_2F_1\left(-m, \frac{\nu}{2} + i\lambda; \nu; 1 - e^{-2i\phi}\right).$$

For $\phi = \frac{\pi}{2}$ the Meixner-Pollaczek polynomials $P_m^{(\frac{\nu}{2})}(\lambda; \frac{\pi}{2})$ are also obtained as Mellin transforms of Laguerre functions. Their main properties follow from this fact: hypergeometric representation above, orthogonality, generating formula, difference equation, and three terms relation.

These polynomials $P_m^{(\frac{\nu}{2})}(\lambda; \frac{\pi}{2})$ have been generalized to the multivariate case. In fact, the multivariable Meixner-Pollaczek (symmetric) polynomials have been essentially considered in the setting of the Fourier analysis on Riemannian symmetric spaces in several papers: [Peetre-Zhang,1992] (Appendix 2: A class of hypergeometric orthogonal polynomials), [Ørsted-Zhang, 1994], section 3.4, [Zhang,2002] and [Davidson-Ólafsson-Zhang,2003]. Also, see [Davidson-Ólafsson,2003] and [Aristidou-Davidson-Ólafsson,2006]. Further, for an arbitrary real value of the multiplicity d , the multivariate Meixner-Pollaczek polynomials are defined in [Sahi-Zhang,2007] in the setting of Heckman-Opdam and Cherednik-Opdam transforms, related to symmetric and non-symmetric Jack polynomials, and generating formulae for them are established. However the case where the parameter ϕ is involved has not been studied so far. Moreover, once we define the multivariate Meixner-Pollaczek polynomials with parameter ϕ , it is also important to clarify a geometric meaning of the parameter. Establishing a natural setting for the study of multivariate Meixner-Pollaczek polynomials with such parameter, one can expect to obtain wider applications such as a study of multi-dimensional Lévi-process, in particular, introducing multi-dimensional Meixner process (see [Schoutens, 2000] for the one dimensional case).

The purpose of this article is to provide a geometric framework for introducing the multivariate Meixner-Pollaczek polynomials (with parameter ϕ) and study their fundamental properties. Our analysis may explain much simpler geometric understanding of several basic properties of the multivariate Meixner-pollaczek polynomials than ever, even in the case $\phi = \frac{\pi}{2}$. For instance, the \mathfrak{S}_n -invariant difference operator of which the multivariate Meixner-Pollaczek polynomials are eigenfunctions can be understood by an image of the Euler operator under the composition of three intertwiners: the Cayley transform, the Laplace transform and the spherical Fourier transform.

Let us present in the one variable case the scheme we will develop.

a) The monomials $\phi_m(z) = z^m$ form an orthogonal basis in the weighted Berman space $\mathcal{H}_\nu^2(D)$ ($\nu > 1$) of holomorphic functions f on the unit disc $D \subset \mathbb{C}$ with

$$\|f\|_\nu^2 := \frac{\nu-1}{\pi} \int_D |f(w)|^2 (1-|w|^2)^{\nu-2} m(dw) < \infty.$$

(m denotes the Lebesgue measure on \mathbb{C} .) Since

$$\|\phi_m\|_\nu^2 = \frac{m!}{(\nu)_m},$$

the reproducing kernel of $\mathcal{H}_\nu^2(D)$ is given by

$$\mathcal{K}_\nu(w, w') = \sum_{m=0}^{\infty} \frac{(\nu)_m}{m!} w^m \bar{w}'^m.$$

It can be written as a generating formula for the functions ϕ_m :

$$\mathcal{G}^{(1)}(\zeta, w) := \sum_{m=0}^{\infty} \frac{(\nu)_m}{m!} \phi_m(\zeta) w^m = (1-w\zeta)^{-\nu}. \quad (0.1)$$

b) The Cayley transform

$$w \mapsto z = c(w) = \frac{1+w}{1-w}$$

maps the unit disc D onto the right half-plane $T = \{z = x + iy \mid x > 0\}$, and its inverse is given by

$$c^{-1}(z) = \frac{z-1}{z+1}.$$

For a holomorphic function f on D define the function $F = C_\nu f$ on T by

$$F(z) = (C_\nu f)(z) = \left(\frac{z+1}{2}\right)^{-\nu} f\left(\frac{z-1}{z+1}\right).$$

Then C_ν maps unitarily $\mathcal{H}_\nu^2(D)$ onto the space $\mathcal{H}_\nu^2(T)$ of holomorphic functions F on T such that

$$\|F\|_\nu^2 := \frac{\nu-1}{4\pi} \int_T |F(x+iy)|^2 x^{\nu-2} m(dz) < \infty.$$

The functions $F_m^{(\nu)} = C_\nu \phi_m$ form an orthogonal basis of $\mathcal{H}_\nu^2(T)$. From the generating formula (0.1), by performing the transform C_ν with respect to the variable ζ , one obtains a generating formula for the functions $F_m^{(\nu)}$:

$$\mathcal{G}^{(2)}(z, w) := \sum_{m=0}^{\infty} \frac{(\nu)_m}{m!} F_m^{(\nu)}(z) w^m = \left(\frac{1-w}{2}\right)^{-\nu} (z + c(w))^{-\nu}. \quad (0.2)$$

c) Every function F in $\mathcal{H}_\nu^2(T)$ admits a Laplace integral representation:

$$F(z) = (\mathcal{L}_\nu)\psi(z) := \frac{2^\nu}{\Gamma(\nu)} \int_0^\infty e^{-zu} \psi(u) u^{\nu-1} du,$$

with $\psi \in L_\nu^2(0, \infty)$, with the norm

$$\|\psi\|_\nu^2 := \frac{2^\nu}{\Gamma(\nu)} \int_0^\infty |\psi(u)|^2 u^{\nu-1} du,$$

normalized in such a way that \mathcal{L}_ν is unitary. Define the Laguerre function $\psi_m^{(\nu)}$ as

$$\psi_m^{(\nu)}(u) = e^{-u} L_m^{(\nu-1)}(2u),$$

where $L_m^{(\nu)}$ denotes the classical Laguerre polynomial of degree m . Then

$$(\mathcal{L}_\nu \psi_m^{(\nu)})(z) = \frac{(\nu)_m}{m!} F_m^{(\nu)}(z).$$

Applying the inverse Laplace transform \mathcal{L}_ν^{-1} to (0.2) one gets the following generating formula for the Laguerre functions:

$$\mathcal{G}^{(3)}(u, w) := \sum_{m=0}^{\infty} \psi_m^{(\nu)}(u) w^m = (1-w)^{-\nu} e^{-uc(w)}. \quad (0.3)$$

d) Finally we perform a modified Mellin transform:

$$\mathcal{M}_\nu \psi(s) := \frac{1}{\Gamma(s + \frac{\nu}{2})} \int_0^\infty \psi(u) u^{s + \frac{\nu}{2} - 1} du.$$

By the classical Plancherel theorem $\psi \mapsto (\mathcal{M}_\nu \psi)(i\lambda)$ is a unitary isomorphism from $L_\nu^2(0, \infty)$ onto $L^2(\mathbb{R}, M_\nu)$, with

$$M_\nu(d\lambda) = \frac{1}{2\pi} \frac{2^\nu}{\Gamma(\nu)} \left| \Gamma\left(i\lambda + \frac{\nu}{2}\right) \right|^2 d\lambda.$$

The function $q_m^{(\nu)} := \mathcal{M}_\nu \psi_m^\nu$ is a Meixner-Pollaczek polynomial. In fact

$$q_m^{(\nu)}(i\lambda) = \frac{(\nu)_m}{m!} {}_2F_1\left(-m, s + \frac{\nu}{2}; \nu; 2\right) = (-i)^m P_m^{(\frac{\nu}{2})}\left(\lambda; \frac{\pi}{2}\right).$$

Hence the Meixner-Pollaczek polynomials $q_m^{(\nu)}$ form an orthogonal basis of $L^2(\mathbb{R}, M_\nu)$, and

$$\|q_m^{(\nu)}\|_\nu^2 := \int_{-\infty}^{\infty} |q_m^{(\nu)}(i\lambda)|^2 M_\nu(d\lambda) = \frac{(\nu)_m}{m!}.$$

If we apply the transform \mathcal{M}_ν to (0.3) with respect to u , we obtain the following generating formula

$$\mathcal{G}_\nu^{(4)}(s, w) := \sum_{m=0}^{\infty} q_m^{(\nu)}(s) w^m = (1-w)^{s-\frac{\nu}{2}} (1+w)^{-s-\frac{\nu}{2}}.$$

(See [Andrews-Askey-Roy,1999], p.348,349, and also [Bump et al.,2000] p.14,15.)

Starting from the Euler equation

$$D_\nu^{(1)} \phi_m := 2w \frac{d}{dw} \phi_m = 2m \psi_m,$$

one obtains a difference equation for the Meixner-Pollaczek polynomial $q_m^{(\nu)}$,

$$\begin{aligned} D_\nu^{(4)} q_m^{(\nu)}(s) &:= \left(s + \frac{\nu}{2}\right) (q_m^{(\nu)}(s+1) - q_m^{(\nu)}(s)) - \left(s - \frac{\nu}{2}\right) (q_m^{(\nu)}(s-1) - q_m^{(\nu)}(s)) \\ &= 2m q_m^{(\nu)}(s), \end{aligned}$$

and the three terms relation

$$2s q_m^{(\nu)}(s) = (m + \nu - 1) q_{m-1}^{(\nu)}(s) - (m + 1) q_{m+1}^{(\nu)}(s).$$

Moreover, by using a Gutzmer formula for the Mellin transform, the orthogonality property extends to the polynomials $P_m^\alpha(\lambda, \phi)$, with $0 < \phi < \pi$.

In the multivariate case we follow the same scheme. Actually, replacing the half-line by a symmetric cone, and the Mellin transform by the spherical Fourier transform, leads to a definition of multivariate Meixner-Pollaczek polynomials together with their properties, analogous to the ones of the one variable Meixner-Pollaczek polynomials.

In Section 1 we recall the basic facts about the spherical Fourier analysis on a symmetric cone. In Section 2 we define the multivariate Meixner-Pollaczek polynomials $Q_{\mathbf{m}}^{(\nu)}(\mathbf{s})$ (the case $\phi = \frac{\pi}{2}$), where \mathbf{m} is a partition, prove that they are orthogonal with respect to a measure M_ν on \mathbb{R}^n , and establish a generating formula.

In Section 3, adding a real parameter θ , we introduce the symmetric polynomials $Q_{\mathbf{m}}^{(\nu, \theta)}(\mathbf{s})$ in the variables $\mathbf{s} = (s_1, \dots, s_n)$ ($Q_{\mathbf{m}}^{(\nu)} = Q_{\mathbf{m}}^{(\nu, 0)}$). In the one variable case

$$\begin{aligned} q_m^{(\nu, \theta)}(s) &= e^{im\theta} \frac{(\nu)_m}{m!} {}_2F_1\left(-m, s + \frac{\nu}{2}; \nu; 2e^{-i\theta} \cos \theta\right) \\ &= (-i)^m P_m^{(\frac{\nu}{2})}\left(-is; \theta + \frac{\pi}{2}\right). \end{aligned}$$

The orthogonality property for the polynomials $Q_{\mathbf{m}}^{(\nu, \theta)}(\mathbf{s})$ is obtained by using a Gutzmer formula for the spherical Fourier transform. A generating formula is obtained for these polynomials. In case of the multiplicity $d = 2$, we establish in Section 4 determinantal formulae for multivariate Laguerre and Meixner-Pollaczek polynomials. The last sections are devoted to a difference equation satisfied by the polynomials $Q_{\mathbf{m}}^{(\nu, \theta)}(\mathbf{s})$. Starting from an Euler-type equation involving the parameter θ , this difference equation is obtained in three steps, corresponding to a Cayley transform, an inverse Laplace transform, and a spherical Fourier transform for symmetric cones. The symmetry $\theta \mapsto -\theta$ in the parameter is related to geometric symmetries and to a generalized Tricomi theorem for the Hankel transform on a symmetric cone. As a byproduct we obtain a differential equation for the multivariate Laguerre polynomials, whose radial part is a special case of an equation in [Baker-Forrester,1997]. In the last section we show that multivariate Meixner-Pollaczek polynomials satisfy a Pieri's formula. In the one variable case it reduces to the three terms relation satisfied by the classical Meixner-Pollaczek polynomials.

1 Spherical Fourier analysis on a symmetric cone

A reference for this preliminary section is [Faraut-Korányi,1994]. We consider an irreducible symmetric cone Ω in a Euclidean Jordan algebra V . We denote by G the identity component in the group $G(\Omega)$ of linear automorphisms of Ω , and $K \subset G$ is the isotropy subgroup of the unit element $e \in V$.

The Gindikin gamma function Γ_Ω of the cone Ω will be the cornerstone of the analysis we will develop. It is defined, for $\mathbf{s} \in \mathbb{C}^n$, with $\operatorname{Re} s_j > \frac{d}{2}(j-1)$, by

$$\Gamma_\Omega(\mathbf{s}) = \int_\Omega e^{-\operatorname{tr}(u)} \Delta_{\mathbf{s}}(u) \Delta(u)^{-\frac{N}{n}} m(du).$$

The notation $\operatorname{tr}(u)$ and $\Delta(u)$ denote the trace and the determinant with respect to the Jordan algebra structure, $\Delta_{\mathbf{s}}$ is the power function, N and n are the dimension and the rank of V , and m is the Euclidean measure associated to the Euclidean structure on V given by $(u|v) = \operatorname{tr}(uv)$. Its evaluation gives

$$\Gamma_\Omega(\mathbf{s}) = (2\pi)^{\frac{N-n}{2}} \prod_{j=1}^n \Gamma(s_j - \frac{d}{2}(j-1)),$$

where d is the multiplicity, related to N and n by the relation $N = n + \frac{d}{2}n(n-1)$.

The spherical function $\varphi_{\mathbf{s}}$, for $\mathbf{s} \in \mathbb{C}^n$, is defined on Ω by

$$\varphi_{\mathbf{s}}(u) = \int_K \Delta_{\mathbf{s}+\rho}(k \cdot u) dk,$$

where $\rho = (\rho_1, \dots, \rho_n)$, $\rho_j = \frac{d}{4}(2j - n - 1)$, and dk is the normalized Haar measure on the compact group K .

The algebra $\mathbb{D}(\Omega)$ of G -invariant differential operators on Ω is commutative, and the spherical function $\varphi_{\mathbf{s}}$ is an eigenfunction of every $D \in \mathbb{D}(\Omega)$:

$$D\varphi_{\mathbf{s}} = \gamma_D(\mathbf{s})\varphi_{\mathbf{s}}.$$

The function γ_D is a symmetric polynomial function, and the map $D \mapsto \gamma_D$ is an algebra isomorphism from $\mathbb{D}(\Omega)$ onto the algebra $\mathcal{P}(\mathbb{C}^n)^{\mathfrak{S}_n}$ of symmetric polynomial functions, a special case of the Harish-Chandra isomorphism. The symbol σ_D of a partial differential operator D on V is defined by

$$De^{(x|\xi)} = \sigma_D(x, \xi)e^{(x|\xi)} \quad (x, \xi \in V)$$

(D acts on the variable x). If $D \in \mathbb{D}(\Omega)$, then σ_D is a G -invariant polynomial on $V \times V$ in the following sense: for $g \in G$,

$$\sigma_D(g \cdot x, \xi) = \sigma_D(x, g^* \cdot \xi).$$

The map $D \mapsto p(\xi) = \sigma_D(e, \xi)$ is a vector space isomorphism from $\mathbb{D}(\Omega)$ onto the space $\mathcal{P}(V)^K$ of K -invariant polynomials on V .

The spherical Fourier transform $\mathcal{F}\psi$ of a K -invariant function ψ on Ω is given by

$$\mathcal{F}\psi(\mathbf{s}) = \int_{\Omega} \psi(u) \varphi_{\mathbf{s}}(u) \Delta^{-\frac{N}{n}}(u) m(du).$$

Hence, for $\psi(u) = e^{-\text{tr} u} \Delta^{\frac{\nu}{2}}$ ($\nu > \frac{d}{2}(n-1)$), then

$$\mathcal{F}\psi(\mathbf{s}) = \Gamma_{\Omega}(\mathbf{s} + \frac{\nu}{2} + \rho) = (2\pi)^{\frac{N-n}{2}} \prod_{j=1}^n \Gamma(s_j + \frac{\nu}{2} - \frac{d}{4}(n-1)).$$

For an invariant differential operator $D \in \mathbb{D}(\Omega)$,

$$\mathcal{F}_{\nu}(D\psi) = \gamma_D(-\mathbf{s}) \mathcal{F}_{\nu}\psi.$$

Recall the spherical Plancherel formula: if the K -invariant function ψ satisfies

$$\int_{\Omega} |\psi(u)|^2 \Delta(u)^{-\frac{N}{n}} m(du) < \infty,$$

then

$$\int_{\Omega} |\psi(u)|^2 \Delta(u)^{-\frac{N}{n}} m(du) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} |\mathcal{F}\psi(i\lambda)|^2 \frac{1}{|c(i\lambda)|^2} m(d\lambda),$$

where c is the Harish-Chandra function:

$$c(\mathbf{s}) = c_0 \prod_{j < k} B(s_j - s_k, \frac{d}{2}).$$

(B is the Euler beta function, the constant c_0 is such that $c(-\rho) = 1$.)

The space $\mathcal{P}(V)$ of polynomials on V decomposes multiplicity free under G as

$$\mathcal{P}(V) = \bigoplus_{\mathbf{m}} \mathcal{P}_{\mathbf{m}},$$

where $\mathcal{P}_{\mathbf{m}}$ is a finite dimensional subspace, irreducible under G . The parameter \mathbf{m} is a partition: $\mathbf{m} = (m_1, \dots, m_n) \in \mathbb{N}^n$, $m_1 \geq \dots \geq m_n$. The subspace $\mathcal{P}_{\mathbf{m}}^K$ of K -invariant polynomials in $\mathcal{P}_{\mathbf{m}}$ is one dimensional, generated by the spherical polynomial $\Phi_{\mathbf{m}}$, normalized by the condition $\Phi_{\mathbf{m}}(e) = 1$. The dimension of $\mathcal{P}_{\mathbf{m}}$ will be denoted by $d_{\mathbf{m}}$.

There is a unique invariant differential operator $D^{\mathbf{m}}$ such that

$$D^{\mathbf{m}}\psi(e) = \left(\Phi_{\mathbf{m}} \left(\frac{\partial}{\partial u} \right) \psi \right) (e).$$

We will write $\gamma_{\mathbf{m}} = \gamma_{D^{\mathbf{m}}}$. If a K -invariant function ψ is analytic in a neighborhood of e , it admits a spherical Taylor expansion near e :

$$\psi(e + v) = \sum_{\mathbf{m}} d_{\mathbf{m}} \frac{1}{\binom{N}{n}_{\mathbf{m}}} D^{\mathbf{m}}\psi(e) \Phi_{\mathbf{m}}(v).$$

For $\alpha \in \mathbb{C}$ and a partition \mathbf{m} , the generalized Pochhammer symbol $(\alpha)_{\mathbf{m}}$ is defined by

$$(\alpha)_{\mathbf{m}} = \frac{\Gamma_{\Omega}(\mathbf{m} + \alpha)}{\Gamma_{\Omega}(\alpha)}.$$

In particular, for $\psi = \varphi_{\mathbf{s}}$, a spherical function,

$$\varphi_{\mathbf{s}}(e + v) = \sum_{\mathbf{m}} d_{\mathbf{m}} \frac{1}{\binom{N}{n}_{\mathbf{m}}} \gamma_{\mathbf{m}}(\mathbf{s}) \Phi_{\mathbf{m}}(v).$$

For $\psi = \Phi_{\mathbf{m}}$, we get the spherical binomial formula

$$\Phi_{\mathbf{m}}(e + v) = \sum_{\mathbf{k} \subset \mathbf{m}} \binom{\mathbf{m}}{\mathbf{k}} \Phi_{\mathbf{k}}(v).$$

In fact the generalized binomial coefficient

$$\binom{\mathbf{m}}{\mathbf{k}} = d_{\mathbf{k}} \frac{1}{\binom{N}{n}_{\mathbf{k}}} \gamma_{\mathbf{k}}(\mathbf{m} - \rho)$$

vanishes if $\mathbf{k} \not\subset \mathbf{m}$.

2 Multivariate Meixner-Pollaczek polynomials $Q_{\mathbf{m}}^{(\nu)}$

For $n = 1$, we define the Meixner-Pollaczek polynomial $q_m^{(\nu)}$ as follows

$$q_m^{(\nu)}(s) = \frac{(\nu)_m}{m!} {}_2F_1\left(-m, s + \frac{\nu}{2}; \nu; 2\right).$$

This definition slightly differs from the classical one $P_m^\alpha(\lambda; \phi)$:

$$q_m^{(\nu)}(i\lambda) = (-i)^m P_m^{\frac{\nu}{2}}(\lambda; \frac{\pi}{2}).$$

(see for instance [Andrews-Askey-Roy,1999], p.348.) Its expansion can be written

$$q_m^{(\nu)}(s) = \frac{(\nu)_m}{m!} \sum_{k=0}^m \frac{[m]_k [-s - \frac{\nu}{2}]_k}{(\nu)_k k!} \frac{1}{k!} 2^k.$$

The polynomials $q_m^{(\nu)}(i\lambda)$ are orthogonal with respect to the weight

$$|\Gamma(i\lambda + \frac{\nu}{2})|^2 \quad (\nu > 0).$$

Observe that for $n = 1$, $\varphi_s(u) = u^s$, and

$$D^m = u^m \left(\frac{d}{du}\right)^m, \quad \gamma_m(s) = [s]_m = s(s-1)\dots(s-m+1).$$

Hence, for higher rank, we see $\gamma_{\mathbf{m}}(\mathbf{s})$ as a multivariate analogue of the Pochhammer symbol $[s]_m$.

We define the multivariate Meixner-Pollaczek polynomial $Q_{\mathbf{m}}^{(\nu)}$ as the following symmetric polynomial in n variables:

$$Q_{\mathbf{m}}^{(\nu)}(\mathbf{s}) = \frac{(\nu)_{\mathbf{m}}}{\binom{N}{n}_{\mathbf{m}}} \sum_{\mathbf{k} \in \mathbf{Cm}} d_{\mathbf{k}} \frac{\gamma_{\mathbf{k}}(\mathbf{m} - \rho) \gamma_{\mathbf{k}}(-\mathbf{s} - \frac{\nu}{2})}{(\nu)_{\mathbf{k}}} \frac{1}{\binom{N}{n}_{\mathbf{k}}} 2^{|\mathbf{k}|}.$$

For $\nu > \frac{d}{2}(n-1)$ let us denote by $M_\nu(d\lambda)$ the probability measure on \mathbb{R}^n given by

$$M_\nu(d\lambda) = \frac{1}{Z_\nu} \prod_{j=1}^n \left| \Gamma(i\lambda_j + \frac{\nu}{2} - \frac{d}{4}(n-1)) \right|^2 \frac{1}{|c(i\lambda)|^2} m(d\lambda),$$

where

$$Z_\nu = \int_{\mathbb{R}^n} \prod_{j=1}^n \left| \Gamma(i\lambda_j + \frac{\nu}{2} - \frac{d}{4}(n-1)) \right|^2 \frac{1}{|c(i\lambda)|^2} m(d\lambda).$$

The constant Z_ν can be evaluated by using the spherical Plancherel formula, applied to the function $\psi(u) = e^{-\text{tr } u} \Delta(u)^{\frac{\nu}{2}}$:

$$\begin{aligned} & \int_{\Omega} e^{-2\text{tr } u} \Delta(u)^{\nu - \frac{N}{n}} m(du) \\ &= (2\pi)^{N-2n} \int_{\mathbb{R}^n} \prod_{j=1}^n \left| \Gamma(i\lambda_j + \frac{\nu}{2} - \frac{d}{4}(n-1)) \right|^2 \frac{1}{|c(i\lambda)|^2} m(d\lambda). \end{aligned}$$

Therefore

$$Z_\nu = (2\pi)^{2n-N} 2^{-n\nu} \Gamma_\Omega(\nu).$$

Next statement involves the geometry of the Hermitian symmetric space of tube type associated to the symmetric cone Ω . The map $z \mapsto (z - e)(z + e)^{-1}$ maps the tube domain $T_\Omega = \Omega + iV \subset V_{\mathbb{C}}$ onto the bounded Hermitian symmetric domain \mathcal{D} . Its inverse is the Cayley transform:

$$c(w) = (e + w)(e - w)^{-1}.$$

Theorem 2.1. *Assume $\nu > \frac{d}{2}(n - 1)$.*

(i) *The multivariate Meixner-Pollaczek polynomials $Q_{\mathbf{m}}^{(\nu)}(i\lambda)$ form an orthogonal basis of $L^2(\mathbb{R}^n, M_\nu)^{\mathfrak{S}_n}$. The norm of $Q_{\mathbf{m}}^{(\nu)}$ can be evaluated:*

$$\int_{\mathbb{R}^n} |Q_{\mathbf{m}}^{(\nu)}(i\lambda)|^2 M_\nu(d\lambda) = \frac{1}{d_{\mathbf{m}}} \frac{(\nu)_{\mathbf{m}}}{\left(\frac{N}{n}\right)_{\mathbf{m}}}.$$

(ii) *The polynomials $Q_{\mathbf{m}}^{(\nu)}$ admit the following generating formula: for $\mathbf{s} \in \mathbb{C}^n$, $w \in \mathcal{D}$,*

$$\sum_{\mathbf{m}} d_{\mathbf{m}} Q_{\mathbf{m}}^{(\nu)}(\mathbf{s}) \Phi_{\mathbf{m}}(w) = \Delta(e - w^2)^{-\frac{\nu}{2}} \varphi_{\mathbf{s}}(c(w)^{-1}).$$

Proof.

a) For $\nu > 2\frac{N}{n} - 1 = 1 + d(n - 1)$, $\mathcal{H}_\nu^2(\mathcal{D})$ denotes the weighted Bergman space of holomorphic functions f on \mathcal{D} such that

$$\|f\|_\nu^2 := a_\nu^{(1)} \int_{\mathcal{D}} |f(w)|^2 h(w)^{\nu - 2\frac{N}{n}} m(dw) < \infty.$$

The constant

$$a_\nu^{(1)} = \frac{1}{\pi^n} \frac{\Gamma_\Omega(\nu)}{\Gamma_\Omega\left(\nu - \frac{N}{n}\right)}$$

is such that the function $\Phi_0 \equiv 1$ has norm 1. The spherical polynomials $\Phi_{\mathbf{m}}$ form an orthogonal basis of the space $\mathcal{H}_\nu^2(\mathcal{D})^K$ of K -invariant functions in $\mathcal{H}_\nu^2(\mathcal{D})$, and

$$\|\Phi_{\mathbf{m}}\|_\nu^2 = \frac{1}{d_{\mathbf{m}}} \frac{\left(\frac{N}{n}\right)_{\mathbf{m}}}{(\nu)_{\mathbf{m}}}. \quad (2.1)$$

The reproducing kernel of $\mathcal{H}_\nu^2(\mathcal{D})$ is given by

$$\mathcal{K}_\nu(w, w') = h(w, w')^{-\nu},$$

where $h(w, w')$ is a polynomial holomorphic in w , antiholomorphic in w' , and, for w invertible,

$$h(w, w') = \Delta(w)\Delta(w^{-1} - \bar{w}').$$

(\bar{w}' is the complex conjugate of w' with respect to the real form V of $V_{\mathbb{C}}$.)
By an integration over K one obtains:

$$\mathcal{G}_\nu^{(1)}(\zeta, w) := \sum_{\mathbf{m}} d_{\mathbf{m}} \frac{\binom{\nu}{\mathbf{m}}}{\binom{N}{\mathbf{m}}} \Phi_{\mathbf{m}}(\zeta) \Phi_{\mathbf{m}}(w) = \int_K h(w, k\bar{\zeta})^{-\nu} dk. \quad (2.2)$$

b) For a function f holomorphic in \mathcal{D} , one defines the function $F = C_\nu f$ on T_Ω by

$$F(z) = (C_\nu f)(z) = \Delta\left(\frac{z+e}{2}\right)^{-\nu} f((z-e)(z+e)^{-1}).$$

The map C_ν is a unitary isomorphism from $\mathcal{H}_\nu^2(\mathcal{D})$ onto the space $\mathcal{H}_\nu^2(T_\Omega)$ of holomorphic functions on T_Ω such that

$$\|F\|_\nu^2 := a_\nu^{(2)} \int_{T_\Omega} |F(z)|^2 \Delta(x)^{\nu-2\frac{N}{n}} m(dz) < \infty.$$

The constant

$$a_\nu^{(2)} = \frac{1}{(4\pi)^n} \frac{\Gamma_\Omega(\nu)}{\Gamma_\Omega(\nu - \frac{N}{n})},$$

is such that the function

$$F_0^{(\nu)} = C_\nu \Phi_0, \quad F_0^{(\nu)}(z) = \Delta\left(\frac{z+e}{2}\right)^{-\nu},$$

has norm 1. The functions $F_{\mathbf{m}}^{(\nu)} = C_\nu \Phi_{\mathbf{m}}$ form an orthogonal basis of the space $\mathcal{H}_\nu^2(T_\Omega)^K$ of K -invariant functions in $\mathcal{H}_\nu^2(T_\Omega)$, and it follows from (2.1) that

$$\|F_{\mathbf{m}}^{(\nu)}\|_\nu^2 = \frac{1}{d_{\mathbf{m}}} \frac{\binom{N}{\mathbf{m}}}{\binom{\nu}{\mathbf{m}}}. \quad (2.3)$$

Performing in (2.2) the transform C_ν with respect to ζ we get a generating formula for the functions $F_{\mathbf{m}}^{(\nu)}$: for $w \in \mathcal{D}$, $z \in T_\Omega$,

$$\begin{aligned} \mathcal{G}_\nu^{(2)}(z, w) &:= \sum_{\mathbf{m}} d_{\mathbf{m}} \frac{(\nu)_{\mathbf{m}}}{\left(\frac{N}{n}\right)_{\mathbf{m}}} \Phi_{\mathbf{m}}(w) F_{\mathbf{m}}^{(\nu)}(z) \\ &= \Delta\left(\frac{e-w}{2}\right)^{-\nu} \int_K \Delta(k \cdot z + c(w))^{-\nu} dk \end{aligned} \quad (2.4)$$

c) The functions in $\mathcal{H}_\nu^2(T_\Omega)$ admit a Laplace integral representation. The modified Laplace transform \mathcal{L}_ν , given, for a function ψ on Ω , by

$$(\mathcal{L}_\nu \psi)(z) = a_\nu^{(3)} \int_\Omega e^{z|u} \psi(u) \Delta(u)^{\nu - \frac{N}{n}} m(du),$$

is an isometric isomorphism from the space $L_\nu^2(\Omega)$ of measurable functions ψ on Ω such that

$$\|\psi\|_\nu^2 := a_\nu^{(3)} \int_\Omega |\psi(u)|^2 \Delta(u)^{\nu - \frac{N}{n}} m(du) < \infty,$$

onto $\mathcal{H}_\nu^2(T_\Omega)$. The constant

$$a_\nu^{(3)} = \frac{2^{n\nu}}{\Gamma_\Omega(\nu)}$$

is such that the function $\Psi_0(u) = e^{-\text{tr } u}$ has norm 1, and then $\mathcal{L}_\nu \Psi_0 = F_0$. By the binomial formula

$$\begin{aligned} F_{\mathbf{m}}^{(\nu)}(z) &= \Delta\left(\frac{z+e}{2}\right)^{-\nu} \Phi_{\mathbf{m}}((z-e)(z+e)^{-1}) \\ &= \Delta\left(\frac{z+e}{2}\right)^{-\nu} \Phi_{\mathbf{m}}(e - 2(z+e)^{-1}) \\ &= \sum_{\mathbf{k} \subset \mathbf{m}} (-1)^{|\mathbf{k}|} \binom{\mathbf{m}}{\mathbf{k}} \Phi_{\mathbf{k}}(2(z+e)^{-1}) \Delta(2(e+z)^{-1})^\nu. \end{aligned}$$

Lemma 2.2.

$$\mathcal{L}_\nu(e^{-\text{tr } u} \Phi_{\mathbf{m}})(z) = (\nu)_{\mathbf{m}} \Phi_{\mathbf{m}}((z+e)^{-1}) \Delta(2(e+z)^{-1})^\nu.$$

(See Lemma XI.2.3 in [Faraud-Korányi, 1994].)

By Lemma 2.2 the function

$$\Psi_{\mathbf{m}}^{(\nu)} = \frac{(\nu)_{\mathbf{m}}}{\left(\frac{N}{n}\right)_{\mathbf{m}}} \mathcal{L}_{\nu}^{-1}(F_{\mathbf{m}}^{(\nu)}).$$

is the Laguerre function given by

$$\Psi_{\mathbf{m}}^{(\nu)}(u) = e^{-\text{tr } u} L_{\mathbf{m}}^{(\nu-1)}(2u),$$

where $L_{\mathbf{m}}^{(\nu-1)}$ is the multivariate Laguerre polynomial

$$\begin{aligned} L_{\mathbf{m}}^{(\nu-1)}(x) &= \frac{(\nu)_{\mathbf{m}}}{\left(\frac{N}{n}\right)_{\mathbf{m}}} \sum_{\mathbf{k} \subset \mathbf{m}} \binom{\mathbf{m}}{\mathbf{k}} \frac{1}{(\nu)_{\mathbf{k}}} \Phi_{\mathbf{k}}(-x) \\ &= \frac{(\nu)_{\mathbf{m}}}{\left(\frac{N}{n}\right)_{\mathbf{m}}} \sum_{\mathbf{k} \subset \mathbf{m}} d_{\mathbf{k}} \frac{\gamma_{\mathbf{k}}(\rho - \mathbf{m})}{(\nu)_{\mathbf{k}}} \frac{1}{\left(\frac{N}{n}\right)_{\mathbf{k}}} \Phi_{\mathbf{k}}(-x). \end{aligned}$$

Proposition 2.3. (i) *The multivariate Laguerre functions $\Psi_{\mathbf{m}}^{(\nu)}$ form an orthogonal basis of $L_{\nu}^2(\Omega)$, and*

$$\|\Psi_{\mathbf{m}}^{(\nu)}\|_{\nu}^2 = \frac{1}{d_{\mathbf{m}}} \frac{(\nu)_{\mathbf{m}}}{\left(\frac{N}{n}\right)_{\mathbf{m}}}. \quad (2.5)$$

(ii) *The functions $\Psi_{\mathbf{m}}^{(\nu)}$ admit the following generating formula: for $u \in \Omega$, $w \in \mathcal{D}$,*

$$\mathcal{G}_{\nu}^{(3)}(u, w) := \sum_{\mathbf{m}} d_{\mathbf{m}} \Psi_{\mathbf{m}}^{(\nu)}(u) \Phi_{\mathbf{m}}(w) = \Delta(e-w)^{-\nu} \int_K e^{-(k \cdot u | c(w))} dk. \quad (2.6)$$

The generating formula can also be written

$$\Delta(e-w)^{-\nu} \int_K e^{(k \cdot x | w(e-w)^{-1})} dk = \sum_{\mathbf{m}} d_{\mathbf{m}} L_{\mathbf{m}}^{(\nu-1)}(x) \Phi_{\mathbf{m}}(w). \quad (2.6')$$

Formula (2.6') is proposed as an exercise in [Faraud-Korányi,1994] (Exercise 3, p.347). It is a special case of formula (4.4) in [Baker-Forrester,1997].

Proof. Part (i) follows from the fact that \mathcal{L}_ν is a unitary isomorphism from $L_\nu^2(\Omega)$ onto $\mathcal{H}_\nu^2(T_\Omega)$, and from (2.3).

The modified Laplace transform of $\mathcal{G}_\nu^{(3)}(u, w)$ with respect to u is equal to $\mathcal{G}_\nu^{(2)}(z, w)$, and one gets (ii) from (2.4). \square

d) We will evaluate the spherical Fourier transform of the Laguerre functions $\Psi_{\mathbf{m}}^{(\nu)}$. We introduce now the modified spherical Fourier transform \mathcal{F}_ν as follows: for a function ψ on Ω ,

$$(\mathcal{F}_\nu \psi)(\mathbf{s}) = \frac{1}{\Gamma_\Omega(\mathbf{s} + \frac{\nu}{2} + \rho)} \int_\Omega \psi(u) \varphi_{\mathbf{s}}(u) \Delta(u)^{\frac{\nu}{2} - \frac{N}{n}} m(du).$$

Observe that $\mathcal{F}_\nu \Psi_0 \equiv 1$.

Lemma 2.4. For $\operatorname{Re} s_j > \frac{d}{4}(n-1) - \frac{\nu}{2}$,

$$\mathcal{F}_\nu(e^{-\operatorname{tr} u} \Phi_{\mathbf{m}})(\mathbf{s}) = (-1)^{|\mathbf{m}|} \gamma_{\mathbf{m}}(-\mathbf{s} - \frac{\nu}{2}).$$

Proof. Let $\sigma_D(u, \xi)$ be the symbol of $D \in \mathbb{D}(\Omega)$, and $p(\xi) = \sigma_D(e, \xi)$. By the invariance property of σ_D , we have $\sigma_D(u, -e) = p(-u)$, and therefore $D e^{-\operatorname{tr} u} = p(-\xi) e^{-\operatorname{tr} u}$. Hence, for $p(\xi) = \Phi_{\mathbf{m}}(\xi)$,

$$\begin{aligned} \mathcal{F}_\nu(e^{-\operatorname{tr} u} \Phi_{\mathbf{m}})(s) &= (-1)^{|\mathbf{m}|} \mathcal{F}_\nu(D^{\mathbf{m}} e^{-\operatorname{tr} u})(s) \\ &= (-1)^{|\mathbf{m}|} \gamma_{\mathbf{m}}(-\mathbf{s} - \frac{\nu}{2}) \mathcal{F}_\nu(e^{-\operatorname{tr} u}) \\ &= (-1)^{|\mathbf{m}|} \gamma_{\mathbf{m}}(-\mathbf{s} - \frac{\nu}{2}). \end{aligned}$$

\square

From Lemma 2.4 we obtain the evaluation of the spherical Fourier transform of the Laguerre functions: For $\operatorname{Re} s_j > \frac{d}{4}(n-1) - \frac{\nu}{2}$,

$$\mathcal{F}_\nu(\Psi_{\mathbf{m}}^\nu)(\mathbf{s}) = Q_{\mathbf{m}}(\mathbf{s}).$$

By the spherical Plancherel formula and part (i) in Proposition 2.3, this proves parts (i) of Theorem 2.1, for $\nu > 1 + d(n-1)$:

$$\int_{\mathbb{R}^n} |Q_{\mathbf{m}}^{(\nu)}(i\lambda)|^2 M_\nu(d\lambda) = \frac{1}{d_{\mathbf{m}}} \frac{(\nu)_{\mathbf{m}}}{\left(\frac{N}{n}\right)_{\mathbf{m}}}. \quad (2.7)$$

By analytic continuation it holds for $\nu > \frac{d}{2}(n-1)$.

For proving part (ii) of Theorem 2.1 one performs the spherical Fourier transform to both handsides of part (ii) in Proposition 2.3:

$$\mathcal{G}_\nu^{(4)} := \sum_{\mathbf{m}} d_{\mathbf{m}} Q_{\mathbf{m}}^{(\nu)}(\mathbf{s}) \Phi_{\mathbf{m}}(w) = \Delta(e-w^2)^{-\frac{\nu}{2}} \varphi_{\mathbf{s}}(c(w)^{-1}). \quad (2.8)$$

This finishes the proof of Theorem 2.1.

We remark that, in [Davidson-Ólafsson-Zang, 2003], a different notation is used for the Meixner-Pollaczek polynomials: their polynomials $p_{\nu, \mathbf{m}}$ (p. 179) are defined through the generating formula above and

$$p_{\nu, \mathbf{m}}(i\mathbf{s}) = d_{\mathbf{m}} Q_{\mathbf{m}}^{(\nu)}(\mathbf{s}).$$

3 Multivariate Meixner-Pollaczek polynomials $Q_{\mathbf{m}}^{(\nu, \theta)}$

The Meixner-Pollaczek polynomials $q_m^{(\nu)}$ we have considered at the beginning of Section 2 correspond to the special value $\phi = \frac{\pi}{2}$ with the classical notation. Using instead $\theta = \phi - \frac{\pi}{2}$, the more general one variable Meixner-Pollaczek polynomials can be written

$$\begin{aligned} q_m^{(\nu, \theta)}(s) &= e^{im\theta} \frac{(\nu)_m}{m!} {}_2F_1\left(-m, s + \frac{\nu}{2}; \nu; 2e^{-i\theta} \cos \theta\right) \\ &= e^{im\theta} \frac{(\nu)_m}{m!} \sum_{k=0}^m \frac{[m]_k \left[-s - \frac{\nu}{2}\right]_k}{(\nu)_k k!} (2e^{-i\theta} \cos \theta)^k. \end{aligned}$$

In terms of the classical notation $P_m^\alpha(\lambda; \phi)$

$$q_m^{(\nu, \theta)}(i\lambda) = (-i)^m P_m^{\frac{\nu}{2}}\left(\lambda; \theta + \frac{\pi}{2}\right).$$

For $\nu > 0$, $|\theta| < \frac{\pi}{2}$, the polynomials $q_m^{(\nu, \theta)}(i\lambda)$ are orthogonal with respect to the weight

$$e^{2\theta\lambda} \left| \Gamma\left(i\lambda + \frac{\nu}{2}\right) \right|^2.$$

In this section we consider the multivariate Meixner-Pollaczek polynomials $Q_{\mathbf{m}}^{(\nu, \theta)}$ defined by

$$Q_{\mathbf{m}}^{(\nu, \theta)}(\mathbf{s}) = e^{i|\mathbf{m}|\theta} \frac{(\nu)_{\mathbf{m}}}{\left(\frac{N}{n}\right)_{\mathbf{m}}} \sum_{\mathbf{k} \subset \mathbf{m}} d_{\mathbf{k}} \frac{\gamma_{\mathbf{k}}(\mathbf{m} - \rho) \gamma_{\mathbf{k}}(-\mathbf{s} - \frac{\nu}{2})}{(\nu)_{\mathbf{k}}} \frac{1}{\left(\frac{N}{n}\right)_{\mathbf{k}}} (2e^{-i\theta} \cos \theta)^{|\mathbf{k}|}.$$

Theorem 3.1. Assume $\nu > \frac{d}{2}(n-1)$, $|\theta| < \frac{\pi}{2}$.

(i) The multivariate Meixner-Pollaczek polynomials $Q_{\mathbf{m}}^{(\nu, \theta)}(i\lambda)$ form an orthogonal basis of $L^2(\mathbb{R}^n, e^{2\theta(\lambda_1 + \dots + \lambda_n)} M_{\nu})^{\mathfrak{S}_n}$. The norm of $Q_{\mathbf{m}}^{(\nu, \theta)}$ can be evaluated:

$$\int_{\mathbb{R}^n} |Q_{\mathbf{m}}^{(\nu, \theta)}(i\lambda)|^2 e^{2\theta(\lambda_1 + \dots + \lambda_n)} M_{\nu}(d\lambda) = (\cos \theta)^{-n\nu} \frac{1}{d_{\mathbf{m}}} \frac{(\nu)_{\mathbf{m}}}{\left(\frac{N}{n}\right)_{\mathbf{m}}}.$$

(ii) The polynomials $Q_{\mathbf{m}}^{(\nu, \theta)}$ admit the following generating formula: for $\mathbf{s} \in \mathbb{C}^n$, $w \in \mathcal{D}$,

$$\sum_{\mathbf{m}} d_{\mathbf{m}} Q_{\mathbf{m}}^{(\nu, \theta)}(\mathbf{s}) \Phi_{\mathbf{m}}(w) = \Delta((e - e^{i\theta}w)(e + e^{-i\theta}))^{-\frac{\nu}{2}} \varphi_{\mathbf{s}}(c_{\theta}(w)^{-1}),$$

where c_{θ} is the modified Cayley transform:

$$c_{\theta}(w) = (e + e^{-i\theta}w)(e - e^{i\theta}w)^{-1}.$$

We will prove Theorem 3.1 in several steps.

a) Let us define the Laguerre functions $\Psi_{\mathbf{m}}^{(\nu, \theta)}$:

$$\Psi_{\mathbf{m}}^{(\nu, \theta)}(u) = e^{i|\mathbf{m}|\theta} e^{-\text{tr } u} L_{\mathbf{m}}^{(\nu-1)}(2e^{-i\theta} \cos \theta u).$$

For functions ψ on V of the form $\psi(u) = e^{-\text{tr } u} p(u)$, where p is a polynomial, define the inner product

$$(\psi_1 | \psi_2)_{(\nu, \theta)} = \frac{2^{n\nu}}{\Gamma_{\Omega}(\nu)} \int_{\Omega} \psi_1(e^{i\theta}u) \overline{\psi_2(e^{i\theta}u)} \Delta(u)^{\nu - \frac{N}{n}} m(du).$$

Proposition 3.2. (i) The Laguerre functions $\Psi_{\mathbf{m}}^{(\nu, \theta)}$ are orthogonal with respect to the inner product $(\cdot | \cdot)_{(\nu, \theta)}$. Furthermore

$$\|\Psi_{\mathbf{m}}^{(\nu, \theta)}\|_{(\nu, \theta)}^2 = (\cos \theta)^{-n\nu} \frac{1}{d_{\mathbf{m}}} \frac{(\nu)_{\mathbf{m}}}{\left(\frac{N}{n}\right)_{\mathbf{m}}}.$$

(ii) The Laguerre functions $\Psi_{\mathbf{m}}^{(\nu, \theta)}$ satisfy the following generating formula: for $u \in \Omega$, $w \in \mathcal{D}$,

$$\mathcal{G}_{\nu, \theta}^{(3)}(u, w) := \sum_{\mathbf{m}} d_{\mathbf{m}} \Psi_{\mathbf{m}}^{(\nu, \theta)}(u) \Phi_{\mathbf{m}}(w) = \Delta(e - e^{i\theta} w)^{-\nu} \int_K e^{\langle k, u | c_{\theta}(w) \rangle} dk.$$

Proof. (i) Put $\alpha = e^{i\theta}$, $\beta = 2e^{-i\theta} \cos \theta$. For two polynomials p_1 and p_2 consider the functions

$$\psi_1^{(\theta)}(u) = e^{-\text{tr } u} p_1(\beta u), \quad \psi_2^{(\theta)}(u) = e^{-\text{tr } u} p_2(\beta u),$$

and their inner product

$$(\psi_1^{(\theta)} | \psi_2^{(\theta)})_{\nu, \theta} = \frac{2^{n\nu}}{\Gamma_{\Omega}(\nu)} \int_{\Omega} e^{-\alpha \text{tr } u} p_1(\beta \alpha u) \overline{e^{-\alpha \text{tr } u} p_2(\beta \alpha u)} \Delta(u)^{\nu - \frac{n}{n}} m(du).$$

Observe that $\beta \alpha = 2 \cos \theta$, $\alpha + \bar{\alpha} = 2 \cos \theta$. Hence

$$\begin{aligned} & (\psi_1^{(\theta)} | \psi_2^{(\theta)})_{\nu, \theta} \\ &= \frac{2^{n\nu}}{\Gamma_{\Omega}(\nu)} \int_{\Omega} e^{-2 \cos \theta \text{tr } u} p_1(2 \cos \theta u) \overline{p_2(2 \cos \theta u)} \Delta(u)^{\nu - \frac{n}{n}} m(du) \\ &= \frac{2^{n\nu}}{\Gamma_{\Omega}(\nu)} (\cos \theta)^{-n\nu} \int_{\Omega} e^{-2 \text{tr } v} p_1(2v) \overline{p_2(2v)} \Delta(v)^{\nu - \frac{n}{n}} m(dv) \\ &= (\cos \theta)^{-n\nu} (\psi_1^{(0)} | \psi_2^{(0)}). \end{aligned}$$

Take

$$p_1(u) = L_{\mathbf{p}}^{(\nu-1)}(u), \quad p_2(u) = L_{\mathbf{q}}^{(\nu-1)}(u).$$

Then, by part (i) of Proposition 2.3, the statement (i) is proven.

(ii) The sum in the generating formula can be written

$$\sum_{\mathbf{m}} d_{\mathbf{m}} e^{-\text{tr } u} L_{\mathbf{m}}^{(\nu-1)}(2e^{-i\theta} \cos \theta u) \Phi_{\mathbf{m}}(e^{i\theta} w).$$

Hence the generating formula follows from part (ii) in Proposition 2.3. \square

b) By Lemma 2.4 we obtain the following evaluation of the spherical Fourier transform of the Laguerre functions $\Psi_{\mathbf{m}}^{(\nu, \theta)}$:

$$\mathcal{F}_{\nu}(\Psi_{\mathbf{m}}^{(\nu, \theta)})(\mathbf{s}) = Q_{\mathbf{m}}^{(\nu, \theta)}(\mathbf{s}).$$

We will need a Gutzmer formula for the spherical Fourier transform on a symmetric cone. Let us first state the following Gutzmer formula for the Mellin transform.

Proposition 3.3. *Let ψ be holomorphic in the following open set in \mathbb{C} :*

$$\{\zeta = re^{i\theta} \mid r > 0, |\theta| < \theta_0\} \quad (0 < \theta_0 < \frac{\pi}{2}).$$

The Mellin transform of ψ is defined by

$$\mathcal{M}\psi(s) = \int_0^\infty \psi(r)r^{s-1}dr.$$

Assume that there is a constant $M > 0$ such that, for $|\theta| < \theta_0$,

$$\int_0^\infty |\psi(re^{i\theta})|^2 r^{-1} dr \leq M.$$

Then

$$\int_0^\infty |\psi(re^{i\theta})|^2 r^{-1} dr = \frac{1}{2\pi} \int_{\mathbb{R}} |\mathcal{M}\psi(i\lambda)|^2 e^{2\theta\lambda} d\lambda.$$

Using the decomposition of the symmetric cone Ω as

$$\Omega =]0, \infty[\times \Omega_1,$$

where $\Omega_1 = \{u \in \Omega \mid \Delta(u) = 1\}$, one gets the following Gutzmer formula for Ω :

Proposition 3.4. *Let ψ be a holomorphic function in the tube $T_\Omega = \Omega + iV$. Assume that there are constants $M > 0$ and $0 < \theta_0 < \frac{\pi}{2}$ such that, for $|\theta| < \theta_0$,*

$$\int_\Omega |\psi(e^{i\theta}u)|^2 \Delta(u)^{-\frac{N}{n}} m(du) \leq M.$$

Then, for $|\theta| < \theta_0$,

$$\begin{aligned} & \int_\Omega |\psi(e^{i\theta}u)|^2 \Delta(u)^{-\frac{N}{n}} du \\ &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} |\mathcal{F}\psi(i\lambda)|^2 e^{2\theta(\lambda_1 + \dots + \lambda_n)} \frac{1}{|c(i\lambda)|^2} m(d\lambda). \end{aligned}$$

From Proposition 3.2 and 3.4 we obtain parts (i) and (ii) of Theorem 3.1.

A more general Gutzmer formula has been established for the spherical Fourier transform on Riemannian symmetric spaces of non compact type [Faraut,2004].

4 Determinantal formulae

In the case $d = 2$, i.e. $V = Herm(n, \mathbb{C})$, $K = U(n)$, there are determinantal formulae for the multivariate Laguerre functions $\Psi_{\mathbf{m}}^{(\nu)}$ and for the multivariate Meixner-Pollaczek polynomials $Q_{\mathbf{m}}^{(\nu, \theta)}$. Consider a Jordan frame $\{c_1, \dots, c_n\}$ in V , and let $\delta = (n - 1, n - 2, \dots, 1, 0)$.

Theorem 4.1. *Assume $d = 2$. The multivariate Laguerre function $\Psi_{\mathbf{m}}^{(\nu)}$ admits the following determinantal formula involving the one variable Laguerre functions $\psi_m^{(\nu)}$: for $u = \sum_{j=1}^n u_j c_j$,*

$$\Psi_{\mathbf{m}}^{(\nu)}(u) = \delta! 2^{-\frac{1}{2}n(n-1)} \frac{\det(\psi_{m_j + \delta_j}^{(\nu-n+1)}(u_i))_{1 \leq i, j \leq n}}{V(u_1, \dots, u_n)},$$

where V denote the Vandermonde polynomial:

$$V(u_1, \dots, u_n) = \prod_{i < j} (u_j - u_i).$$

As a result one obtains the following determinantal formula for the multivariate Laguerre polynomials:

$$\mathbf{L}_{\mathbf{m}}^{\nu}(u) = \delta! \frac{\det(L_{m_j + \delta_j}^{(\nu-n+1)}(u_i))}{V(u_1, \dots, u_n)}.$$

Proof. We start from the generating formula for the multivariate Laguerre functions (Proposition 2.3):

$$\begin{aligned} \mathcal{G}_{\nu}^{(3)}(u, w) &= \sum_{\mathbf{m}} d_{\mathbf{m}} \Phi_{\mathbf{m}}(w) \Psi_{\mathbf{m}}^{(\nu)}(u) \\ &= \Delta(e - w)^{-\nu} \int_K e^{-(ku|(e+w)(e-w)^{-1})} dk. \end{aligned}$$

In the case $d = 2$, the evaluation of this integral is classical: for $x = \sum_{i=1}^n x_i c_i$, $y = \sum_{j=1}^n y_j c_j$, then

$$\mathcal{I}(x, y) = \int_K e^{(kx|y)} dk = \delta! \frac{\det(e^{x_i y_j})}{V(x_1, \dots, x_n) V(y_1, \dots, y_n)}.$$

Therefore, for $u = \sum_{i=1}^n u_i c_i$, $w = \sum_{j=1}^n w_j c_j$,

$$\mathcal{G}_\nu^{(3)}(u, w) = \delta! \prod_{j=1}^n (1 - w_j)^{-\nu} \frac{\det\left(e^{-u_i \frac{1+w_j}{1-w_j}}\right)}{V(u_1, \dots, u_n) V\left(\frac{1+w_1}{1-w_1}, \dots, \frac{1+w_n}{1-w_n}\right)}.$$

Noticing that

$$\frac{1 + w_j}{1 - w_j} - \frac{1 + w_k}{1 - w_k} = 2 \frac{w_j - w_k}{(1 + w_j)(1 + w_k)},$$

we obtain

$$\mathcal{G}_\nu^{(3)}(u, w) = \delta! 2^{-\frac{1}{2}n(n-1)} \frac{\det\left((1 - w_j)^{-(\nu-n+1)} e^{-u_i \frac{1+w_j}{1-w_j}}\right)}{V(u_1, \dots, u_n) V(w_1, \dots, w_n)}.$$

We will expand the above expression in Schur function series by using a formula due to Hua.

Lemma 4.2. *Consider n power series*

$$f_i(w) = \sum_{m=0}^{\infty} c_m^{(i)} w^m \quad (i = 1, \dots, n).$$

Then

$$\frac{\det(f_i(w_j))}{V(w_1, \dots, w_n)} = \sum_{\mathbf{m}} a_{\mathbf{m}} s_{\mathbf{m}}(w_1, \dots, w_n),$$

where $s_{\mathbf{m}}$ is the Schur function associated to the partition \mathbf{m} , and

$$a_{\mathbf{m}} = \det(c_{m_j + \delta_j}^{(i)}).$$

(See [Hua,1963], Theorem 1.2.1, p.22).

Let $\nu' = \nu - n + 1$, and consider the n power series

$$f_i(w) = (1 - w)^{-\nu'} e^{-u_i \frac{1+w}{1-w}} = \sum_{m=0}^{\infty} \psi_m^{(\nu')}(u_i) w^m.$$

Since

$$d_{\mathbf{m}} \Phi_{\mathbf{m}}\left(\sum_{j=1}^n w_j c_j\right) = s_{\mathbf{m}}(w_1, \dots, w_n),$$

we obtain

$$\Psi_{\mathbf{m}}^{(\nu)}(u) = \delta! 2^{-\frac{1}{2}n(n-1)} \frac{\det(\psi_{m_j+\delta_j}^{(\nu-n+1)}(u_i))}{V(u_1, \dots, u_n)}.$$

□

By using the same method we will obtain a determinantal formula for the multivariate Meixner-Pollaczek polynomials $Q_{\mathbf{m}}^{(\nu, \theta)}$.

Theorem 4.3. *Assume $d = 2$. Then*

$$Q_{\mathbf{m}}^{(\nu, \theta)}(\mathbf{s}) = (-2 \cos \theta)^{-\frac{1}{2}n(n-1)} \delta! \frac{\det\left(q_{m_j+\delta_j}^{(\nu-n+1, \theta)}(s_i)\right)_{1 \leq i, j \leq n}}{V(s_1, \dots, s_n)},$$

where $q_m^{(\nu, \theta)}$ denotes the one variable Meixner-Pollaczek polynomial.

Proof. We start from the generating formula for the multivariate Meixner-Pollaczek polynomials $Q_{\mathbf{m}}^{(\nu, \theta)}$ (Theorem 3.1, (ii)):

$$\sum_{\mathbf{m}} d_{\mathbf{m}} Q_{\mathbf{m}}^{(\nu, \theta)}(\mathbf{s}) \Phi_{\mathbf{m}}(w) = \Delta((e - e^{i\theta}w)(e + e^{-i\theta}w))^{-\frac{\nu}{2}} \varphi_{\mathbf{s}}(c_{\theta}(w))^{-1}.$$

For $x = \sum_{i=1}^n x_i c_i$, the spherical function $\varphi_{\mathbf{s}}(x)$ is essentially a Schur function in the variables x_1, \dots, x_n :

$$\varphi_{\mathbf{s}}(x) = \delta! (x_1 x_2 \dots x_n)^{\frac{1}{2}(n-1)} \frac{\det(x_j^{s_i})}{V(s_1, \dots, s_n) V(x_1, \dots, x_n)}.$$

Let us compute now, for $w = \sum_{j=1}^n w_j c_j$,

$$\begin{aligned} & \Delta((e - e^{i\theta}w)(e + e^{-i\theta}w))^{-\frac{\nu}{2}} \varphi_{\mathbf{s}}(c_{\theta}(w))^{-1} \\ &= \delta! \prod_{j=1}^n (1 - 2i \sin \theta w_j - w_j^2)^{-\frac{\nu}{2}} \\ & \prod_{j=1}^n (c_{\theta}(w_j))^{\frac{1}{2}(n-1)} \frac{\det\left((c_{\theta}(w_j))^{s_i}\right)}{V(s_1, \dots, s_n) V(c_{\theta}(w_1), \dots, c_{\theta}(w_n))}. \end{aligned}$$

In the same way as for the proof of Theorem 4.1, we obtain

$$\begin{aligned} & \Delta((e - e^{i\theta}w)(e + e^{-i\theta}))^{-\frac{\nu}{2}} \varphi_{\mathbf{s}}(c_{\theta}(w))^{-1} \\ &= (-2 \cos \theta)^{-\frac{1}{2}n(n-1)} \delta! \\ & \frac{\det\left((1 - e^{i\theta}w_j)^{s_i - \frac{\nu}{2} + \frac{1}{2}(n-1)} (1 + e^{-i\theta}w_j)^{-s_i - \frac{\nu}{2} + \frac{1}{2}(n-1)}\right)}{V(s_1, \dots, s_n) V(w_1, \dots, w_n)}. \end{aligned}$$

We apply once more Lemma 4.2 to the n power series

$$f_i(w) = (1 - e^{i\theta}w)^{s_i - \frac{\nu'}{2}} (1 + e^{-i\theta}w)^{-s_i - \frac{\nu'}{2}} = \sum_m^{\infty} q_m^{(\nu', \theta)}(s_i) w^m$$

with $\nu' = \nu - n + 1$, and obtain finally:

$$Q_{\mathbf{m}}^{(\nu, \theta)}(\mathbf{s}) = (-2 \cos \theta)^{-\frac{1}{2}n(n-1)} \delta! \frac{\det\left(q_{m_j + \delta_j}^{(\nu - n + 1, \theta)}(s_i)\right)}{V(s_1, \dots, s_n)}.$$

□

5 Difference equation for the Meixner-Pollaczek polynomials $Q_{\mathbf{m}}^{(\nu, \theta)}$

We will introduce a difference operator acting on functions in n variables. We recall first the following Pieri's formula for the spherical functions.

Proposition 5.1.

$$(\text{tr } \mathbf{x}) \varphi_{\mathbf{s}}(\mathbf{x}) = \sum_{j=1}^n \alpha_j(\mathbf{s}) \varphi_{\mathbf{s} + \varepsilon_j}(\mathbf{x}),$$

with

$$\alpha_j(\mathbf{s}) = \prod_{k \neq j} \frac{s_j - s_k + \frac{d}{2}}{s_j - s_k}.$$

($\{\varepsilon_j\}$ denotes the canonical basis of \mathbb{C}^n .)

See [Dib, 1990], Proposition 6.1 (with a minor correction), where it is called Kushner's formula. See also [Zhang, 1995], Theorem 1. One observes that

$$\alpha_j(\mathbf{s}) = \frac{c(\mathbf{s})}{c(\mathbf{s} + \varepsilon_j)},$$

in agreement with the asymptotic behaviour of the spherical function $\varphi_{\mathbf{s}}$: for $\mathbf{s} = (s_1, \dots, s_n) \in \mathbb{C}^n$, with $\operatorname{Re} s_1 > \dots > \operatorname{Re} s_n$, and $a = \sum_{j=1}^n a_j c_j$ with $a_1 > \dots > a_n$ ((c_1, \dots, c_n) is a Jordan frame in V),

$$\varphi_{\mathbf{s}}(\exp ta) \sim c(\mathbf{s})e^{(\mathbf{s}+\rho|a)t} \quad (t \rightarrow \infty).$$

For a partition \mathbf{m} , by letting $\mathbf{m} = \mathbf{s} + \rho$, one gets

$$(\operatorname{tr} x)\Phi_{\mathbf{m}}(x) = \sum_{j=1}^n a_j(\mathbf{m})\Psi_{\mathbf{m}+\varepsilon_j}(x),$$

with

$$a_j(\mathbf{m}) = \prod_{k \neq j} \frac{m_j - m_k - \frac{d}{2}(j - k - 1)}{m_j - m_k - \frac{d}{2}(j - k)}$$

(in agreement with Lassalle's results [1998], p.320, l.-4).

The difference operator $D_{\nu, \theta}$ is defined by

$$\begin{aligned} & D_{\nu, \theta} f(\mathbf{s}) \\ &= e^{-i\theta} \sum_{j=1}^n \left(s_j + \frac{\nu}{2} - \frac{d}{4}(n-1) \right) \alpha_j(\mathbf{s}) (f(\mathbf{s} + \varepsilon_j) - f(\mathbf{s})) \\ &+ e^{i\theta} \sum_{j=1}^n \left(-s_j + \frac{\nu}{2} - \frac{d}{4}(n-1) \right) \alpha_j(-\mathbf{s}) (f(\mathbf{s} - \varepsilon_j) - f(\mathbf{s})). \end{aligned}$$

Theorem 5.2. *The Meixner-Pollaczek polynomial $Q_{\mathbf{m}}^{(\nu, \theta)}$ is an eigenfunction of the difference operator $D_{\nu, \theta}$:*

$$D_{\nu, \theta} Q_{\mathbf{m}}^{(\nu, \theta)} = 2|\mathbf{m}| \cos \theta Q_{\mathbf{m}}^{(\nu, \theta)}.$$

For the proof we will use the scheme we have used in the proof of Theorem 2.1. For $i = 1, 2, 3, 4$, we define the operators $D_{\nu, \theta}^{(i)}$. The operator $D_{\nu, \theta}^{(1)} = D_{\theta}^{(1)}$ is a first order differential operator on the domain \mathcal{D} :

$$D_{\theta}^{(1)} f = e^{i\theta} \langle w + e, \nabla f \rangle + e^{-i\theta} \langle w - e, \nabla f \rangle.$$

(For $W_1, w_2 \in V_{\mathbb{C}}$, $\langle w_1, w_2 \rangle = \text{tr}(w_1 w_2)$.) The operators $D_{\nu, \theta}^{(i)}$, for $i = 2, 3, 4$ are defined by the relations:

$$\begin{aligned} D_{\nu, \theta}^{(2)} C_{\nu} &= C_{\nu} D_{\nu, \theta}^{(1)}, \\ \mathcal{L}_{\nu} D_{\nu, \theta}^{(3)} &= D_{\nu, \theta}^{(2)} \mathcal{L}_{\nu}, \\ \mathcal{F}_{\nu} D_{\nu, \theta}^{(3)} &= D_{\nu, \theta}^{(4)} \mathcal{F}_{\nu}. \end{aligned}$$

The operator $D_{\nu, \theta}^{(2)}$ is a first order differential operator on the tube T_{Ω} . In Section 7 we will see that $D_{\nu, \theta}^{(3)}$ is a second order differential operator on the cone Ω , and prove that $D_{\nu, \theta}^{(4)}$ is the difference operator $D_{\nu, \theta}$ we have introduced above.

The function

$$\Phi_{\mathbf{m}}^{(\theta)}(w) = \Phi_{\mathbf{m}}(w \cos \theta + ie \sin \theta)$$

is an eigenfunction of the operator $D_{\theta}^{(1)}$:

$$D_{\theta}^{(1)} \Phi_{\mathbf{m}}^{(\theta)} = 2|\mathbf{m}| \cos \theta \Phi_{\mathbf{m}}^{(\theta)}.$$

In fact $\Phi_{\mathbf{m}}$ is homogeneous of degree $|\mathbf{m}|$, and satisfies the Euler equation

$$\langle w, \nabla \Phi_{\mathbf{m}} \rangle = |\mathbf{m}| \Phi_{\mathbf{m}}.$$

Hence $F_{\mathbf{m}}^{(\nu, \theta)} = C_{\nu} \Phi_{\mathbf{m}}^{(\theta)}$ is an eigenfunction of $D_{\nu, \theta}^{(2)}$:

$$D_{\nu, \theta}^{(2)} F_{\mathbf{m}}^{(\nu, \theta)} = 2|\mathbf{m}| \cos \theta F_{\mathbf{m}}^{(\nu, \theta)}.$$

Further, since $\mathcal{L}_{\nu} \Psi_{\mathbf{m}}^{(\nu, \theta)} = \frac{(\nu)_{\mathbf{m}}}{\left(\frac{N}{n}\right)_{\mathbf{m}}} F_{\mathbf{m}}^{(\nu, \theta)}$, we get

$$D_{\nu, \theta}^{(3)} \Psi_{\mathbf{m}}^{(\nu, \theta)} = 2|\mathbf{m}| \cos \theta \Psi_{\mathbf{m}}^{(\nu, \theta)}.$$

Finally, since $Q_{\mathbf{m}}^{(\nu,\theta)} = \mathcal{F}_\nu \Psi_{\mathbf{m}}^{(\nu,\theta)}$,

$$D_{\nu,\theta}^{(4)} Q_{\mathbf{m}}^{(\nu,\theta)} = 2|\mathbf{m}| \cos \theta Q_{\mathbf{m}}^{(\nu,\theta)}.$$

Hence the proof of Theorem 5.2 amounts to showing that $D_{\nu,\theta}^{(4)} = D_{\nu,\theta}$.

The symmetries $S_\nu^{(i)}$ we will introduce in next Section will be useful for the computation of the operators $D_{\nu,\theta}^{(i)}$.

6 The symmetries $S_\nu^{(i)}$ ($i = 1, 2, 3, 4$) and the Hankel transform

We start from the symmetry $w \mapsto -w$ of the domain \mathcal{D} . Its action on functions is given by

$$S^{(1)} f(w) = f(-w).$$

We carry this symmetry over the tube T_Ω through the Cayley transform and obtain the inversion $z \mapsto z^{-1}$. We define $S_\nu^{(2)}$ such that

$$S_\nu^{(2)} C_\nu = C_\nu S^{(1)}.$$

Hence, for a function F on T_Ω ,

$$S_\nu^{(2)} F(z) = \Delta(z)^{-\nu} F(z^{-1}).$$

Further $S_\nu^{(3)}$ is defined by the relation

$$\mathcal{L}_\nu S_\nu^{(3)} = S_\nu^{(2)} \mathcal{L}_\nu.$$

By a generalized theorem of Tricomi (Theorem XV.4.1 in [Faraut-Korányi,1994]), the unitary isomorphism $S_\nu^{(3)}$ of $L_\nu^2(\Omega)$ is the Hankel transform: $S_\nu^{(3)} = U_\nu$,

$$U_\nu \psi(u) = \int_\Omega H_\nu(u, v) \psi(v) \Delta(v)^{\nu - \frac{N}{n}} m(dv).$$

The kernel $H_\nu(u, v)$ has the following invariance property:

$$H_\nu(g \cdot u, v) = H_\nu(u, g^* \cdot v) \quad (g \in G),$$

and

$$H_\nu(u, e) = \frac{1}{\Gamma_\Omega(\nu)} \mathcal{J}_\nu(u),$$

where \mathcal{J}_ν is a multivariate Bessel function.

Finally we define $S_\nu^{(4)}$ acting on symmetric polynomials in n variables such that

$$S_\nu^{(4)} \mathcal{F}_\nu = \mathcal{F}_\nu S_\nu^{(3)}.$$

Proposition 6.1. *For a symmetric polynomial p ,*

$$S_\nu^{(4)} p(\mathbf{s}) = p(-\mathbf{s}).$$

Proof. We will evaluate the spherical Fourier transform $\mathcal{F}_\nu(U_\nu \psi)$. By the invariance property, the kernel $H_\nu(u, v)$ can be written

$$H_\nu(u, v) = h_\nu(P(v^{\frac{1}{2}})u) \Delta(u)^{-\frac{\nu}{2}} \Delta(v)^{-\frac{\nu}{2}},$$

with $h_\nu(u) = H_\nu(u, e) \Delta(u)^{\frac{\nu}{2}}$, and P is the so-called quadratic representation of the Jordan algebra V . Let us compute first

$$\begin{aligned} & \int_\Omega H_\nu(u, v) \varphi_{\mathbf{s}}(u) \Delta(u)^{\frac{\nu}{2} - \frac{N}{n}} m(du) \\ &= \Delta(v)^{-\frac{\nu}{2}} \int_\Omega h_\nu(P(v^{\frac{1}{2}})u) \varphi_{\mathbf{s}}(u) \Delta(u)^{-\frac{N}{n}} m(du). \end{aligned}$$

By letting $P(v^{\frac{1}{2}})u = u'$, we get

$$\begin{aligned} & \int_\Omega H_\nu(u, v) \varphi_{\mathbf{s}}(u) \Delta(u)^{\frac{\nu}{2} - \frac{N}{n}} m(du) \\ &= \Delta(v)^{-\frac{\nu}{2}} \int_\Omega h_\nu(u') \varphi_{\mathbf{s}}(P(v^{-\frac{1}{2}})u') \Delta(u)^{-\frac{N}{n}} m(du). \end{aligned}$$

By using K -invariance, and the functional equation of the spherical function $\varphi_{\mathbf{s}}$:

$$\int_K \varphi_{\mathbf{s}}(P(v^{-\frac{1}{2}})ku') dk = \varphi_{\mathbf{s}}(v^{-1}) \varphi_{\mathbf{s}}(u'),$$

we get

$$\int_\Omega H_\nu(u, v) \varphi_{\mathbf{s}}(u) \Delta(u)^{\frac{\nu}{2} - \frac{N}{n}} m(du) = \varphi_{\mathbf{s}}(v^{-1}) \Delta(v)^{-\frac{\nu}{2}} \mathcal{F}(h_\nu)(\mathbf{s}).$$

Recall that $\varphi_{\mathbf{s}}(v^{-1}) = \varphi_{-\mathbf{s}}(v)$. We multiply both sides by $\psi(v)$ and get by integrating with respect to v :

$$\Gamma_{\Omega}(\mathbf{s} + \frac{\nu}{2} + \rho) \mathcal{F}_{\nu}(U_{\nu}\psi)(\mathbf{s}) = \mathcal{F}h_{\nu}(\mathbf{s}) \Gamma_{\Omega}(-\mathbf{s} + \frac{\nu}{2} + \rho) \mathcal{F}_{\nu}\psi(-\mathbf{s}).$$

Consider the special case $\psi(u) = \Psi_0(u) = e^{-\text{tr } u}$. Since $U_{\nu}\Psi_0 = \Psi_0$, and $\mathcal{F}_{\nu}\Psi_0 \equiv 1$, we get

$$\mathcal{F}(h_{\nu}) = \frac{\Gamma_{\Omega}(\mathbf{s} + \frac{\nu}{2} + \rho)}{\Gamma_{\Omega}(-\mathbf{s} + \frac{\nu}{2} + \rho)}.$$

Finally

$$\mathcal{F}_{\nu}(U_{\nu}\psi)(\mathbf{s}) = \mathcal{F}_{\nu}\psi(-\mathbf{s}).$$

It follows that $S_{\nu}^{(4)}p(\mathbf{s}) = p(-\mathbf{s})$. □

Corollary 6.2.

$$Q_{\mathbf{m}}^{(\nu, \theta)}(-\mathbf{s}) = (-1)^{|\mathbf{m}|} Q_{\mathbf{m}}^{(\nu, -\theta)}(\mathbf{s}).$$

Proof. This relation follows from

$$S^{(1)}\Phi_{\mathbf{m}}^{(\theta)} = \Phi_{\mathbf{m}}^{(\theta)}(-w) = (-1)^{|\mathbf{m}|} \Phi_{\mathbf{m}}^{(-\theta)}(w),$$

which is easy to check, and Proposition 6.1. □

The operator $D_{\nu, \theta}^{(i)}$ ($i = 1, 2, 3, 4$) can be written

$$D_{\nu, \theta}^{(i)} = e^{i\theta} D_{\nu}^{(i, +)} + e^{-i\theta} D_{\nu}^{(i, -)}.$$

For $i = 1$, $D_{\nu}^{(1, \pm)}$ does not depend on ν , $D_{\nu}^{(1, \pm)} = D^{(1, \pm)}$:

$$D^{(1, +)}f(w) = \langle w + e, \nabla f(w) \rangle, \quad D^{(1, -)}f(w) = \langle w - e, \nabla f(w) \rangle.$$

Observe that

$$D^{(1, -)} = S^{(1)}D^{(1, +)}S^{(1)}.$$

It follows that, for $i = 2, 3, 4$,

$$D_{\nu}^{(i, -)} = S_{\nu}^{(i)}D_{\nu}^{(i, +)}S_{\nu}^{(i)}.$$

In next Section we will compute first $D_{\nu}^{(i, -)}$. The operator $D_{\nu}^{(i, +)}$ is then obtained by using the above relation. For $i = 3$, we will use the following property of the Hankel transform

Proposition 6.3.

$$U_{\nu}(\text{tr } v \psi) = -\left(\langle \mathbf{u}, \left(\frac{\partial}{\partial \mathbf{u}} \right)^2 \rangle + \nu \text{tr} \left(\frac{\partial}{\partial \mathbf{u}} \right) \right) U_{\nu}\psi.$$

This is a consequence of Proposition XV.2.3 in [Faraud-Korányi, 1994].

7 Proof of Theorem 5.2

a) Recall that $D^{(1,-)}$ is the first order differential operator on the domain \mathcal{D} given by

$$D^{(1,-)}f(w) = \langle w - e, \nabla f(w) \rangle,$$

and $D_\nu^{(2,-)}$ is the first order differential operator on the tube T_Ω such that

$$D_\nu^{(2,-)}C_\nu = C_\nu D^{(1,-)}.$$

Lemma 7.1.

$$D_\nu^{(2,-)}F(z) = -\langle z + e, \nabla F(z) \rangle - n\nu F(z).$$

Proof. Recall that, for a function F on the tube T_Ω ,

$$f(w) = (C_\nu^{-1}F)(w) = \Delta(e - w)^{-\nu}F(c(w)),$$

where c is the Cayley transform

$$c(w) = (e + w)(e - w)^{-1} = 2(e - w)^{-1} - e.$$

Its differential is given by

$$(Dc)_w = 2P((e - w)^{-1}).$$

We get

$$\nabla f(w) = \nabla(\Delta(e - w)^{-\nu})F(c(w)) + \Delta(e - w)^{-\nu}2P(e - w)^{-1}(\nabla F(c(w))).$$

By using

$$\begin{aligned} \nabla(\Delta(x)^\alpha) &= \alpha\Delta(x)^\alpha x^{-1}, \\ \langle e - w, (e - w)^{-1} \rangle &= n, \\ P((e - w)^{-1})(e - w) &= (e - w)^{-1}, \end{aligned}$$

we obtain

$$\begin{aligned} \langle w - e, \nabla f(w) \rangle &= \Delta(e - w)^{-\nu} \left(-n\nu F(c(w)) + 2\langle (w - e)^{-1}, \nabla F(c(w)) \rangle \right) \\ &= (C_\nu^{-1}G)(z), \end{aligned}$$

with

$$G(z) = -\langle z + e, \nabla F(z) \rangle - n\nu F(z).$$

□

b) Consider now the differential operator $D_\nu^{(3,-)}$ on the cone Ω such that

$$\mathcal{L}_\nu D_\nu^{(3,-)} = D_\nu^{(2,-)} \mathcal{L}_\nu.$$

Recall that the modified Laplace transform $\mathcal{L}_\nu \psi$ of a function ψ , defined on Ω , is given by

$$F(z) = \mathcal{L}_\nu \psi(z) = \frac{2^{n\nu}}{\Gamma_\Omega(\nu)} \int_\Omega e^{-\langle z|u \rangle} \psi(u) \Delta(u)^{\nu - \frac{N}{n}} m(du).$$

Lemma 7.2.

$$D_\nu^{(3,-)} \psi(u) = \langle u, \nabla \psi(u) \rangle + \text{tr } u \psi(u).$$

Proof. For $a \in V_\mathbb{C}$,

$$\langle a, \nabla F(z) \rangle = \frac{2^{n\nu}}{\Gamma_\Omega(\nu)} \int_\Omega e^{-\langle z|u \rangle} (-\langle a, u \rangle) \psi(u) \Delta(u)^{\nu - \frac{N}{n}} m(du).$$

Observe that

$$\langle z|u \rangle e^{-\langle z|u \rangle} = \langle u, \nabla_u \rangle e^{-\langle z|u \rangle}.$$

Therefore

$$\langle z, \nabla F(z) \rangle = \frac{2^{n\nu}}{\Gamma_\Omega(\nu)} \int_\Omega (-\langle u, \nabla_u \rangle e^{-\langle z|u \rangle}) \psi(u) \Delta(u)^{\nu - \frac{N}{n}} m(du).$$

An integration by parts gives

$$= \frac{2^{n\nu}}{\Gamma_\Omega(\nu)} \int_\Omega e^{-\langle z|u \rangle} (\langle u, \nabla \rangle + n\nu) \psi(u) \Delta(u)^{\nu - \frac{N}{n}} m(du).$$

Finally

$$(D_\nu^{(2,-)} F)(z) = \mathcal{L}_\nu (\langle u, \nabla \psi \rangle + \text{tr } u \psi).$$

□

c) The operator $D_\nu^{(4,-)}$ acting on symmetric functions on \mathbb{C}^n is such that

$$D_\nu^{(4,-)} \mathcal{F}_\nu = \mathcal{F}_\nu D_\nu^{(3,-)}.$$

Recall that the spherical Fourier transform $f = \mathcal{F}_\nu \psi$ of a function ψ , defined on Ω , is given by

$$f(\mathbf{s}) = (\mathcal{F}_\nu \psi)(\mathbf{s}) = \frac{1}{\Gamma_\Omega(\mathbf{s} + \frac{\nu}{2} + \rho)} \int_\Omega \varphi_{\mathbf{s}}(u) \psi(u) \Delta(u)^{\frac{\nu}{2} - \frac{N}{n}} m(du).$$

Proposition 7.3. *The operator $D_\nu^{(4,-)}$ is the following difference operator: for a function f on \mathbb{C}^n ,*

$$D_\nu^{(4,-)} f(\mathbf{s}) = \sum_{j=1}^n \left(s_j + \frac{\nu}{2} - \frac{d}{4}(n-1)\alpha_j(\mathbf{s}) \right) (f(\mathbf{s} + \varepsilon_j) - f(\mathbf{s})).$$

Proof. We will compute $\mathcal{F}_\nu(D_\nu^{(3,-)}\psi) = \mathcal{F}_\nu(\langle u, \nabla \psi \rangle + \text{tr } u \psi)$. Consider first

$$\mathcal{F}_\nu(\langle u, \nabla \psi \rangle)(\mathbf{s}) = \frac{1}{\Gamma_\Omega(\mathbf{s} + \frac{\nu}{2} + \rho)} \int_\Omega \langle u, \nabla \psi(u) \rangle \varphi_{\mathbf{s} + \frac{\nu}{2}}(u) \Delta(u)^{-\frac{N}{n}} m(du).$$

An integration by parts gives, since the function $\varphi_{\mathbf{s}}$ is homogeneous of degree $\sum_{j=1}^n s_j$ (observe that $\sum_{j=1}^n \rho_j = 0$),

$$\begin{aligned} &= \frac{1}{\Gamma_\Omega(\mathbf{s} + \frac{\nu}{2} + \rho)} \int_\Omega \psi(u) (-\langle u, \nabla_u \rangle \varphi_{\mathbf{s} + \frac{\nu}{2}}(u)) \Delta(u)^{-\frac{N}{n}} m(du) \\ &= \frac{1}{\Gamma_\Omega(\mathbf{s} + \frac{\nu}{2} + \rho)} \int_\Omega \psi(u) \left(-\sum_{j=1}^n \left(s_j + \frac{\nu}{2} \right) \varphi_{\mathbf{s}}(u) \Delta(u)^{\frac{\nu}{2} - \frac{N}{n}} m(du) \right) \\ &= -\sum_{j=1}^n \left(s_j + \frac{\nu}{2} \right) \mathcal{F}_\nu \psi(\mathbf{s}). \end{aligned}$$

Recall the Pieri's formula (Proposition 5.1):

$$\text{tr } u \varphi_{\mathbf{s}}(u) = \sum_{j=1}^n \alpha_j(\mathbf{s}) \varphi_{\mathbf{s} + \varepsilon_j}(u).$$

Hence

$$\begin{aligned}
& \mathcal{F}_\nu(\text{tr u } \psi)(\mathbf{s}) \\
&= \frac{1}{\Gamma_\Omega(\mathbf{s} + \frac{\nu}{2} + \rho)} \int_\Omega \psi(u) \left(\sum_{j=1}^n \alpha_j(\mathbf{s}) \varphi_{\mathbf{s} + \varepsilon_j}(u) \right) \Delta(u)^{\frac{\nu}{2} - \frac{N}{n}} m(du) \\
&= \sum_{j=1}^n \frac{\Gamma_\Omega(\mathbf{s} + \varepsilon_j + \frac{\nu}{2} + \rho)}{\Gamma_\Omega(\mathbf{s} + \frac{\nu}{2} + \rho)} \alpha_j(\mathbf{s}) \\
&\quad \frac{1}{\Gamma_\Omega(\mathbf{s} + \varepsilon_j + \frac{\nu}{2} + \rho)} \int_\Omega \psi(u) \varphi_{\mathbf{s} + \varepsilon_j}(u) \Delta(u)^{\frac{\nu}{2} - \frac{N}{n}} m(du) \\
&= \sum_{j=1}^n \left(s_j + \frac{\nu}{2} - \frac{d}{4}(n-1) \right) \alpha_j(\mathbf{s}) \mathcal{F}_\nu \psi(\mathbf{s} + \varepsilon_j).
\end{aligned}$$

Finally

$$\begin{aligned}
& \mathcal{F}_\nu(D_\nu^{(3,-)} \psi)(\mathbf{s}) \\
&= \sum_{j=1}^n \left(s_j + \frac{\nu}{2} - \frac{d}{4}(n-1) \right) \alpha_j(\mathbf{s}) f(\mathbf{s} + \varepsilon_j) - \sum_{j=1}^n \left(s_j + \frac{\nu}{2} \right) f(\mathbf{s}),
\end{aligned}$$

with $f = \mathcal{F}_\nu(\psi)$. From $D_\nu^{(3,-)} \Psi_0 = 0$ and $\mathcal{F}_\nu(\Psi_0) = 1$, we get

$$\sum_{j=1}^n \left(s_j + \frac{\nu}{2} - \frac{d}{4}(n-1) \right) \alpha_j(\mathbf{s}) = \sum_{j=1}^n \left(s_j + \frac{\nu}{2} \right).$$

Therefore

$$\begin{aligned}
& \mathcal{F}_\nu(D_\nu^{(3,-)} \psi)(\mathbf{s}) \\
&= \sum_{j=1}^n \left(s_j + \frac{\nu}{2} - \frac{d}{4}(n-1) \right) \alpha_j(\mathbf{s}) (f(\mathbf{s} + \varepsilon_j) - f(\mathbf{s})).
\end{aligned}$$

□

We finish now the proof of Theorem 5.2. Recall that

$$D_\nu^{(4,+)} = S_\nu^{(4)} D_\nu^{(4,-)} S_\nu^{(4)}, \quad \text{and} \quad S_\nu^{(4)} f(\mathbf{s}) = f(-\mathbf{s}).$$

Therefore, by Proposition 7.3,

$$D_{\nu}^{(4,+)} f(\mathbf{s}) = \sum_{j=1}^n \left(-s_j + \frac{\nu}{2} - \frac{d}{4}(n-1)\right) \alpha_j(-\mathbf{s}) (f(\mathbf{s} - \varepsilon_j) - f(\mathbf{s})).$$

We have establish the formula of Theorem 5.2 since

$$D_{\nu,\theta} = D_{\nu,\theta}^{(4)} = e^{i\theta} D_{\nu}^{(4,+)} + e^{-i\theta} D_{\nu}^{(4,-)}.$$

8 Differential equation for the Laguerre polynomials $L_{\mathbf{m}}^{(\nu-1)}$

Theorem 8.1. *The Laguerre polynomial $L = L_{\mathbf{m}}^{(\nu-1)}$ is a solution of the differential equation*

$$\langle x, \left(\frac{\partial}{\partial x}\right)^2 \rangle L + \langle \nu e - x, \left(\frac{\partial}{\partial x}\right) \rangle L + |\mathbf{m}|L = 0.$$

Observe that, for $n = 1$, this is the classical Laguerre differential equation for the ordinary Laguerre polynomial $y = L_m^{(\nu-1)}$:

$$xy'' + (\nu - x)y' + my = 0.$$

An equivalent formula is given in [Davidson-Ólafsson,2003], Theorem 6.1, and in [Aristidou et al.,2007], Theorem 6.3.

Proof. Recall the relation

$$\Psi_{\mathbf{m}}^{(\nu)}(u) = e^{-\text{tr } u} L_{\mathbf{m}}^{(\nu-1)}(2u),$$

and that

$$D_{\nu,0}^{(3)} \Psi_{\mathbf{m}}^{(\nu)} = 2|\mathbf{m}| \Psi_{\mathbf{m}}^{(\nu)}.$$

Furthermore

$$D_{\nu,0}^{(3)} = D_{\nu}^{(3,+)} + D_{\nu}^{(3,-)}, \quad D_{\nu}^{(3,+)} = U_{\nu} D_{\nu}^{(3,-)} U_{\nu},$$

where $U_{\nu} = S_{\nu}^{(3)}$ is the Hankel transform. By Proposition 7.2,

$$D_{\nu}^{(3,-)} \psi = \langle u, \nabla \psi(u) \rangle + \text{tr } u \psi.$$

By using the relation

$$U_\nu(\langle v, \nabla \psi \rangle) = -(\langle u, \nabla \rangle + n\nu)U_\nu \psi,$$

and Proposition 6.3 we obtain

$$D_\nu^{(3,+)} = -\left(\langle u, \left(\frac{\partial}{\partial u}\right)^2 \right) + \nu \operatorname{tr} \left(\frac{\partial}{\partial u}\right) + \langle u, \left(\frac{\partial}{\partial u}\right) \rangle + n\nu),$$

and also

$$\begin{aligned} D_{\nu,0}^{(3)} &= D_\nu^{(3,+)} + D_\nu^{(3,-)} \\ &= -\langle u, \left(\frac{\partial}{\partial u}\right)^2 \rangle - \nu \operatorname{tr} \left(\frac{\partial}{\partial u}\right) + \operatorname{tr} u - n\nu. \end{aligned}$$

This formula and the relation

$$\Psi_{\mathbf{m}}^{(\nu)}(u) = e^{-\operatorname{tr} u} L_{\mathbf{m}}^{(\nu-1)}(2u),$$

gives Theorem 8.1. □

A K -invariant function f on V only depends on the eigenvalues. Define

$$F(x_1, \dots, x_n) = f(x_1 c_1 + \dots + x_n c_n),$$

where (c_1, \dots, c_n) is a Jordan frame. Hence F is a symmetric function on \mathbb{R}^n .

Corollary 8.2. *The multivariate Laguerre polynomial $L_{\mathbf{m}}^{(\nu-1)}(x) = L(x_1, \dots, x_n)$ is solution of the following equation*

$$\begin{aligned} &\sum_{i=1}^n x_i \frac{\partial^2 L}{\partial x_i^2} + d \sum_{i < j} \frac{1}{x_i - x_j} \left(x_i \frac{\partial L}{\partial x_i} - x_j \frac{\partial L}{\partial x_j} \right) \\ &+ \sum_{i=1}^n \left(\nu - \frac{d}{2}(n-1) - x_i \right) \frac{\partial L}{\partial x_i} + |\mathbf{m}|L = 0. \end{aligned}$$

This is essentially the differential operator (2.1b) in [Baker-Forrester,1997].

One follows the same lines as in the proof of Proposition VI.4.2 in [Faraud-Korányi,1994].

9 Pieri's formula for the Meixner-Pollaczek polynomials $Q_{\mathbf{m}}^{(\nu, \theta)}$

Theorem 9.1. *The Meixner-Pollaczek polynomials $Q_{\mathbf{m}}^{(\nu, \theta)}$ satisfy the following Pieri's formula:*

$$\begin{aligned} & (2|\mathbf{s}| \cos \theta - 2i|2\mathbf{m} + \nu| \sin \theta) Q_{\mathbf{m}}^{(\nu, \theta)}(\mathbf{s}) \\ &= \sum_{j=1}^n (m_j + \nu - 1 - \frac{d}{4}(j-1)) \alpha_j(\mathbf{m} - \varepsilon_j - \rho) d_{\mathbf{m} - \varepsilon_j} Q_{\mathbf{m} - \varepsilon_j}^{(\nu, \theta)}(\mathbf{s}) \\ & \quad - \sum_{j=1}^n (m_j + 1 + \frac{d}{4}(n-j)) \alpha_j(-\mathbf{m} - \varepsilon_j - \rho) d_{\mathbf{m} + \varepsilon_j} Q_{\mathbf{m}}^{(\nu, \theta)}(\mathbf{s}). \end{aligned}$$

Proof. The generating formula (Theorem 3.1 (ii)), with $\mathbf{s} = \mathbf{m} + \frac{\nu}{2} - \rho$ can be written:

$$\begin{aligned} & \sum_{\mathbf{k}} d_{\mathbf{k}} Q_{\mathbf{k}}^{(\nu, \theta)}(\mathbf{m} + \frac{\nu}{2} - \rho) \Phi_{\mathbf{k}}(w) \\ &= \Delta(e + e^{-i\theta}w)^{-\nu} \Phi_{\mathbf{m}}((e - e^{i\theta}w)(e + e^{-i\theta}w)^{-1}). \end{aligned}$$

Since

$$\begin{aligned} & F_{\mathbf{m}}^{(\nu, \theta)}(e^{-i\theta}w) \\ &= 2^{n\nu} \Delta(e + e^{-i\theta}w)^{-\nu} (-1)^{|\mathbf{m}|} e^{-i|\mathbf{m}|\theta} \Phi_{\mathbf{m}}((e - e^{i\theta}w)(e + e^{-i\theta}w)^{-1}), \end{aligned}$$

we obtain

$$\sum_{\mathbf{k}} Q_{\mathbf{k}}^{(\nu, \theta)}(\mathbf{m} + \frac{\nu}{2} - \rho) e^{i|\mathbf{k}|\theta} \Phi_{\mathbf{k}}(w) = 2^{-n\nu} (-1)^{|\mathbf{m}|} e^{i|\mathbf{m}|\theta} F_{\mathbf{m}}^{(\nu, \theta)}(w).$$

Recall that the function $F_{\mathbf{m}}^{(\nu, \theta)}$ is an eigenfunction of the differential operator $D_{\nu, \theta}^{(2)}$:

$$D_{\nu, \theta}^{(2)} F_{\mathbf{m}}^{(\nu, \theta)}(w) = 2|\mathbf{m}| \cos \theta F_{\mathbf{m}}^{(\nu, \theta)}(w).$$

It follows that

$$\begin{aligned} & \sum_{\mathbf{k}} d_{\mathbf{k}} Q_{\mathbf{k}}^{(\nu, \theta)}(\mathbf{m} + \frac{\nu}{2} - \rho) e^{i|\mathbf{k}|\theta} D_{\nu, \theta}^{(2)} \Phi_{\mathbf{k}}(w) \\ &= 2|\mathbf{m}| \cos \theta \sum_{\mathbf{k}} d_{\mathbf{k}} Q_{\mathbf{k}}^{(\nu, \theta)}(\mathbf{m} + \frac{\nu}{2} - \rho) \Phi_{\mathbf{k}}(w). \quad (9.1) \end{aligned}$$

□

Lemma 9.2. (i)

$$\operatorname{tr}(\nabla\varphi_{\mathbf{s}}(z)) = \sum_{j=1}^n \left(s_j + \frac{d}{4}(n-1)\right) \alpha_j(-\mathbf{s}) \varphi_{\mathbf{s}-\varepsilon_j}(z).$$

(ii)

$$\begin{aligned} & D_{\nu,\theta}^{(2)}\varphi_{\mathbf{s}}(z) \\ = & e^{i\theta} \left(\sum_{j=1}^n \left(s_j - \frac{d}{4}(n-1) + \nu\right) \alpha_j(\mathbf{s}) \varphi_{\mathbf{s}+\varepsilon_j}(z) + \left(\sum_{j=1}^n s_j\right) \varphi_{\mathbf{s}}(z) \right) \\ & - e^{-i\theta} \left(\sum_{j=1}^n \left(s_j + \frac{d}{4}(n-1)\right) \alpha_j(-\mathbf{s}) \varphi_{\mathbf{s}-\varepsilon_j}(z) + \left(\sum_{j=1}^n s_j\right) \varphi_{\mathbf{s}}(z) + n\nu\varphi_{\mathbf{s}}(z) \right). \end{aligned}$$

((i) is in agreement with Lassalle's results [1998], p.321, first line of (14.1).)

Proof. (i) For $t > 0$ we consider the following Laplace integral:

$$\int_{\Omega} e^{-(x|y)} e^{-t\operatorname{tr}y} \varphi_{\mathbf{s}}(y) \Delta(y)^{-\frac{N}{n}} m(dy) = \Gamma_{\Omega}(\mathbf{s} + \rho) \varphi_{-\mathbf{s}}(te + x).$$

Taking the derivatives with respect to t for $t = 0$, one gets:

$$- \int_{\Omega} e^{-(x|y)} \operatorname{tr}y \varphi_{\mathbf{s}}(y) \Delta(y)^{-\frac{N}{n}} m(dy) = \Gamma_{\Omega}(\mathbf{s} + \rho) \operatorname{tr}(\nabla\varphi_{-\mathbf{s}}(x)).$$

By using Proposition 5.1:

$$\operatorname{tr}y \varphi_{\mathbf{s}}(y) = \sum_{j=1}^n \alpha_j(\mathbf{s}) \varphi_{\mathbf{s}+\varepsilon_j}(y),$$

and since

$$\begin{aligned} & \sum_{j=1}^n \alpha_j(\mathbf{s}) \int_{\Omega} e^{-(x|y)} \varphi_{\mathbf{s}+\varepsilon_j}(y) \Delta(y)^{-\frac{N}{n}} m(dy) \\ = & \sum_{j=1}^n \alpha_j(\mathbf{s}) \Gamma_{\Omega}(\mathbf{s} + \varepsilon_j + \rho) \varphi_{-\mathbf{s}-\varepsilon_j}(x), \end{aligned}$$

one obtains

$$\begin{aligned}\mathrm{tr}(\nabla\varphi_{-\mathbf{s}}(x)) &= -\sum_{j=1}^n \alpha_j(\mathbf{s}) \frac{\Gamma_{\Omega}(\mathbf{s} + \varepsilon_j + \rho)}{\Gamma_{\Omega}(\mathbf{s} + \rho)} \varphi_{-\mathbf{s} - \varepsilon_j}(x) \\ &= -\sum_{j=1}^n \alpha_j(\mathbf{s}) \left(s_j - \frac{d}{4}(n-1) \right) \varphi_{-\mathbf{s} - \varepsilon_j}(x),\end{aligned}$$

or

$$\mathrm{tr}(\nabla\varphi_{\mathbf{s}}(x)) = \sum_{j=1}^n \alpha_j(-\mathbf{s}) \left(s_j + \frac{d}{4}(n-1) \right) \varphi_{\mathbf{s} - \varepsilon_j}(x).$$

In fact the explicit formula for Γ_{Ω} ,

$$\Gamma_{\Omega}(\mathbf{s} + \rho) = (2\pi)^{N-n} \prod_{j=1}^n \Gamma\left(s_j - \frac{d}{4}(n-1)\right),$$

gives

$$\frac{\Gamma_{\Omega}(\mathbf{s} + \varepsilon_j + \rho)}{\Gamma_{\Omega}(\mathbf{s} + \rho)} = \frac{\Gamma\left(s_j + 1 - \frac{d}{4}(n-1)\right)}{\Gamma\left(s_j - \frac{d}{4}(n-1)\right)} = s_j - \frac{d}{4}(n-1).$$

(ii) Recall that

$$D_{\nu}^{(2,-)}F(z) = -\langle z + e, \nabla F(z) \rangle - n\nu F(z).$$

From (i) we obtain

$$D_{\nu}^{(2,-)}\varphi_{\mathbf{s}}(z) = \sum_{j=1}^n \left(s_j + \frac{d}{4}(n-1) \right) \alpha_j(-\mathbf{s}) \varphi_{\mathbf{s} - \varepsilon_j}(z) - \left(\sum_{j=1}^n s_j + n\nu \right) \varphi_{\mathbf{s}}(z).$$

By using $D_{\nu}^{(2,+)} = S_{\nu}^{(2)} D_{\nu}^{(2,-)} S_{\nu}^{(2)}$ and $S_{\nu}^{(2)}\varphi_{\mathbf{s}}(z) = \varphi_{-\mathbf{s} - \nu}(z)$, we get (ii). \square

We continue the proof of Theorem 9.1. Let us write (ii) of Lemma 9.2 with $\mathbf{s} = \mathbf{k} - \rho$:

$$\begin{aligned}& D_{\nu,k}^{(2)}\Phi_{\mathbf{k}}(w) \\ &= e^{i\theta} \left(\sum_{j=1}^n \left(k_j + \nu - \frac{d}{2}(j-1) \right) \alpha_j(\mathbf{k} - \rho) \Phi_{\mathbf{k} + \varepsilon_j}(w) + |\mathbf{k}| \Phi_{\mathbf{k}}(w) \right) \\ & \quad - e^{-i\theta} \left(\sum_{j=1}^n \left(k_j + \frac{d}{2}(n-j) \right) \alpha_j(-\mathbf{k} + \rho) \Phi_{\mathbf{k} - \varepsilon_j}(w) + (|\mathbf{k}| + n\nu) \Phi_{\mathbf{k}}(w) \right).\end{aligned}$$

(Observe that $\sum_{j=1}^n \rho_j = 0$.) Now, equaling the coefficient of $\Phi_{\mathbf{k}}(z)$ in both sides of (9.1), we obtain the formula of Theorem 9.1 for all $\mathbf{s} = \mathbf{m} + \frac{z}{2} - \rho$. Since both sides are polynomial functions in \mathbf{s} , the equality holds for every \mathbf{s} .

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