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# ANALYSE OF THE BRYLINSKI-KOSTANT MODEL FOR SPHERICAL MINIMAL REPRESENTATIONS

Dehbia Achab & Jacques Faraut

**Abstract** *We revisit with another view point the construction by R. Brylinski and B. Kostant of minimal representations of simple Lie groups. We start from a pair* $(V,Q)$*, where* $V$ *is a complex vector space and* $Q$ *a homogeneous polynomial of degree 4 on* $V$*. The manifold* $\Xi$ *is an orbit of a covering of* $\mathrm{Conf}(V,Q)$*, the conformal group of the pair* $(V,Q)$*, in a finite dimensional representation space. By a generalized Kantor-Koecher-Tits construction we obtain a complex simple Lie algebra* $\mathfrak{g}$*, and furthermore a real form* $\mathfrak{g}_{\mathbb{R}}$*. The connected and simply connected Lie group* $G_{\mathbb{R}}$ *with* $\mathrm{Lie}(G_{\mathbb{R}})=\mathfrak{g}_{\mathbb{R}}$ *acts unitarily on a Hilbert space of holomorphic functions defined on the manifold* $\Xi$*.*

*Key words*: Minimal representation, Kantor-Koecher-Tits construction, Jordan algebra, Bernstein identity, Meijer $G$-function.

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**Contents**

The construction of a realization for the minimal unitary representation of a simple Lie group by using geometric quantization has been the topic of many papers during the last thirty years: [Rawnsley-Sternberg,1982], [Torasso,1983], and more recently [Kobayashi-Ørsted,2003]. In a series of papers R. Brylinski and B. Kostant have introduced and studied a geometric quantization of minimal nilpotent orbits for simple real Lie groups which are not of Hermitian type: [Brylinski-Kostant,1994,1995], [Brylinski, 1997,1998]. They have constructed the associated irreducible unitary representation on a Hilbert space of half forms on the minimal nilpotent orbit. This can be considered as a Fock model for the minimal representation. In this paper we revisit this construction with another point of view. We start from a pair $(V,Q)$ where $V$ is a complex vector space and $Q$ is a homogeneous polynomial on $V$ of degree 4. The structure group $\mathrm{Str}(V,Q)$, for which $Q$ is a semi-invariant, is assumed to have a symmetric open orbit. The conformal group $\mathrm{Conf}(V,Q)$ consists of rational transformations of $V$ whose differential belongs to $\mathrm{Str}(V,Q)$. The main geometric object is the orbit $\Xi$ of $Q$ under $K$, a covering of $\mathrm{Conf}(V,Q)$, on a space $\mathcal{W}$ of polynomials on $V$. Then, by a generalized Kantor-Koecher-Tits construction, starting from the Lie algebra $\mathfrak{k}$ of $K$, we obtain a simple Lie algebra $\mathfrak{g}$ such that the pair $(\mathfrak{g},\mathfrak{k})$ is non Hermitian. As a vector space $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$, with $\mathfrak{p}=\mathcal{W}$. The main point is to define a bracket

$$\mathfrak{p}\oplus\mathfrak{p}\to\mathfrak{k},\quad (X,Y)\mapsto[X,Y],$$

such that $\mathfrak{g}$ becomes a Lie algebra. The Lie algebra $\mathfrak{g}$ is 5-graded:

$$\mathfrak{g}=\mathfrak{g}_{-2}\oplus\mathfrak{g}_{-1}\oplus\mathfrak{g}_{0}\oplus\mathfrak{g}_{1}\oplus\mathfrak{g}_{2}.$$

In the fourth part one defines a representation $\rho$ of $\mathfrak{g}$ on the space $\mathcal{O}(\Xi)_{\mathrm{fin}}$ of polynomial functions on $\Xi$. In a first step one defines a representation of an $\mathfrak{sl}_2$-triple $(E,F,H)$. It turns out that this is only possible under a condition (T). In such a case one obtains an irreducible unitary representation of the connected and simply connected group $\widetilde{G}_{\mathbb{R}}$ whose Lie algebra is a real form of $\mathfrak{g}$. The representation is spherical. It is realized on a Hilbert space of holomorphic functions on $\Xi$. There is an explicit formula for the reproducing kernel of $\mathcal{H}$ involving a hypergeometric function ${}_1F_2$. Further the space $\mathcal{H}$ is a weighted Bergman space with a weight taking in general both positive and negative values.

The pairs satisfying (T) are the following ones:

$$\begin{array}{ll} \text{Classical pairs} & \big((\mathfrak{sl}(n,\mathbb{R}),\mathfrak{so}(n)\big),\ \big(\mathfrak{so}(p,p),\mathfrak{so}(p)\oplus\mathfrak{so}(p)\big), \\ \text{Exceptional pairs} & \big(\mathfrak{e}_{6(6)},\mathfrak{sp}(8)\big),\ \big(\mathfrak{e}_{7(7)},\mathfrak{su}(8)\big),\ \big(\mathfrak{e}_{8(8)},\mathfrak{so}(16)\big). \end{array}$$

If $Q = R^2$ or $Q = R^4$ where $R$ is a semi-invariant, then by considering a covering of order 2 or 4 of the orbit $\Xi$, one can obtain one or 3 other unitary representations of $\widetilde{G}_{\mathbb{R}}$. They are not spherical. If the condition T is not satisfied, by a modified construction, one still obtains an irreducible representation of $\widetilde{G}_{\mathbb{R}}$ which is not spherical. This last point is the subject of a paper in preparation by the first author.

The construction of a Schrödinger model for the minimal representation of the group $O(p,q)$ is the subject of a recent book by T. Kobayashi and G. Mano [2008]. We should not wonder that there is a link between both models: the Fock and the Schrödinger models, and that there is an analogue of the Bargmann transform in this setting.

# 1 The conformal group and the representation $\kappa$

Let $V$ be a finite dimensional complex vector space and $Q$ a homogeneous polynomial on $V$. Define

$$L = \mathrm{Str}(V,Q) = \{g \in GL(V) \mid \exists\gamma = \gamma(g), Q(g\cdot x) = \gamma(g)Q(x)\}.$$

Assume that there exists $e \in V$ such that

(1) The symmetric bilinear form

$$\langle x, y\rangle = -D_x D_y \log Q(e),$$

is non-degenerate.

(2) The orbit $\Omega = L\cdot e$ is open.

(3) The orbit $\Omega = L\cdot e$ is symmetric, i.e. the pair $(L, L_0)$, with $L_0 = \{g \in L \mid g\cdot e = e\}$, is symmetric, which means that there is an involutive automorphism $\nu$ of $L$ such that $L_0$ is open in $\{g \in L \mid \nu(g) = g\}$.

We will equip the vector space $V$ with a Jordan algebra structure. The Lie algebra $\mathfrak{l} = \mathrm{Lie}(L)$ of $L = \mathrm{Str}(V,Q)$ decomposes into the $+1$ and $-1$

eigenspaces of the differential of $\nu : \mathfrak{l} = \mathfrak{l}_0 + \mathfrak{q}$, where $\mathfrak{l}_0 = \{X \in \mathfrak{l} \mid X \cdot e = e\} = \mathrm{Lie}(L_0)$. Since the orbit $\Omega$ is open, the map

$$\mathfrak{q} \to V, \quad X \mapsto X \cdot e,$$

is a linear isomorphism. If $X \cdot e = x \quad (X \in \mathfrak{q}, x \in V)$ one writes $X = T_x$. The product on $V$ is defined by

$$xy = T_x \cdot y = T_x \circ T_y \cdot e.$$

**Theorem 1.1.** *This product makes $V$ into a semi-simple complex Jordan algebra:*

(J1) *For $x, y \in V, xy = yx$.*
(J2) *For $x, y \in V, x^2(xy) = x(x^2y)$.*
(J3) *The symmetric bilinear form $\langle .,. \rangle$ is associative:*

$$\langle xy, z \rangle = \langle x, yz \rangle.$$

*Proof.* (a) This product is commutative. In fact

$$xy - yx = [T_x, T_y] \cdot e = 0,$$

since $[\mathfrak{q}, \mathfrak{q}] \subset \mathfrak{l}_0$.

(b) Let $\tau$ be the differential of $\gamma$ at the identity element of $L$: for $X \in \mathfrak{l}$,

$$\tau(X) = \frac{d}{dt}\Big|_{t=0} \gamma(\exp tX).$$

**Lemma 1.2.**

$$\begin{array}{ll} \text{(i)} & (D_x \log Q)(e) = \tau(T_x), \\ \text{(ii)} & (D_x D_y \log Q)(e) = -\tau(T_{xy}), \\ \text{(iii)} & (D_x D_y D_z \log Q)(e) = \dfrac{1}{2}\tau(T_{(xy)z}). \end{array}$$

The proof amounts to differentiating at $e$ the relation

$$\log Q(\exp T_x \cdot e) = \tau(T_x) + \log Q(e),$$

up to third order. (See Exercise 5 in [Satake, 1980], p.38.) Hence, by (ii), $\langle x, y \rangle = \tau(T_{xy})$, and, by (iii), the symmetric bilinear form $\langle .,. \rangle$ is associative.

(c) Define the associator of three elements $x, y, z$ in $V$ by

$$[x, y, z] = x(zy) - (xz)y = [L(x), L(y)]z.$$

Identity (J2) can be written: $[x^2, y, x] = 0$ for all $x, y \in V$. It can be shown by following the proof of Theorem 8.5 in [Satake,1980], p.34, which is also the proof of Theorem III.3.1 in [Faraut-Koranyi,1994], p.50. □

The Jordan algebra $V$ is a direct sum of simple ideals:

$$V = \bigoplus_{i=1}^{s} V_i,$$

and

$$Q(x) = \prod_{i=1}^{s} \Delta_i(x_i)^{k_i} \quad (x = (x_1, \ldots, x_s)),$$

where $\Delta_i$ is the determinant polynomial of the simple Jordan algebra $V_i$ and the $k_i$ are positive integers. The degree of $Q$ is equal to $\sum_{i=1}^{s} k_i r_i$, where $r_i$ is the rank of $V_i$.

The conformal group $\mathrm{Conf}(V, Q)$ is the group of rational transformations $g$ of $V$ generated by: the translations $\tau_a : \ z \mapsto z + a$ $(a \in V)$, the dilations $z \mapsto \ell \cdot z$ $(\ell \in L)$, and the inversion $j : z \mapsto -z^{-1}$. A transformation $g \in \mathrm{Conf}(V, Q)$ is conformal in the sense that the differential $Dg(z)$ belongs to $L \in \mathrm{Str}(V, Q)$ at any point $z$ where $g$ is defined.

Let $\mathcal{W}$ be the space of polynomials on $V$ generated by the translated $Q(z - a)$ of $Q$. We will define a representation $\kappa$ on $\mathcal{W}$ of $\mathrm{Conf}(V, Q)$ or of a covering of order two of it.

*Case 1*

In case there exists a character $\chi$ of $\mathrm{Str}(V, Q)$ such that $\chi^2 = \gamma$, then let $K = \mathrm{Conf}(V, Q)$. Define the cocycle

$$\mu(g, z) = \chi(Dg(z)^{-1}) \quad (g \in K, \ z \in V),$$

and the representation $\kappa$ of $K$ on $\mathcal{W}$,

$$(\kappa(g)p)(z) = \mu(g^{-1}, z)p(g^{-1} \cdot z).$$

The function $\kappa(g)p$ belongs actually to $\mathcal{W}$. In fact the cocycle $\mu(g,z)$ is a polynomial in $z$ of degree $\leq \deg Q$ and

$$\begin{aligned}
(\kappa(\tau_a)p)(z) &= p(z-a) \quad (a \in V), \\
(\kappa(\ell)p)(z) &= \chi(\ell)p(\ell^{-1}\cdot z) \quad (\ell \in L), \\
(\kappa(j)p)(z) &= Q(z)p(-z^{-1}).
\end{aligned}$$

*Case 2*

Otherwise the group $K$ is defined as the set of pairs $(g,\mu)$ with $g \in \mathrm{Conf}(V,Q)$, and $\mu$ is a rational function on $V$ such that

$$\mu(z)^2 = \gamma(Dg(z))^{-1}.$$

We consider on $K$ the product $(g_1,\mu_1)(g_2,\mu_2) = (g_1g_2,\mu_3)$ with $\mu_3(z) = \mu_1(g_2\cdot z)\mu_2(z)$. For $\tilde{g} = (g,\mu) \in K$, define $\mu(\tilde{g},z) := \mu(z)$. Then $\mu(\tilde{g},z)$ is a cocycle:

$$\mu(\tilde{g}_1\tilde{g}_2,z) = \mu(\tilde{g}_1,\tilde{g}_2\cdot z)\mu(\tilde{g}_2,z),$$

where $\tilde{g}\cdot z = g\cdot z$ by definition.

**Proposition 1.3.** (i) *The map*

$$K \to \mathrm{Conf}(V,Q), \quad \tilde{g} = (g,\mu) \mapsto g$$

*is a surjective group morphism.*

(ii) *For $g \in K$, $\mu(g,z)$ is a polynomial in $z$ of degree $\leq \deg Q$.*

*Proof.* It is clearly a group morphism. We will show that the image contains a set of generators of $\mathrm{Conf}(V,Q)$. If $g$ is a translation, then $(g,1)$ and $(g,-1)$ are elements in $K$. If $g = \ell \in L$, then $Dg(z) = \ell$, and $(\ell,\alpha),(\ell,-\alpha)$, with $\alpha^2 = \gamma(\ell)^{-1}$, are elements in $K$. If $g\cdot z = j(z) := -z^{-1}$, then $Dg(z)^{-1} = P(z)$, where $P(z)$ denotes the quadratic representation of the Jordan algebra $V$: $P(z) = 2T_z^2 - T_{z^2}$, and $\gamma(P(z)) = Q(z)^2$. Then $(j,Q(z)),(j,Q(-z))$ are elements in $K$. ∎

Let $P_{\max}$ denote the preimage in $K$ of the maximal parabolic subgroup $L \ltimes N \subset \mathrm{Conf}(V,Q)$, where $N$ is the subgroup of translations. For $g \in P_{\max}$, $\mu(g,z)$ does not depend on $z$, and $\chi(g) = \mu(g^{-1},z)$ is a character of $P_{\max}$. For $g = (\ell,\alpha)$ $(\ell \in L)$, $\chi(g)^2 = \gamma(\ell)$.

Observe that the inverse in $K$ of $\sigma = (j, Q(z))$ is $\sigma^{-1} = (j, Q(-z))$. If $K$ is connected, then $K$ is a covering of order 2 of $\mathrm{Conf}(V, Q)$. If not, the identity component $K_0$ of $K$ is homeomorphic to $\mathrm{Conf}(V, Q)$.

The representation $\kappa$ of $K$ on $\mathcal{W}$ is then given by

$$\big(\kappa(g)p\big)(z) = \mu(g^{-1}, z)p(g^{-1} \cdot z).$$

In particular

$$\begin{aligned} \big(\kappa(g)p\big)(z) &= \chi(g)p(g^{-1} \cdot z) \quad (g \in P_{\max}), \\ \big(\kappa(\sigma)p\big)(z) &= Q(-z)p(-z^{-1}). \end{aligned}$$

Hence $p_0 \equiv 1$ is a highest weight vector with respect to the parabolic subgroup $P_{\max}$, and $Q = \kappa(\sigma)p_0$ is a lowest weight vector. The representation $\kappa$ is irreducible since every highest weight vector in $\mathcal{W}$ is proportional to $p_0$.

*Example 1*

If $V = \mathbb{C}$, $Q(z) = z^n$, then $\mathrm{Str}(V, Q) = \mathbb{C}^*$, $\gamma(\ell) = \ell^n$, and $\mathrm{Conf}(V, Q) \simeq PSL(2, \mathbb{C})$ is the group of fractional linear transformations

$$z \mapsto g \cdot z = \frac{az + b}{cz + d}, \text{ with } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{C}).$$

Furthermore

$$Dg(z) = \frac{1}{(cz + d)^2}, \ \gamma\big(Dg(z)^{-1}\big) = (cz + d)^{2n}, \ \mu(g, z) = (cz + d)^n.$$

Hence, if $n$ is even, then $K = PSL(2, \mathbb{C})$, and, if $n$ is odd, then $K = SL(2, \mathbb{C})$.

The space $\mathcal{W}$ is the space of polynomials of degree $\leq n$ in one variable. The representation $\kappa$ of $K$ on $\mathcal{W}$ is given by

$$\big(\kappa(g)p\big)(z) = (cz + d)^n p\Big(\frac{az + b}{cz + d}\Big), \text{ if } g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

*Example 2*

If $V = M(n, \mathbb{C})$, $Q(z) = \det z$, then $\mathrm{Str}(V, Q) = GL(n, \mathbb{C}) \times GL(n, \mathbb{C})$, acting on $V$ by

$$\ell \cdot z = \ell_1 z \ell_2^{-1} \quad \ell = (\ell_1, \ell_2).$$

Then $\gamma(\ell) = \det \ell_1 \det \ell_2^{-1}$, and $\gamma$ is not the square of a character of $\mathrm{Str}(V, Q)$. Furthermore $\mathrm{Conf}(V, Q) = PSL(2n, \mathbb{C})$ is the group of the rational transformations

$$z \mapsto g \cdot z = (az + b)(cz + d)^{-1}, \text{ with } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2n, \mathbb{C}),$$

decomposed in $n \times n$-blocks. To determine the differential of such a transformation, let us write (assuming $c$ to be invertible)

$$g \cdot z = (az + c)(cz + d)^{-1} = ac^{-1} - (ac^{-1}d - b)(cz + d)^{-1},$$

and we get

$$Dg(z)w = (ac^{-1}d - b)(cz + d)^{-1}cw(cz + d)^{-1}.$$

Notice that $Dg(z) \in \mathrm{Str}(V, Q)$:

$$Dg(z)w = \ell_1 w \ell_2^{-1}, \text{ with } \ell_1 = (ac^{-1}d - b)(cz + d)^{-1}c, \ \ell_2 = (cz + d).$$

Since $\det(ac^{-1}d - b) \det c = \det g = 1$,

$$\gamma\big(Dg(z)^{-1}\big) = \det(cz + d)^2.$$

It follows that $K = SL(2n, \mathbb{C})$, and $\mu(g, z) = \det(cz + d)$.

The space $\mathcal{W}$ is a space of polynomials of an $n \times n$ matrix variable, with degree $\leq n$. The representation $\kappa$ of $K$ on $\mathcal{W}$ is given by

$$\big(\kappa(g)p\big)(z) = \det(cz + d)p\big((az + b)(cz + d)^{-1}\big), \text{ if } g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

## 2 The orbit $\Xi$, and the irreducible $K$-invariant Hilbert subspaces of $\mathcal{O}(\Xi)$

Let $\Xi$ be the $K$-orbit of $Q$ in $\mathcal{W}$:

$$\Xi = \{\kappa(g)Q \mid g \in K\}.$$

Then $\Xi$ is a conical variety. In fact, if $\xi = \kappa(g)Q$, then, for $\lambda \in \mathbb{C}^*$, $\lambda\xi = \kappa(g \circ h_t)Q$, where $h_t \cdot z = e^{-t}z$ $(t \in \mathbb{C})$ with $\lambda = e^{2t}$.

A polynomial $\xi \in \mathcal{W}$ can be written

$$\xi(v) = wQ(v) + \text{ terms of degree } < N = \deg Q \quad (w \in \mathbb{C}),$$

and $w = w(\xi)$ is a linear form on $\mathcal{W}$ which is invariant under the parabolic subgroup $P_{\max}$. The set $\Xi_0 = \{\xi \in \Xi \mid w(\xi) \neq 0\}$ is open and dense in $\Xi$. A polynomial $\xi \in \Xi_0$ can be written

$$\xi(v) = wQ(v - z) \quad (w \in \mathbb{C}^*, z \in V).$$

Hence we get a coordinate system $(w, z) \in \mathbb{C}^* \times V$ for $\Xi_0$.

**Proposition 2.1.** *In this system, the action of $K$ is given by*

$$\kappa(g) : (w, z) \mapsto \big(\mu(g, z)w, g \cdot z\big).$$

Observe that the orbit $\Xi$ can be seen as a line bundle over the conformal compactification of $V$.

*Proof.* Recall that, for $\xi \in \Xi$,

$$\big(\kappa(g)\xi\big)(v) = \mu(g^{-1}, v)\xi(g^{-1} \cdot v),$$

and, if $\xi(v) = wQ(v - z)$, then

$$= \mu(g^{-1}, v)wQ(g^{-1} \cdot v - z) = \mu(g^{-1}, v)wQ(g^{-1} \cdot v - g^{-1}g \cdot z).$$

By Lemma 6.6 in [Faraut-Gindikin,1996],

$$\mu(g, z)\mu(g, z')Q(g \cdot z - g' \cdot z') = Q(z - z').$$

Therefore

$$(\kappa(g)\xi)(v) = \mu(g^{-1}, g \cdot z)^{-1}wQ(v - g.z) = \mu(g, z)wQ(v - g \cdot z),$$

by the cocycle property. ∎

The group $K$ acts on the space $\mathcal{O}(\Xi)$ of holomorphic functions on $\Xi$ by:

$$\big(\pi(g)f\big)(\xi) = f\big(\kappa(g)^{-1}\xi\big).$$

If $\xi \in \Xi_0$, i.e. $\xi(v) = wQ(v - z)$, and $f \in \mathcal{O}(\Xi)$, we will write $f(\xi) = \phi(w, z)$ for the restriction of $f$ to $\Xi_0$. In the coordinates $(w, z)$, the representation $\pi$ is given by

$$(\pi(g)\phi)(w, z) = \phi(\mu(g^{-1}, z)w, g^{-1} \cdot z).$$

Let $\mathcal{O}_m(\Xi)$ denote the space of holomorphic functions $f$ on $\Xi$, homogeneous of degree $m \in \mathbb{Z}$:

$$f(\lambda\xi) = \lambda^m f(\xi) \quad (\lambda \in \mathbb{C}^*).$$

The space $\mathcal{O}_m(\Xi)$ is invariant under the representation $\pi$. If $f \in \mathcal{O}_m(\Xi)$, then its restriction $\phi$ to $\Xi_0$ can be written $\phi(w, z) = w^m \psi(z)$, where $\psi$ is a holomorphic function on $V$. We will write $\tilde{\mathcal{O}}_m(V)$ for the space of the functions $\psi$ corresponding to the functions $f \in \mathcal{O}_m(\Xi)$, and denote by $\tilde{\pi}_m$ the representation of $K$ on $\tilde{\mathcal{O}}_m(V)$ corresponding to the restriction $\pi_m$ of $\pi$ to $\mathcal{O}_m(\Xi)$. The representation $\tilde{\pi}_m$ is given by

$$\big(\tilde{\pi}_m(g)\psi\big)(z) = \mu(g^{-1}, z)^m \psi(g^{-1} \cdot z).$$

Observe that $(\tilde{\pi}_m(\sigma)1)(z) = Q(-z)^m$.

**Theorem 2.2.** (i) $\mathcal{O}_m(\Xi) = \{0\}$ *for* $m < 0$.

(ii) *The space* $\mathcal{O}_m(\Xi)$ *is finite dimensional, and the representation* $\pi_m$ *is irreducible.*

(iii) *The functions* $\psi$ *in* $\tilde{\mathcal{O}}_m(V)$ *are polynomials.*

*Proof.* (i) Assume $\mathcal{O}_m(\Xi) \neq \{0\}$. Let $f \in \mathcal{O}_m(\Xi)$, $f \not\equiv 0$, and $\phi(w, z) = \psi(z)w^m$ its restriction to $\Xi_0$. Then $\psi$ is holomorphic on $V$, and

$$\big(\tilde{\pi}_m(\sigma)\psi\big)(z) = Q(-z)^m \psi(-z^{-1}),$$

is holomorphic as well. We may assume $\psi(e) \neq 0$. The function $h(\zeta) = \psi(\zeta e) \quad (\zeta \in \mathbb{C})$ is holomorphic on $\mathbb{C}$,

$$h(\zeta) = \sum_{k=0}^{\infty} a_k \zeta^k,$$

together with the function

$$Q(\zeta e)^m \psi(-\frac{1}{\zeta}e) = \zeta^{mN} h(-\frac{1}{\zeta}) = \zeta^{mN} \sum_{k=0}^{\infty} a_k (-\frac{1}{\zeta})^k \quad (N = \deg Q).$$

It follows that $m \geq 0$, and that $a_k = 0$ for $k > mN$.

(ii) The subspace

$$\{f \in \mathcal{O}_m(\Xi) \mid \forall a \in V, \pi(\tau_a)f = f\}$$

reduces to the functions $Cw^m$, hence is one dimensional. By the theorem of the highest weight [Goodman,2008], it follows that $\mathcal{O}_m(\Xi)$ is finite dimensional and irreducible.

(iii) Furthermore it follows that the functions in $\mathcal{O}_m(\Xi)$ are of the form $w^m\psi(z)$, where $\psi$ is a polynomial on $V$ of degree $\leq m \cdot \deg Q$. ∎

We fix a Euclidean real form $V_{\mathbb{R}}$ of the complex Jordan algebra $V$, denote by $z \mapsto \bar{z}$ the conjugation of $V$ with respect to $V_{\mathbb{R}}$, and then consider the involution $g \mapsto \bar{g}$ of $\mathrm{Conf}(V, Q)$ given by: $\bar{g} \cdot z = \overline{g \cdot \bar{z}}$. For $(g, \mu) \in K$ define

$$\overline{(g, \mu)} = (\bar{g}, \bar{\mu}), \text{ where } \bar{\mu}(z) = \overline{\mu(\bar{z})}.$$

The involution $\alpha$ defined by $\alpha(g) = \sigma \circ \bar{g} \circ \sigma^{-1}$ is a Cartan involution of $K$ (see Proposition 1.1. in [Pevzner,2002]), and

$$K_{\mathbb{R}} := \{g \in K \mid \alpha(g) = g\}$$

is a compact real form of $K$.

*Example 1.*

If $V = \mathbb{C}$, $Q(z) = z^n$. Then $V_{\mathbb{R}} = \mathbb{R}$, and $z \mapsto \bar{z}$ is the usual conjugation. We saw that $K = PSL(2, \mathbb{C})$ if $n$ is even, and $SL(2, \mathbb{C})$ if $n$ is odd. For $g \in SL(2, \mathbb{C})$,

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

we get

$$\alpha(g) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \bar{d} & -\bar{c} \\ -\bar{b} & \bar{a} \end{pmatrix}.$$

Hence $K_{\mathbb{R}} = PSU(2)$ if $n$ is even, and $K_{\mathbb{R}} = SU(2)$ if $n$ is odd.

*Example 2.*

If $V = M(n,\mathbb{C})$, $Q(z) = \det z$, then $V_{\mathbb{R}} = Herm(n,\mathbb{C})$ and the conjugation is $z \mapsto z^*$. We saw that $K = SL(2n,\mathbb{C})$. For $g \in SL(2n,\mathbb{C})$,

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

we get

$$\alpha(g) = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} a^* & b^* \\ c^* & d^* \end{pmatrix} \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} = \begin{pmatrix} d^* & -c^* \\ -b^* & a^* \end{pmatrix}.$$

Hence $K_{\mathbb{R}} = SU(2n)$.

We will define on $\mathcal{O}_m(\Xi)$ a $K_{\mathbb{R}}$-invariant inner product. Define the subgroup $K_0$ of $K$ as $K_0 = L$ in Case 1, and the preimage of $L$ in Case 2, relatively to the covering map $K \to \mathrm{Conf}(V,Q)$, and also $(K_0)_{\mathbb{R}} = K_0 \cap K_{\mathbb{R}}$. The coset space $M = K_{\mathbb{R}}/(K_0)_{\mathbb{R}}$, is a compact Hermitian space and is the conformal compactification of $V$. There is on $M$ a $K_{\mathbb{R}}$-invariant probability measure, for which $M \setminus V$ has measure 0. Its restriction $m_0$ to $V$ is a probability measure with a density which can be computed by using the decomposition of $V$ into simple Jordan algebras.

Let $H(z,z')$ be the polynomial on $V \times V$, holomorphic in $z$, anti-holomorphic in $z'$ such that

$$H(x,x) = Q(e + x^2) \quad (x \in V_{\mathbb{R}}).$$

Put $H(z) = H(z,z)$. If $z$ is invertible, then $H(z) = Q(\bar{z})Q(\bar{z}^{-1} + z)$.

**Proposition 2.3.** *For $g \in K_{\mathbb{R}}$,*

$$H(g\cdot z_1, g\cdot z_2)\mu(g,z_1)\overline{\mu(g,z_2)} = H(z_1,z_2),$$

*and*

$$H(g\cdot z)|\mu(g,z)|^2 = H(z).$$

*Proof.* Recall that an element $g \in K_{\mathbb{R}}$ satisfies $\sigma\circ\bar{g}\circ\sigma^{-1} = g$, or $\sigma\circ\bar{g} = g\circ\sigma$. Recall also the cocycle property: for $g_1, g_2 \in K$,

$$\mu(g_1 g_2, z) = \mu(g_1, g_2\cdot z)\mu(g_2, z).$$

Since $\mu(\sigma, z) = Q(z)$, it follows that, for $g \in K_{\mathbb{R}}$,

$$\mu(g, \sigma\cdot z)Q(z) = Q(\bar{g}\cdot z)\mu(\bar{g}, z). \tag{1}$$

By Lemma 6.6 in [Faraut-Gindikin,1996], for $g \in K$,

$$Q(g \cdot z_1 - g \cdot z_2)\mu(g, z_1)\mu(g, z_2) = Q(z_1 - z_2). \tag{2}$$

For $g \in K_{\mathbb{R}}$,

$$\begin{aligned} H(g \cdot z_1, g \cdot z_2) &= Q(\bar{g} \cdot z_2)Q(g \cdot z_1 - \sigma\bar{g} \cdot \bar{z}_2) \\ &= Q(\bar{g} \cdot \bar{z}_2)Q(g \cdot z_1 - g\sigma\bar{z}_2), \end{aligned}$$

and, by (2),

$$= Q(\bar{g} \cdot \bar{z}_2)\mu(g, z_1)^{-1}\mu(g, \sigma \cdot \bar{z}_2)^{-1}Q(z_1 - \sigma \cdot \bar{z}_2).$$

Finally, by (1),

$$= \mu(g, z_1)^{-1}\mu(\bar{g}, \bar{z}_2)^{-1}H(z_1, z_2).$$

∎

We define the norm of a function $\psi \in \tilde{\mathcal{O}}_m(V)$ by

$$\|\psi\|_m^2 = \frac{1}{a_m}\int_V |\psi(z)|^2 H(z)^{-m} m_0(dz),$$

with

$$a_m = \int_V H(z)^{-m} m_0(dz).$$

**Proposition 2.4.** (i) *This norm is $K_{\mathbb{R}}$-invariant. Hence, $\tilde{\mathcal{O}}_m(V)$ is a Hilbert subspace of $\mathcal{O}(V)$.*

(ii) *The reproducing kernel of $\tilde{\mathcal{O}}_m(V)$ is given by*

$$\tilde{\mathcal{K}}_m(z, z') = H(z, z')^m.$$

*Proof.* (i) From Proposition 2.3 it follows that, for $g \in K_{\mathbb{R}}$,

$$\begin{aligned} \|\tilde{\pi}_m(g^{-1})\psi\|_m^2 &= \frac{1}{a_m}\int_V |\mu(g, z)|^{2m}|\psi(g^{-1} \cdot z)|^2 H(z)^{-m} m_0(dz) \\ &= \frac{1}{a_m}\int_V |\psi(g^{-1} \cdot z)|^2 H(g^{-1} \cdot z)^{-m} m_0(dz) \\ &= \frac{1}{a_m}\int_V |\psi(z)|^2 H(z)^{-m} m_0(dz) = \|\psi\|_m^2. \end{aligned}$$

(ii) There is a unique function $\psi_0 \in \tilde{\mathcal{O}}_m(V)$ such that, for $\psi \in \tilde{\mathcal{O}}_m(V)$,

$$(\psi \mid \psi_0) = \psi(0).$$

The function $\psi_0$ is $K_0$-invariant, therefore constant: $\psi_0(z) = C$. Taking $\psi = \psi_0$, one gets $C^2 = C$, hence $C = 1$. It means that, if $\tilde{\mathcal{K}}_m(z, z')$ denotes the reproducing kernel of $\tilde{\mathcal{O}}_m(V)$,

$$\tilde{\mathcal{K}}_m(z, 0) = \tilde{\mathcal{K}}_m(0, z') = 1.$$

Since $\tilde{\mathcal{K}}_m(z, z')$ and $H(z, z')$ satisfy the following invariance properties: for $g \in K_{\mathbb{R}}$,

$$\begin{aligned} \tilde{\mathcal{K}}_m(g \cdot z, g \cdot z')\mu(g, z)^m \overline{\mu(g, z')}^m &= \tilde{\mathcal{K}}_m(z, z'), \\ H(g \cdot z, g \cdot z')\mu(g, z)\overline{\mu(g, z')} &= H(z, z'), \end{aligned}$$

it follows that

$$\tilde{\mathcal{K}}_m(z, z') = H(z, z')^m.$$

∎

Since $\mathcal{O}_m(\Xi)$ is isomorphic to $\tilde{\mathcal{O}}_m(V)$, the space $\mathcal{O}_m(\Xi)$ becomes an invariant Hilbert subspace of $\mathcal{O}(\Xi)$, with reproducing kernel

$$\mathcal{K}_m(\xi, \xi') = \Phi(\xi, \xi')^m,$$

where

$$\Phi(\xi, \xi') = H(z, z')w\overline{w'} \qquad (\xi = (w, z), \xi' = (w', z')).$$

**Theorem 2.5.** *The group $K_{\mathbb{R}}$ acts multiplicity free on $\mathcal{O}(\Xi)$. The irreducible $K_{\mathbb{R}}$-invariant subspaces of $\mathcal{O}(\Xi)$ are the spaces $\mathcal{O}_m(\Xi)$ ($m \in \mathbb{N}$). If $\mathcal{H} \subset \mathcal{O}(\Xi)$ is a $K_{\mathbb{R}}$-invariant Hilbert subspace, the reproducing kernel of $\mathcal{H}$ can be written*

$$\mathcal{K}(\xi, \xi') = \sum_{m=0}^{\infty} c_m \Phi(\xi, \xi')^m,$$

*with $c_m \geq 0$, such that the series $\sum_{m=0}^{\infty} c_m \Phi(\xi, \xi')^m$ converges uniformly on compact subsets in $\Xi$.*

This multiplicity free property means that $K_{\mathbb{R}}$ acts multiplicity free on every $K_{\mathbb{R}}$-invariant Hilbert space $\mathcal{H} \subset \mathcal{O}(\Xi)$.

*Proof.* The representation $\pi$ of $K_{\mathbb{R}}$ on $\mathcal{O}(\Xi)$ commutes with the $\mathbb{C}^*$-action by dilations and the spaces $\mathcal{O}_m(\Xi)$ are irreducible, and mutually inequivalent. It follows that $K_{\mathbb{R}}$ acts multiplicity free. ∎

In case of a weighted Bergman space there is an integral formula for the numbers $c_m$. For a positive function $p(\xi)$ on $\Xi$, consider the subspace $\mathcal{H} \subset \mathcal{O}(\Xi)$ of functions $\phi$ such that

$$\|\phi\|^2 = \int_{\mathbb{C}\times V} |\phi(w,z)|^2 p(w,z) m(dw) m_0(dz) < \infty,$$

where $m(dw)$ denotes the Lebesgue measure on $\mathbb{C}$.

**Theorem 2.6.** *Let $F$ be a positive function on $[0,\infty[$, and define*

$$p(w,z) = F(H(z)|w|^2)H(z).$$

(i) *Then $\mathcal{H}$ is $K_{\mathbb{R}}$-invariant.*
(ii) *If*

$$\phi(w,z) = \sum_{m=0}^{\infty} w^m \psi_m(z),$$

*then*

$$\|\phi\|^2 = \sum_{m=0}^{\infty} \frac{1}{c_m} \|\psi_m\|_m^2,$$

*with*

$$\frac{1}{c_m} = \pi a_m \int_0^{\infty} F(u) u^m du.$$

(iii) *The reproducing kernel of $\mathcal{H}$ is given by*

$$\mathcal{K}(\xi,\xi') = \sum_{m=0}^{\infty} c_m \Phi(\xi,\xi')^m.$$

*Proof.* a) Observe first that the function defined on $\Xi$ by

$$(w,z) \mapsto |w|^2 H(z),$$

is $K_{\mathbb{R}}$-invariant. In fact, for $g \in K$,

$$\kappa(g) : (w,g) \mapsto \big(\mu(g,z)w, g\cdot z\big),$$

and, by Propositiion 2.3, for $g \in K_{\mathbb{R}}$,

$$|\mu(g,z)|^2 H(g\cdot z) = H(z).$$

Furthermore the measure $h(z)m(dw)m_0(dz)$ is also invariant under $K_{\mathbb{R}}$. In fact, under the transformation $z = g \cdot z', w = \mu(g, z')w'$ $(g \in K_{\mathbb{R}})$, we get

$$\begin{aligned} H(z)m(dw)m_0(dz) &= H(g \cdot z')|\mu(g, z')|^2 m(dw')m_0(dz') \\ &= H(z')m(dw')m_0(dz'). \end{aligned}$$

b) Assume that $p(w, z) = F\big(H(z)|w|^2\big)H(z)$. Then

$$\|\pi(g)\phi\|^2 = \int_{\mathbb{C}\times V} |\phi(\mu(g^{-1}, z)w, g^{-1} \cdot z)|^2 F\big(H(z)|w|^2\big)H(z)m(dw)m_0(dz).$$

We put

$$g^{-1} \cdot z = z' \quad , \quad \mu(g^{-1}, z)w = w'.$$

By the invariance of the measure $H(z)m(dw)m_0(dz)$, we obtain

$$\|\pi(g)\phi\|^2 = \int_{\mathbb{C}\times V} |\phi(w', z')|^2 F\big(H(g \cdot z')|\mu(g^{-1}, g \cdot z')|^{-2}|w'|^2\big)H(z')m(dw')m_0(dz').$$

Furthermore

$$H(g \cdot z')|\mu(g^{-1}, g \cdot z')|^{-2} = H(g \cdot z')|\mu(g, z')|^2 = H(z'),$$

and, finally, $\|\pi(g)\phi\| = \|\phi\|$.

c) If $\phi(w, z) = w^m\psi(z)$, then

$$\|\phi\|^2 = \int_{\mathbb{C}\times V} |w|^{2m}|\psi(z)|^2 F\big(H(z)|w|^2\big)H(z)m(dw)m_0(dz).$$

We put $w' = \sqrt{H(z)}w$, then

$$\begin{aligned} \|\phi\|^2 &= \int_{\mathbb{C}\times V} H(z)^{-m}|w'|^{2m}|\psi(z)|^2 F(|w'|^2)m(dw')m_0(dz) \\ &= a_m\|\psi\|_m^2 \int_{\mathbb{C}} F(|w'|^2)|w'|^{2m}m(dw') \\ &= a_m\|\psi\|_m^2 \pi \int_0^\infty F(u)u^m du. \end{aligned}$$

□

# 3 Decomposition into simple Jordan algebras

Let us decompose the semi-simple Jordan algebra $V$ into simple ideals:

$$V = \bigoplus_{i=1}^{s} V_i.$$

Denote by $n_i$ and $r_i$ the dimension and the rank of the simple Jordan algebra $V_i$, and $\Delta_i$ the determinant polynomial. Then

$$Q(z) = \prod_{i=1}^{s} \Delta_i(z_i)^{k_i}.$$

Let $H_i(z, z')$ be the polynomial on $V_i \times V_i$, holomorphic in $z$, antiholomorphic in $z'$, such that

$$H_i(x, x) = \Delta_i(e_i + x^2) \quad \big(x \in (V_i)_{\mathbb{R}}\big),$$

and put $H_i(z) = H_i(z, z)$. The measure $m_0$ has a density with respect to the Lebesgue measure $m$ on $V$:

$$m_0(dz) = \frac{1}{C_0} H_0(z) m(dz),$$

with

$$\begin{aligned} H_0(z) &= \prod_{i=1}^{s} H_i(z_i)^{-2\frac{ni}{r_i}}, \\ C_0 &= \int_V H_0(z) m(dz). \end{aligned}$$

The Lebesgue measure $m$ will be chosen such that $C_0 = 1$.

**Proposition 3.1.** (i) *The polynomial $Q$ satisfies the following Bernstein identity*

$$Q\Big(\frac{\partial}{\partial z}\Big) Q(z)^{\alpha} = B(\alpha) Q(z)^{\alpha - 1} \quad (z \in \mathbb{C}),$$

*where the Bernstein polynomial $B$ is given by*

$$B(\alpha) = \prod_{i=1}^{s} b_i(k_i\alpha) b_i(k_i\alpha - 1) \ldots b_i(k_i\alpha - k_i + 1),$$

*and $b_i$ is the Bernstein polynomial relative to the determinant polynomial $\Delta_i$.*

(ii) *Furthermore*

$$Q\Big(\frac{\partial}{\partial z}\Big)H(z)^\alpha = B(\alpha)\overline{Q(z)}H(z)^{\alpha-1}.$$

*Proof.* (i) The Bernstein identity for $Q$ follows from Proposition VII.1.4 in [Faraut-Korányi,1994].

(ii) For $z$ invertible

$$H(z) = Q(\bar z)Q(\bar z^{-1} + z),$$

and then, by (i),

$$\begin{aligned} Q\Big(\frac{\partial}{\partial z}\Big)H(z)^\alpha &= Q(\bar z)^\alpha B(\alpha)Q(\bar z^{-1}+z)^{\alpha-1} \\ &= Q(\bar z)B(\alpha)H(z)^{\alpha-1}. \end{aligned}$$

∎

*Example 1*

If $V=\mathbb{C}$, $Q(z)=z^n$, then

$$\Big(\frac{d}{dz}\Big)^n z^{n\alpha} = B(\alpha)z^{n(\alpha-1)},$$

with

$$B(\alpha) = n\alpha(n\alpha-1)\dots(n\alpha-n+1).$$

*Example 2*

If $V=M(n,\mathbb{C})$, $Q(z)=\det z$, then

$$\det\Big(\frac{\partial}{\partial z}\Big)(\det z)^\alpha = B(\alpha)(\det z)^{\alpha-1},$$

with

$$B(\alpha) = \alpha(\alpha+1)\dots(\alpha+n-1).$$

Recall that we have introduced the numbers

$$a_m = \int_V H(z)^{-m} m_0(dz).$$

**Proposition 3.2.**

$$a_m = \prod_{i=1}^{s} \frac{\Gamma_{\Omega_i}(2\frac{n_i}{r_i})}{\Gamma_{\Omega_i}(\frac{n_i}{r_i})} \prod_{i=1}^{s} \frac{\Gamma_{\Omega_i}(mk_i + \frac{n_i}{r_i})}{\Gamma_{\Omega_i}(mk_i + 2\frac{n_i}{r_i})},$$

*where $\Gamma_{\Omega_i}$ is the Gindikin gamma function of the symmetric cone $\Omega_i$ in the Euclidean Jordan algebra $(V_i)_{\mathbb{R}}$.*

*Proof.* If the Jordan algebra $V$ is simple and $Q = \Delta$, the determinant polynomial, by Proposition X.3.4 in [Faraut-Korányi,1994],

$$\begin{aligned} a_m &= \int_V H(z)^{-m} m_0(dz) = \frac{1}{C_0} \int_V H(z)^{-m-2\frac{n}{r}} m(dz) \\ &= C \int_\Omega \Delta(e+x)^{-m-2\frac{n}{r}} m(dx). \end{aligned}$$

By Exercice 4 of Chapter VII in [Faraut-Korányi,1994] we obtain

$$a_m = C' \frac{\Gamma_\Omega(m + \frac{n}{r})}{\Gamma_\Omega(m + 2\frac{n}{r})}.$$

In the general case

$$a_m = \frac{1}{C_0} \prod_{i=1}^{s} \int_{V_i} H_i(z_i)^{-mk_i - 2\frac{n_i}{r_i}} m_i(dz_i),$$

and the formula of the proposition follows. ∎

# 4 Generalized Kantor–Koecher–Tits construction

From now on, $Q$ is assumed to be of degree 4. The group of dilations of $V$ : $h_t \cdot z = e^{-t} z$ $(t \in \mathbb{C})$ is a one parameter subgroup of $L$, and $\chi(h_t) = e^{-2t}$. Put $h_t = \exp(tH)$. Then $\mathrm{ad}(H)$ defines a grading of the Lie algebra $\mathfrak{k}$ of $K$:

$$\mathfrak{k} = \mathfrak{k}_{-1} + \mathfrak{k}_0 + \mathfrak{k}_1,$$

with $\mathfrak{k}_j = \{X \in \mathfrak{k} \mid \mathrm{ad}(H)X = jX\}$, $(j = -1, 0, 1)$. Notice that

$$\mathfrak{k}_{-1} = Lie(N) \simeq V, \quad \mathfrak{k}_0 = Lie(L), \quad \mathrm{Ad}(\sigma) : \mathfrak{k}_j \to \mathfrak{k}_{-j},$$

and also that $H$ belongs to the centre $\mathfrak{z}(\mathfrak{k}_0)$ of $\mathfrak{k}_0$. The element $H$ defines also a grading of $\mathfrak{p} := \mathcal{W}$:

$$\mathfrak{p} = \mathfrak{p}_{-2} + \mathfrak{p}_{-1} + \mathfrak{p}_0 + \mathfrak{p}_1 + \mathfrak{p}_2,$$

where

$$\mathfrak{p}_j = \{p \in \mathfrak{p} \mid d\kappa(H)p = jp\}$$

is the set of polynomials in $\mathfrak{p}$, homogeneous of degree $j+2$. The subspaces $\mathfrak{p}_j$ are invariant under $K_0$. Furthermore $\kappa(\sigma) : \mathfrak{p}_j \to \mathfrak{p}_{-j}$, and

$$\mathfrak{p}_{-2} = \mathbb{C}, \quad \mathfrak{p}_2 = \mathbb{C}\, Q, \quad \mathfrak{p}_{-1} \simeq V, \quad \mathfrak{p}_1 \simeq V.$$

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Put $E = Q$, $F = 1$.

**Theorem 4.1.** *There exists on $\mathfrak{g}$ a unique Lie algebra structure such that:*

$$\begin{array}{rrcll} (i) & [X, X'] & = & [X, X']_{\mathfrak{k}} & (X, X' \in \mathfrak{k}), \\ (ii) & [X, p] & = & d\kappa(X)p & (X \in \mathfrak{k}, p \in \mathfrak{p}), \\ (iii) & [E, F] & = & H. & \end{array}$$

*Proof.* Observe that $(E, F, H)$ is an $\mathfrak{sl}_2$-triple, and that $H$ defines a grading of

$$\mathfrak{g} = \mathfrak{g}_{-2} + \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1 + \mathfrak{g}_2,$$

with

$$\mathfrak{g}_{-2} = \mathfrak{p}_{-2}, \quad \mathfrak{g}_{-1} = \mathfrak{k}_{-1} + \mathfrak{p}_{-1}, \quad \mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0, \quad \mathfrak{g}_1 = \mathfrak{k}_1 + \mathfrak{p}_1, \quad \mathfrak{g}_2 = \mathfrak{p}_2.$$

It is possible to give a direct proof of Theorem 4.1 (see Theorem 3.1. in [Achab,2011]). It is also possible to see this statement as a special case of constructions of Lie algebras by Allison and Faulkner [1984]. We describe below this construction in our case.

a) *Cayley-Dickson process.*

Let $x \mapsto x^*$ denote the symmetry with respect to the one dimensional subspace $\mathbb{C}e$:

$$x^* = \frac{1}{2}\langle x, e\rangle\, e - x.$$

Observe that

$$\langle x, e\rangle = \tau(T_x) = D_x \log Q(e), \quad \langle e, e\rangle = 4.$$

On the vector space $W = V \oplus V$, one defines an algebra structure: if $z_1 = (x_1, y_1), z_2 = (x_2, y_2)$, then $z_1 z_2 = z = (x, y)$ with

$$x = x_1 x_2 - (y_1 y_2^*)^*, \quad y = x_1^* y_2 + (y_1^* x_2^*)^*,$$

and an involution

$$\bar{z} = \overline{(x, y)} = (x, -y^*).$$

This involution is an antiautomorphism: $\overline{z_1 z_2} = \bar{z}_2 \bar{z}_1$. For $a, b \in W$, one introduces the endomorphisms $V_{a,b}$ and $T_a$ given by

$$\begin{aligned} V_{a,b} z &= \{a, b, z\} := (a\bar{b})z + (z\bar{b})a - (z\bar{a})b, \\ T_a z &= V_{a,e} z = az + z(a - \bar{a}). \end{aligned}$$

By Theorem 6.6 in [Allison-Faulkner, 1984] the algebra $W$ is structurable. This means that, for $a, b, c, d \in W$,

$$[V_{a,b}, V_{c,d}] = V_{V_{a,b}c,d} - V_{c,V_{b,a}d}. \tag{$*$}$$

Moreover the structurable algebra $W$ is simple. By $(*)$, the vector space spanned by the endomorphisms $V_{a,b}$ $(a, b \in W)$ is a Lie algebra denoted by $Instrl(W)$. This algebra is the Lie algebra $\mathfrak{g}_0$ in the grading, and its subalgebra $\mathfrak{k}_0$ is the structure algebra of the Jordan algebra $V$. The space $S$ of skew-Hermitian elements in $W$, $S = \{z \in W \mid \bar{z} = -z\}$, has dimension one. Its elements are proportionnal to $s_0 = (0, e)$. The subspace $\{(x, 0) \mid x \in V\}$ of $W$ is identified to $V$, and any element $z = (x, y) \in W$ can be written $z = x + s_0 y$.

b) *Generalized Kantor-Koecher-Tits construction.*

One defines a bracket on the vector space

$$\mathcal{K}(W) = \tilde{S} \oplus \tilde{W} \oplus Instrl(W) \oplus W \oplus S,$$

where $\tilde{S}$ is a second copy of $S$, and $\tilde{W}$ of $W$. This construction is described in [Allison,1979], and, by Corollary 6 in that paper, $\mathcal{K}(W)$ is a simple Lie algebra. On the subspace $\mathcal{K}(V) = \tilde{V} \oplus \mathfrak{str}(V) \oplus V$, this construction agrees with the classical Kantor-Koecher-Tits construction, which produces the Lie algebra $\mathfrak{k} = \mathfrak{k}_{-1} \oplus \mathfrak{k}_0 \oplus \mathfrak{k}_1$. This algebra $\mathcal{K}(W)$ satisfies property (i): the restriction of the bracket of $\mathcal{K}(W)$ to $\mathcal{K}(V)$ coincides to the one of $\mathcal{K}(V)$. It satisfies (iii) as well: $[s_0, \tilde{s}_0] = I$, the identity of $End(W)$. It remains to check property (ii). This can be seen as a consequence of the theorem of the highest weight for irreducible finite dimensional representations of reductive Lie algebras. In fact, the representation $d\kappa$ of $\mathfrak{k}$ on $\mathfrak{p}$ is irreducible with highest weight vector $Q$, with respect to any Borel subalgebra $\mathfrak{b} \subset \mathfrak{k}_0 + \mathfrak{k}_1$ :

- If $X \in \mathfrak{k}_1$, then $\mathrm{d}\kappa(X)Q = 0$.
- If $X \in \mathfrak{k}_0$, such that $d\gamma(X) = 0$, then $d\kappa(X)Q = 0$, and $d\kappa(H)Q = 2Q$.

On the other hand, for the bracket of $\mathcal{K}(W)$,

- If $u \in V, [u, s_0] = 0$.
- If $X \in \mathfrak{str}(V)$, such that $\mathrm{tr}(X) = 0$, then $[X, s_0] = 0$ and $[H, s_0] = 2s_0$.

It follows that the adjoint representation of $\mathcal{K}(V) = \tilde{V} \oplus \mathfrak{str}(V) \oplus V$ on

$$\tilde{S} \oplus \tilde{s}_0\tilde{V} \oplus T_W \oplus s_0V \oplus S,$$

where $T_W = \{T_w = V_{w,e} \mid w \in W\}$, agrees with the representation $d\kappa$ of $\mathfrak{k}$ on $\mathfrak{p}$. In the present case, $T_w = L(w) + \frac{1}{2}\langle v, e\rangle Id$, if $w = u + s_0 v$ $(u, v \in V)$.

On the vector space

$$\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2,$$

with

$$\mathfrak{g}_1 = W, \quad \mathfrak{g}_{-1} = W, \quad \mathfrak{g}_2 = \mathbb{C}\,E, \quad \mathfrak{g}_{-2} = \mathbb{C}\,F, \quad \mathfrak{g}_0 = Instrl(W),$$

one defines a bracket satisfying the following properties:

(1) $\mathfrak{g}_1 + \mathfrak{g}_2$ is a Heisenberg Lie algebra:

$$\mathfrak{g}_1 \times \mathfrak{g}_1 \longrightarrow \mathfrak{g}_2, \quad (w_1, w_2) \mapsto w_1\bar{w}_2 - w_2\bar{w}_1 = \psi(w_1, w_2)s_0.$$

The bilinear form $\psi$ is skew symmetric, and $[w_1, w_2] = \psi(w_1, w_2)E$.

(2) $\mathfrak{g}_1 \times \mathfrak{g}_{-1} \longrightarrow \mathfrak{g}_0, \quad (w, \tilde{w}) \mapsto V_{w,\tilde{w}}$.

(3) $\mathfrak{g}_2 \times \mathfrak{g}_{-1} \longrightarrow \mathfrak{g}_1, \quad (\lambda E, \tilde{w}) \mapsto \lambda\tilde{w}$. □

With a different point of view the above construction is closely related to the paper [Clerc,2003].

bigskip

We introduce now a real form $\mathfrak{g}_{\mathbb{R}}$ of $\mathfrak{g}$ which will be considered in the sequel. In Section 2 we have considered the involution $\alpha$ of $K$ given by

$$\alpha(g) = \sigma \circ \bar{g} \circ \sigma^{-1} \quad (g \in K),$$

and the compact real form $K_{\mathbb{R}}$ of $K$:

$$K_{\mathbb{R}} = \{g \in K \mid \alpha(g) = g\}.$$

Recall that $\mathfrak{p}$ has been defined as a space of polynomial functions on $V$. For $p \in \mathfrak{p}$, define

$$\bar{p} = \overline{p(\bar{z})},$$

and consider the antilinear involution $\beta$ of $\mathfrak{p}$ given by

$$\beta(p) = \kappa(\sigma)\bar{p}.$$

Observe that $\beta(E) = F$. The involution $\beta$ is related to the involution $\alpha$ of $K$ by the relation

$$\kappa\big(\alpha(g)\big) \circ \beta = \beta \circ \kappa(g) \quad (g \in K).$$

Hence, for $g \in K_{\mathbb{R}}$, $\kappa(g) \circ \beta = \beta \circ \kappa(g)$. Define

$$\mathfrak{p}_{\mathbb{R}} = \{p \in \mathfrak{p} \mid \beta(p) = p\}.$$

The real subspace $\mathfrak{p}_{\mathbb{R}}$ is invariant under $K_{\mathbb{R}}$, and irreducible for that action. The space $\mathfrak{p}$, as a real vector space, decomposes under $K_{\mathbb{R}}$ into two irreducible subspaces

$$\mathfrak{p} = \mathfrak{p}_{\mathbb{R}} \oplus i\mathfrak{p}_{\mathbb{R}}.$$

One checks that $E + F \in \mathfrak{p}_{\mathbb{R}}$ (and hence $i(E - F)$ as well).

Let $\mathfrak{u}$ be a compact real form of $\mathfrak{g}$ such that $\mathfrak{k} \cap \mathfrak{u} = \mathfrak{k}_{\mathbb{R}}$, the Lie algebra of $K_{\mathbb{R}}$. Then $\mathfrak{p}$ decomposes as

$$\mathfrak{p} = \mathfrak{p} \cap (i\mathfrak{u}) \oplus \mathfrak{p} \cap \mathfrak{u}$$

into two irreducible $K_{\mathbb{R}}$-invariant real subspaces. Looking at the subalgebra $\mathfrak{g}^0$ isomorphic to $\mathfrak{sl}(2, \mathbb{C})$ generated by the triple $(E, F, H)$, one sees that $E + F \in \mathfrak{p} \cap (i\mathfrak{u})$. Therefore $\mathfrak{p}_{\mathbb{R}} = \mathfrak{p} \cap (i\mathfrak{u})$, and

$$\mathfrak{g}_{\mathbb{R}} = \mathfrak{k}_{\mathbb{R}} \oplus \mathfrak{p}_{\mathbb{R}}$$

is a Lie algebra, real form of $\mathfrak{g}$, and the above decomposition is a Cartan decomposition of $\mathfrak{g}_{\mathbb{R}}$. This real form $\mathfrak{g}_{\mathbb{R}}$ is not Hermitian since the adjoint action of $K$ on $\mathfrak{p}$ is irreducible.

For the table of next page we have used the notation:

$$\varphi_n(z) = z_1^2 + \cdots + z_n^2, \quad (z \in \mathbb{C}^n).$$

In case of an exceptional Lie algebra $\mathfrak{g}$, the real form $\mathfrak{g}_{\mathbb{R}}$ has been identified by computing the Cartan signature.

| $V$ | $Q$ | $\mathfrak{k}$ | $\mathfrak{g}$ | $\mathfrak{g}_{\mathbb{R}}$ |
|---|---|---|---|---|
| $\mathbb{C}^n$ | $\varphi_n(z)^2$ | $\mathfrak{so}(n+2,\mathbb{C})$ | $\mathfrak{sl}(n+2,\mathbb{C})$ | $\mathfrak{sl}(n+2,\mathbb{R})$ |
| $\mathbb{C}^p \oplus \mathbb{C}^q$ | $\varphi_p(z)\varphi_q(z')$ | $\mathfrak{so}(p+2,\mathbb{C}) \oplus \mathfrak{so}(q+2,\mathbb{C})$ | $\mathfrak{so}(p+q+4,\mathbb{C})$ | $\mathfrak{so}(p+2,q+2)$ |
| $Sym(4,\mathbb{C})$ | $\det z$ | $\mathfrak{sp}(8,\mathbb{C})$ | $\mathfrak{e}_6$ | $\mathfrak{e}_{6(6)}$ |
| $M(4,\mathbb{C})$ | $\det z$ | $\mathfrak{sl}(8,\mathbb{C})$ | $\mathfrak{e}_7$ | $\mathfrak{e}_{7(7)}$ |
| $Skew(8,\mathbb{C})$ | $\mathrm{Pfaff}(z)$ | $\mathfrak{so}(16,\mathbb{C})$ | $\mathfrak{e}_8$ | $\mathfrak{e}_{8(8)}$ |
| $Sym(3,\mathbb{C}) \oplus \mathbb{C}$ | $\det z \cdot z'$ | $\mathfrak{sp}(6,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$ | $\mathfrak{f}_4$ | $\mathfrak{f}_{4(4)}$ |
| $M(3,\mathbb{C}) \oplus \mathbb{C}$ | $\det z \cdot z'$ | $\mathfrak{sl}(6,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$ | $\mathfrak{e}_6$ | $\mathfrak{e}_{6(2)}$ |
| $Skew(6,\mathbb{C}) \oplus \mathbb{C}$ | $\mathrm{Pfaff}(z) \cdot z'$ | $\mathfrak{so}(12,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$ | $\mathfrak{e}_7$ | $\mathfrak{e}_{7(-5)}$ |
| $Herm(3,\mathbb{O})_{\mathbb{C}} \oplus \mathbb{C}$ | $\det z \cdot z'$ | $\mathfrak{e}_7 \oplus \mathfrak{sl}(2,\mathbb{C})$ | $\mathfrak{e}_8$ | $\mathfrak{e}_{8(-24)}$ |
| $\mathbb{C} \oplus \mathbb{C}$ | $z^3 \cdot z'$ | $\mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$ | $\mathfrak{g}_2$ | $\mathfrak{g}_{2(2)}$ |

# 5 Representation of the generalized Kantor-Koecher-Tits Lie algebra

Following the method of R. Brylinski and B. Kostant, we will construct a representation $\rho$ of $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ on the space of finite sums

$$\mathcal{O}(\Xi)_{\text{fin}} = \sum_{m=0}^{\infty} \mathcal{O}_m(\Xi),$$

such that, for all $X \in \mathfrak{k}$, $\rho(X) = d\pi(X)$. We define first a representation $\rho$ of the subalgebra generated by $E, F, H$, isomorphic to $\mathfrak{sl}(2, \mathbb{C})$. In particular

$$\rho(H) = d\pi(H) = \frac{d}{dt}\Big|_{t=0} \pi(\exp tH).$$

Hence, for $\phi \in \mathcal{O}_m(\Xi)$, $\rho(H)\phi = (\mathcal{E} - 2m)\phi$, where $\mathcal{E}$ is the Euler operator

$$\mathcal{E}\phi(w, z) = \frac{d}{dt}\Big|_{t=0} \phi(w, e^t z).$$

One introduces two operators $\mathcal{M}$ and $\mathcal{D}$. The operator $\mathcal{M}$ is a multiplication operator:

$$(\mathcal{M}\phi)(w, z) = w\phi(w, z),$$

which maps $\mathcal{O}_m(\Xi)$ into $\mathcal{O}_{m+1}(\Xi)$, and $\mathcal{D}$ is a differential operator:

$$(\mathcal{D}\phi)(w, z) = \frac{1}{w}\left(Q\Big(\frac{\partial}{\partial z}\Big)\phi\right)(w, z),$$

which maps $\mathcal{O}_m(\Xi)$ into $\mathcal{O}_{m-1}(\Xi)$. (Recall that $\mathcal{O}_{-1}(\Xi) = \{0\}$.) We denote by $\mathcal{M}^\sigma$ and $\mathcal{D}^\sigma$ the conjugate operators:

$$\mathcal{M}^\sigma = \pi(\sigma)\mathcal{M}\pi(\sigma)^{-1}, \quad \mathcal{D}^\sigma = \pi(\sigma)\mathcal{D}\pi(\sigma)^{-1}.$$

Given a sequence $(\delta_m)_{m\in\mathbb{N}}$ one defines the diagonal operator $\delta$ on $\mathcal{O}(\Xi)_{fin}$ by

$$\delta(\sum_m \phi_m) = \sum_m \delta_m \phi_m,$$

and put

$$\begin{aligned} \rho(F) &= \mathcal{M} - \delta \circ \mathcal{D}, \\ \rho(E) &= \pi(\sigma)\rho(F)\pi(\sigma)^{-1} = \mathcal{M}^\sigma - \delta \circ \mathcal{D}^\sigma. \end{aligned}$$

(Observe that, since $\deg Q = 4$, then $Q$ is even, and $\sigma = \sigma^{-1}$.)

**Lemma 5.1.**

$$\begin{aligned} [\rho(H), \rho(E)] &= 2\rho(E), \\ [\rho(H), \rho(F)] &= -2\rho(F). \end{aligned}$$

*Proof.* Since

$$\begin{aligned} \rho(H)\mathcal{M} &: \psi(z)w^m \mapsto (\mathcal{E} - 2(m+1))\psi(z)w^{m+1}, \\ \mathcal{M}\rho(H) &: \psi(z)w^m \mapsto (\mathcal{E} - 2m)\psi(z)w^{m+1}, \end{aligned}$$

one obtains $[\rho(H), \mathcal{M}] = -2\mathcal{M}$. Since

$$\begin{aligned} \rho(H)\delta\mathcal{D} &: \psi(z)w^m \mapsto \delta_{m-1}(\mathcal{E} - 2(m-1))Q\Big(\frac{\partial}{\partial z}\Big)\psi(z)w^{m-1}, \\ \delta\mathcal{D}\rho(H) &: \psi(z)w^m \mapsto \delta_{m-1}Q\Big(\frac{\partial}{\partial z}\Big)(\mathcal{E} - 2m)\psi(z)w^{m-1}, \end{aligned}$$

and, by using the identity

$$[Q\Big(\frac{\partial}{\partial z}\Big), \mathcal{E}] = 4Q\Big(\frac{\partial}{\partial z}\Big),$$

one gets

$$[\rho(H), \delta\mathcal{D}] : \psi(z)w^m \mapsto 2\delta_{m-1}Q\Big(\frac{\partial}{\partial z}\Big)\psi(z)w^{m-1}.$$

Finally $[\rho(H), \rho(F)] = -2\rho(F)$. Since the operator $\delta$ commutes with $\pi(\sigma)$, and $\pi(\sigma)\rho(H)\pi(\sigma)^{-1} = -\rho(H)$, we get also $[\rho(H), \rho(E)] = 2\rho(E)$. ∎

Let $\mathbb{D}(V)^L$ denote the algebra of $L$-invariant differential operators on $V$. This algebra is commutative. In fact it is isomorphic to the algebra of invariant differential operators on the symmetric cone in the Euclidean real form $V_{\mathbb{R}}$. If $V$ is simple and $Q = \Delta$, the determinant polynomial, then $\mathbb{D}(V)^L$ is isomorphic to the algebra $\mathcal{P}(\mathbb{C}^r)^{\mathfrak{S}_r}$ of symmetric polynomials in $r$ variables. The map

$$D \mapsto \gamma(D), \quad \mathbb{D}(V)^L \to \mathcal{P}(\mathbb{C}^r)^{\mathfrak{S}_r},$$

is the Harish-Chandra isomorphism (see Theorem XIV.1.7 in [Faraut-Korányi,1994]). In general $V$ decomposes into simple ideals,

$$V = \bigoplus_{i=1}^{s} V_i,$$

and $\mathbb{D}(V)^L$ is isomorphic to the algebra

$$\prod_{i=1}^{s} \mathcal{P}(\mathbb{C}^{r_i})^{\mathfrak{S}_{r_i}}.$$

The isomorphism is given by

$$D \mapsto \gamma(D) = \big(\gamma_1(D), \ldots, \gamma_s(D)\big),$$

where $\gamma_i$ is the isomorphism relative to the algebra $V_i$. For $D \in \mathbb{D}(V)^L$, we define the adjoint $D^*$ by $D^* = J \circ D \circ J$, where $Jf(z) = f \circ j(z) = f(-z^{-1})$. Then $\gamma(D^*)(\lambda) = \gamma(D)(-\lambda)$. (See Proposition XIV.1.8 in [Faraut-Korányi,1994].)

In our setting we define the Maass operator $\mathbf{D}_\alpha$ as

$$\mathbf{D}_\alpha = Q(z)^{1+\alpha} Q\Big(\frac{\partial}{\partial z}\Big) Q(z)^{-\alpha}.$$

It is $L$-invariant. We write

$$\gamma_\alpha(\lambda) = \gamma(\mathbf{D}_\alpha)(\lambda).$$

If $V$ is simple and $Q = \Delta$, then

$$\gamma_\alpha(\lambda) = \prod_{i=1}^{r} \Big(\lambda_j - \alpha + \frac{1}{2}(\frac{n}{r} - 1)\Big),$$

([Faraut-Korányi,1994], p.296). If $V$ is simple and $Q = \Delta^k$, then

$$\begin{aligned} \mathbf{D}_\alpha &= \Delta^{k+k\alpha} \Delta\Big(\frac{\partial}{\partial z}\Big)^k \Delta(z)^{-k\alpha} \\ &= \prod_{j=1}^{k} \Delta^{k\alpha+k-j+1} \Delta\Big(\frac{\partial}{\partial z}\Big) \Delta^{-(k\alpha+k-j)}, \end{aligned}$$

and

$$\gamma_\alpha(\lambda) = \prod_{j=1}^{r} \big[\lambda_j - k\alpha + \frac{1}{2}(\frac{n}{r} - 1)\big]_k.$$

(We have used the Pochhammer symbol $[a]_k = a(a-1)\ldots(a-k+1)$.)

**Proposition 5.2.** *In general*

$$\gamma_\alpha(\lambda) = \prod_{i=1}^{s} \prod_{j=1}^{r_i} \left[\lambda_j^{(i)} - k_i\alpha + \frac{1}{2}\left(\frac{n_i}{r_i} - 1\right)\right]_{k_i},$$

*for* $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(s)})$, $\lambda^{(i)} \in \mathbb{C}^{r_i}$.

We say that the pair $(V, Q)$ has property (T) if there is a constant $\eta$ such that, for $X \in \mathbf{l} = Lie(L)$,

$$\mathrm{Tr}(X) = \eta\tau(X).$$

In such a case, for $g \in L$,

$$\mathrm{Det}(g) = \gamma(g)^\eta,$$

and, for $x \in V$,

$$\mathrm{Det}\big(P(x)\big) = Q(x)^{2\eta}.$$

Furthermore $Q(x)^{-\eta} m(dx)$ is an $L$-invariant measure on the symmetric cone $\Omega \subset V_{\mathbb{R}}$, and $H_0(z) = H(z)^{-2\eta}$.

Let $V = \oplus_{i=1}^{s} V_i$ be the decomposition of $V$ into simple ideals. Property (T) is equivalent to the following: there is a constant $\eta$ such that

$$\frac{n_i}{r_i} = \eta k_i \quad (i = 1, \ldots, s).$$

In fact, for $x \in V$,

$$\mathrm{Tr}(T_x) = \sum_{i=1}^{s} \frac{n_i}{r_i} \,\mathrm{tr}_i(x_i), \quad \tau(T_x) = \sum_{i=1}^{s} k_i \,\mathrm{tr}_i(x_i),$$

with $x = (x_1, \ldots, x_s)$, $x_i \in V_i$.

Property (T) is satisfied either if $V$ is simple, or if $V = \mathbb{C}^p \oplus \mathbb{C}^p$, and

$$Q(z) = (z_1^2 + \cdots + z_p^2)(z_{p+1}^2 + \cdots + z_{2p}^2).$$

Hence we get the following cases with property (T):

(1) $V = \mathbb{C}^n$, $Q(z) = (z_1^2 + \cdots + z_n^2)^2$, and then

$$\mathbf{g} = \mathbf{sl}(n+2, \mathbb{C}), \quad \mathbf{k} = \mathbf{so}(n+2, \mathbb{C}).$$

(2) $V = \mathbb{C}^p \oplus \mathbb{C}^p$, and then

$$\mathbf{g} = \mathbf{so}(2p+4, \mathbb{C}), \quad \mathbf{k} = \mathbf{so}(p+2, \mathbb{C}) \oplus \mathbf{so}(p+2, \mathbb{C}).$$

(3) $V$ is simple of rank 4, and $Q = \Delta$, the determinant polynomial. Then

$$(\mathbf{g}, \mathbf{k}) = \big(\mathbf{e}_6, \mathbf{sp}(8, \mathbb{C})\big), \quad \big(\mathbf{e}_7, \mathbf{sl}(8, \mathbb{C})\big), \quad \big(\mathbf{e}_8, \mathbf{so}(16, \mathbb{C})\big).$$

Observe that the case $V = \mathbb{C}^2$, $Q(z_1, z_2) = (z_1 z_2)^2 = z_1^2 z_2^2$ belongs both to (1) and (2). This corresponds to the isomorphisms:

$$\mathbf{sl}(4, \mathbb{C}) \simeq \mathbf{so}(6, \mathbb{C}), \ \mathbf{so}(4, \mathbb{C}) \simeq \mathbf{so}(3, \mathbb{C}) \oplus \mathbf{so}(3, \mathbb{C}).$$

**Proposition 5.3.** *The subspaces $\mathcal{O}_m(\Xi)$ are invariant under $[\rho(E), \rho(F)]$, and the restriction of $[\rho(E), \rho(F)]$ to $\mathcal{O}_m(\Xi)$ commutes with the L-action:*

$$[\rho(E), \rho(F)] : \mathcal{O}_m(\Xi) \to \mathcal{O}_m(\Xi), \quad \psi(z) w^m \mapsto (P_m \psi)(z) w^m,$$

*where $P_m$ is an L-invariant differential operator on $V$ of degree $\leq 4$. It is given by*

$$P_m = \delta_m(\mathbf{D}_{-1} - \mathbf{D}^*_{-m-1}) + \delta_{m-1}(\mathbf{D}^*_{-m} - \mathbf{D}_0).$$

*Proof.* Restricted to $\mathcal{O}_m(\Xi)$,

$$\mathcal{M}^\sigma \mathcal{D} = \mathbf{D}_0, \quad \mathcal{D}\mathcal{M}^\sigma = \mathbf{D}_{-1}, \quad \mathcal{M}\mathcal{D}^\sigma = \mathbf{D}^*_{-m}, \quad \mathcal{D}^\sigma \mathcal{M} = \mathbf{D}^*_{-m-1}.$$

It follows that the restriction of the operator $[\rho(E), \rho(F)]$ to $\mathcal{O}_m(\Xi)$ is given by

$$\begin{aligned}
[\rho(E), \rho(F)] &= [\mathcal{M}^\sigma - \delta \circ \mathcal{D}^\sigma, \mathcal{M} - \delta \circ \mathcal{D}] \\
&= [\mathcal{M}, \delta \circ \mathcal{D}^\sigma] + [\delta \circ \mathcal{D}, \mathcal{M}^\sigma] \\
&= \mathcal{M}\delta\mathcal{D}^\sigma - \delta\mathcal{D}^\sigma\mathcal{M} + \delta\mathcal{D}\mathcal{M}^\sigma - \mathcal{M}^\sigma\delta \circ \mathcal{D} \\
&= \delta_m(\mathcal{D}\mathcal{M}^\sigma - \mathcal{D}^\sigma\mathcal{M}) + \delta_{m-1}(\mathcal{M}\mathcal{D}^\sigma - \mathcal{M}^\sigma\mathcal{D}) \\
&= \delta_m(\mathbf{D}_{-1} - \mathbf{D}^*_{-m-1}) + \delta_{m-1}(\mathbf{D}^*_{-m} - \mathbf{D}_0).
\end{aligned}$$

∎

By the Harish-Chandra isomorphism the operator $P_m$ corresponds to the polynomial $p_m = \gamma(P_m)$,

$$p_m(\lambda) = \delta_m\big(\gamma_{-1}(\lambda) - \gamma_{-m-1}(-\lambda)\big) + \delta_{m-1}\big(\gamma_{-m}(-\lambda) - \gamma_0(\lambda)\big).$$

The question is now whether it is possible to choose the sequence $(\delta_m)$ in such a way that $[\rho(E), \rho(F)] = \rho(H)$. Recall that restricted to $\mathcal{O}_m(\Xi)$,

$$\rho(H) = \mathcal{E} - 2m,$$

where $\mathcal{E}$ is the Euler operator

$$\mathcal{E}\phi(w,z) = \frac{d}{dt}\Big|_{t=0}\phi(w, e^t z).$$

Then, by Proposition 5.3, it amounts to checking that, for every $m$,

$$p_m(\lambda) = \gamma(\mathcal{E})(\lambda) - 2m.$$

**Theorem 5.4.** *It is possible to choose the sequence* $(\delta_m)$ *such that*

$$[\rho(H), \rho(E)] = 2\rho(E), \quad [\rho(H), \rho(F)] = -2\rho(F), \quad [\rho(E), \rho(F)] = \rho(H),$$

*if and only if* $(V, Q)$ *has property (T), and then*

$$\delta_m = \frac{A}{(m+\eta)(m+\eta+1)},$$

*where* $A$ *is a constant depending on* $(V, Q)$*.*

(This corresponds to Theorem 6.3 in [Brylinski,1998].)

*Proof.* a) Let us assume first that the Jordan algebra $V$ is simple of rank 4. In such a case

$$\gamma_\alpha(\lambda) = \prod_{j=1}^{4}\Big(\lambda_j - \alpha + \frac{1}{2}(\eta - 1)\Big) \quad (\eta = \frac{n}{r})$$

(Proposition 5.2) . With $X_j = \lambda_j + \frac{1}{2}(\eta - 1)$, the polynomial $p_m$ can be written

$$\begin{aligned} p_m(\lambda) &= \delta_m\Big(\prod_{j=1}^{4}(X_j + 1) - \prod_{j=1}^{4}(X_j - m - \eta)\Big) \\ &\quad + \delta_{m-1}\Big(\prod_{j=1}^{4}(X_j - m + 1 - \eta) - \prod_{j=1}^{4} X_j\Big). \end{aligned}$$

Furthermore

$$\gamma(\mathcal{E})(\lambda) - 2m = \sum_{j=1}^{4}\lambda_j - 2m = \sum_{j=1}^{4} X_j - 2(m + \eta - 1).$$

**Lemma 5.5.** *The identity in the four variables* $X_j$

$$\alpha\Big(\prod_{j=1}^{4}(X_j+1)-\prod_{j=1}^{4}(X_j-b_j-1)\Big)+\beta\Big(\prod_{j=1}^{4}(X_j-b_j)-\prod_{j=1}^{4}X_j\Big)$$
$$=\sum_{j=1}^{4}X_j+c$$

*holds if and only if there is a constant* $b$ *such that*

$$b_1=b_2=b_3=b_4=b,\ \ c=-2b,$$
$$\alpha=\frac{1}{(b+1)(b+2)},\ \ \beta=\frac{1}{b(b+1)}.$$

Hence we apply the lemma, and get $b=m+\eta-1$. ∎

b) In the general case

$$\begin{aligned}\gamma_\alpha(\lambda) &= \prod_{i=1}^{s}\prod_{j=1}^{r_i}\big[\lambda_j^{(i)}-k_i\alpha+\frac{1}{2}(\frac{n_i}{r_i}-1)\big]_{k_i}\\ &= \prod_{i=1}^{s}\prod_{j=1}^{r_i}\prod_{k=1}^{k_i}\Big(\lambda_j^{(i)}-k_i\alpha+\frac{1}{2}\big(\frac{n_i}{r_i}-1\big)-(k-1)\Big)\\ &= A\prod_{i=1}^{s}\prod_{j=1}^{r_i}\prod_{k=1}^{k_i}\Big(\frac{\lambda_j^{(i)}}{k_i}-\alpha+\frac{1}{2k_i}\big(\frac{n_i}{r_i}-1\big)-\frac{k-1}{k_i}\Big),\end{aligned}$$

with $A=\prod_{i=1}^{s}k_i^{k_ir_i}$. We introduce the notation

$$\begin{aligned}X_{jk}^{(i)} &= \frac{\lambda_j^{(i)}}{k_i}+\frac{1}{2k_i}\big(\frac{n_i}{r_i}-1\big)-\frac{k-1}{k_i},\\ b_m^{(i)} &= m+\frac{n_i}{k_ir_i}-1.\end{aligned}$$

Then we obtain

$$p_m(\lambda)=\quad A\delta_m\Big(\prod_{i=1}^{s}\prod_{j=1}^{r_i}\prod_{k=1}^{k_i}(X_{jk}^{(i)}+1)-\prod_{i=1}^{s}\prod_{j=1}^{r_i}\prod_{k=1}^{k_i}(X_{jk}^{(i)}-b_m^{(i)}-1)\Big)$$

$$+A\delta_{m-1}\Big(\prod_{i=1}^{s}\prod_{j=1}^{r_i}\prod_{k=1}^{k_i}(X_{jk}^{(i)}-b_m^{(i)})-\prod_{i=1}^{s}\prod_{j=1}^{r_i}\prod_{k=1}^{k_i}(X_{jk}^{(i)})\Big),$$

and

$$\gamma(\mathcal{E})(\lambda)=\sum_{i=1}^{s}\sum_{j=1}^{r_i}\sum_{k=1}^{k_i}X_{jk}^{(i)}-\frac{1}{2}\sum_{i=1}^{s}\sum_{j=1}^{r_i}\sum_{k=1}^{k_i}b_m^{(i)}.$$

If the rank of $V$ is equal to 4, then the $k_i$ are equal to 1, and the four variables $X_{j1}^{(i)}$ are independant. By Lemma 5.5, Theorem 5.4 is proven in that case.

If the rank $r$ of $V$ is $< 4$, then

$$X_{jk}^{(i)}=X_{j1}^{(i)}-\frac{k-1}{k_i},$$

and there are only $r$ independant variables: $X_{j1}^{(i)}$. In that case Theorem 5.4 is proven by using an alternative form of Lemma 5.5: ∎

**Lemma 5.6.** *To a partition $k=(k_1,\dots,k_\ell)$ of 4 and length $\ell$:*

$$k_1+\cdots+k_\ell=4,$$

*and the numbers $\gamma_{ij}$ ($1\le i\le\ell$, $1\le j\le k_i-1$), one associates the polynomial $F$ in the $\ell$ variables $T_1,\dots,T_\ell$:*

$$F(T_1,\dots,T_\ell)=\prod_{i=1}^{\ell}T_i\prod_{j=1}^{k_i-1}(T_i+\gamma_{ij}).$$

*Given $\alpha,\beta,c\in\mathbb{R}$, and $b_1,\dots b_\ell\in\mathbb{R}$, then*

$$\alpha\big(F(T_1+1,\dots,T_\ell+1)-F(T_1-b_1-1,\dots,T_\ell-b_\ell-1)\big)$$
$$+\beta\big(F(T_1-b_1,\dots,T_\ell-b_\ell)-F(T_1,\dots,T_\ell\big)=\sum_{i=1}^{\ell}T_i+c$$

*is an identity in the variables $T_1,\dots,T_\ell$ if and only if there exists $b$ such that*

$$b_1=\cdots=b_\ell=b,\ \ \alpha=\frac{1}{(b+1)(b+2)},\ \ \beta=\frac{1}{b(b+1)},$$

*and*

$$c=\sum_{i=1}^{\ell}\sum_{j=1}^{k_i-1}\gamma_{ij}-2b.$$

For $p \in \mathbf{p}$, define the multiplication operator $\mathcal{M}(p)$ given by

$$\big(\mathcal{M}(p)\phi\big)(w,z) = wp(z)\phi(w,z).$$

Observe that $\mathcal{M}(1) = \mathcal{M}$. Then, for $g \in K$,

$$\mathcal{M}\big(\kappa(g)p\big) = \pi(g)\mathcal{M}(p)\pi(g^{-1}).$$

In fact

$$\big(\mathcal{M}(p)\pi(g^{-1})\phi\big)(w,z) = wp(z)\phi\big(\mu(g,z)w, g\cdot z\big),$$

and

$$\begin{aligned}&\big(\pi(g)\mathcal{M}(p)\pi(g^{-1})\phi\big)(w,z)\\ &= \mu(g^{-1},z)wp(g^{-1}\cdot z)\phi\big(\mu(g^{-1},z)\mu(g,g^{-1}\cdot z)w, g^{-1}g\cdot z\big)\\ &= w\big(\kappa(z)p\big)(z)\phi(w,z) = \mathcal{M}\big(\kappa(g)p\big)\phi(w,z).\end{aligned}$$

**Proposition 5.7.** *There is a unique map*

$$\mathbf{p} \to \mathrm{End}\big(\mathcal{O}_{\mathrm{fin}}(\Xi)\big), \quad p \mapsto \mathcal{D}(p),$$

*such that* $\mathcal{D}(1) = \mathcal{D}$*, and, for* $g \in K$*,*

$$\mathcal{D}\big(\kappa(g)p\big) = \pi(g)\mathcal{D}(p)\pi(g^{-1}).$$

(This corresponds to part of Theorem 6.1 in [Brylinski,1998].)

*Proof.* Recall that, for $g \in P_{\max}$,

$$\big(\kappa(g)p\big)(z) = \chi(g)p(g^{-1}\cdot z),$$

and

$$\big(\pi(g)\phi\big)(w,z) = \phi\big(\chi(g)w, g^{-1}\cdot g\big).$$

Let us show that, for $g \in P_{\max}$,

$$\pi(g)\mathcal{D}\pi(g^{-1}) = \chi(g)\mathcal{D}.$$

Observe first that, for $\ell \in L$ and a smooth function $\psi$ on $V$,

$$Q\Big(\frac{\partial}{\partial z}\Big)\big(\psi(\ell\cdot z)\big) = \gamma(\ell)\Big(Q\big(\frac{\partial}{\partial z}\big)\psi\Big)(\ell\cdot z).$$

Therefore, for $g \in P_{\max}$,

$$\begin{aligned}\mathcal{D}\pi(g^{-1})\phi(w,z) &= \frac{1}{w}Q\Big(\frac{\partial}{\partial z}\Big(\phi\big(\chi(g^{-1})w, g\cdot z\big)\Big) \\ &= \frac{1}{w}\chi(g)^2\Big(Q\big(\frac{\partial}{\partial z}\big)\phi\Big)\big(\chi(g^{-1})w, g\cdot z\big),\end{aligned}$$

and

$$\big(\pi(g)\mathcal{D}\pi(g^{-1})\phi\big)(w,z) = \frac{1}{\chi(g)w}\chi(g)^2\Big(Q\big(\frac{\partial}{\partial z}\big)\phi\Big)(w,z) = \chi(g)\mathcal{D}\phi(w,z).$$

It follows that the vector subspace in $\mathrm{End}\big(\mathcal{O}_{\mathrm{fin}}(\Xi)\big)$ generated by the endomorphisms $\pi(g)\mathcal{D}\pi(g^{-1})$ $(g \in K)$ is a representation space for $K$ equivalent to $\mathbf{p}$. (See Theorem 3.10 in [Brylinski-Kostant,1994].) Hence there exists a unique $K$-equivariant map $p \mapsto \mathcal{D}(p)$ such that $\mathcal{D}(1) = \mathcal{D}$.

For $p \in \mathbf{p}$, define

$$\rho(p) = \mathcal{M}(p) - \delta\mathcal{D}(p).$$

Observe that this definition is consistent with the definition of $\rho(E)$ and $\rho(F)$. Recall that, for $X \in \mathbf{k}$, $\rho(X) = d\pi(X)$. Hence we get a map

$$\rho : \mathbf{g} = \mathbf{k} \oplus \mathbf{p} \to \mathrm{End}\big(\mathcal{O}(\Xi)_{\mathrm{fin}}\big).$$

**Theorem 5.8.** *Assume that Property* $(T)$ *holds. Fix* $(\delta_m)$ *as in Theorem 5.4.*
(i) $\rho$ *is a representation of the Lie algebra* $\mathbf{g}$ *on* $\mathcal{O}(\Xi)_{\mathrm{fin}}$.
(ii) *The representation* $\rho$ *is irreducible.*

*Proof.* (i) Since $\pi$ is a representation of $K$, for $X, X' \in \mathbf{k}$,

$$[\rho(X), \rho(X')] = \rho([X, X']).$$

It follows from Proposition 5.7 that, for $X \in \mathbf{k}, p \in \mathbf{p}$,

$$[\rho(X), \rho(p)] = \rho([X, p]).$$

It remains to show that, for $p, p' \in \mathbf{p}$,

$$[\rho(p), \rho(p')] = \rho([p, p']).$$

By Theorem 5.4, $[\rho(E), \rho(F)] = \rho(H)$. Then this follows from Lemma 3.6 in [Brylinski-Kostant,1995]: consider the map

$$\tau : \bigwedge^2 \mathbf{p} \to \mathrm{End}\big(\mathcal{O}(\Xi)_{\text{fin}},$$

defined by

$$\tau(p \wedge p') = [\rho(p), \rho(p')] - \rho([p, p']).$$

We know that $\tau(E \wedge F) = 0$. It follows that, for $g \in K$,

$$\tau\big(\kappa(g)E \wedge \kappa(g)F\big) = 0.$$

Since the representation $\kappa$ is irreducible, and $E$ and $F$ are highest and lowest vectors with respect to $P$, the vector $E \wedge F$ is cyclic in $\bigwedge^2 \mathbf{p}$ for the action of $K$. Therefore $\tau \equiv 0$.

(ii) Let $\mathcal{V} \neq \{0\}$ be a $\rho(\mathbf{g})$-invariant subspace of $\mathcal{O}(\Xi)_{\text{fin}}$. Then $\mathcal{V}$ is $\rho(\mathbf{k})$-invariant. As $\mathcal{O}(\Xi)_{\text{fin}} = \sum_{m=0}^{\infty} \mathcal{O}_m(\Xi)$ and as the subspaces $\mathcal{O}_m(\Xi)$ are $\rho(\mathbf{k})$-irreducible, then there exists $\mathcal{I} \subset \mathbb{N}$ ($\mathcal{I} \neq \emptyset$) such that $\mathcal{V} = \sum_{m \in \mathcal{I}} \mathcal{O}_m(\Xi)$. Observe that if $\mathcal{V}$ contains $\mathcal{O}_m(\Xi)$, then it contains $\mathcal{O}_{m+1}(\Xi)$ too. In fact denote by $\phi_m$ the function in $\mathcal{O}_m(\Xi)$ defined by $\phi_m(w, z) = w^m$. As $\mathcal{D}\phi_m = 0$, it follows that

$$\rho(F)\phi_m = \mathcal{M}\phi_m = \phi_{m+1},$$

and $\rho(F)\phi_m$ belongs to $\mathcal{O}_{m+1}(\Xi)$, therefore $\mathcal{O}_{m+1}(\Xi) \subset \mathcal{V}$. Denote by $m_0$ the minimum of the $m$ such that $\mathcal{O}_m(\Xi) \subset \mathcal{V}$, then

$$\mathcal{V} = \bigoplus_{m=m_0}^{\infty} \mathcal{O}_m(\Xi).$$

The function $\phi(w, z) = Q(z)^m w^m$ belongs to $\mathcal{O}_m(\Xi)$, and

$$\rho(F)\phi(w, z) = Q(z)^m w^{m+1} - \delta_{m-1} Q\Big(\frac{\partial}{\partial z}\Big) Q(z)^m w^{m-1}.$$

By the Bernstein identity (Proposition 3.1)

$$Q\Big(\frac{\partial}{\partial z}\Big) Q(z)^m = B(m) Q(z)^{m-1},$$

and since $B(m) > 0$ for $m > 0$, it follows that, if $\mathcal{O}_m(\Xi) \subset \mathcal{V}$ with $m > 0$, then $\mathcal{O}_{m-1}(\Xi) \subset \mathcal{V}$. Therefore $m_0 = 0$ and $\mathcal{V} = \mathcal{O}(\Xi)_{\text{fin}}$. ∎

# 6 The unitary representation of the Kantor-Koecher-Tits group

We consider, for a sequence $(c_m)$ of positive numbers, an inner product on $\mathcal{O}(\Xi)_{\rm fin}$ such that

$$\|\phi\|^2 = \sum_{m=0}^{\infty} \frac{1}{c_m} \|\psi_m\|_m^2,$$

for

$$\phi(w,z) = \sum_{m=0}^{\infty} \psi_m(z) w^m.$$

This inner product is invariant under $K_{\mathbb{R}}$. We assume that Property (T) holds, and we will determine the sequence $(c_m)$ such that this inner product is invariant under the representation $\rho$ restricted to $\mathbf{g}_{\mathbb{R}}$. We denote by $\mathcal{H}$ the Hilbert space completion of $\mathcal{O}(\Xi)_{\rm fin}$ with respect to this inner product. We will assume $c_0 = 1$.

The Bernstein polynomial $B$ is of degree 4, and vanishes at 0 and $\alpha_1 = 1 - \eta$. Let $\alpha_2$ and $\alpha_3$ be the two remaining roots:

$$B(\alpha) = A\alpha(\alpha - \alpha_1)(\alpha - \alpha_2)(\alpha - \alpha_3).$$

(1) $V = \mathbb{C}^n$, $Q(z) = (z_1^2 + \cdots + z_n^2)^2$. Then

$$B(\alpha) = A\alpha\Big(\alpha - \frac{1}{2}\Big)\Big(\alpha + \frac{n-4}{4}\Big)\Big(\alpha + \frac{n-2}{4}\Big).$$

$A = 2^4$ if $n \geq 2$, $A = 4^4$ if $n = 1$.

(2) $V = \mathbb{C}^{2p}$, $Q(z) = (z_1^2 + \cdots + z_p^2)(z_{p+1}^2 + \cdots + z_{2p}^2)$. Then

$$B(\alpha) = \alpha^2\Big(\alpha + \frac{p-2}{2}\Big)^2.$$

(3) $V$ is simple of rank 4, complexification of $V_{\mathbb{R}} = Herm(4, \mathbb{F})$, $Q(z) = \Delta(z)$, the determinant polynomial. Then

$$B(\alpha) = \alpha\Big(\alpha + \frac{d}{2}\Big)\Big(\alpha + 2\frac{d}{2}\Big)\Big(\alpha + 3\frac{d}{2}\Big),$$

where $d = \dim_{\mathbb{R}} \mathbb{F}$.

Here are the non zero roots of the Bernstein polynomial:

| | $\eta$ | $\alpha_1$ | $\alpha_2$ | $\alpha_3$ |
|---|---|---|---|---|
| (1) | $\frac{n}{4}$ | $-\frac{n-4}{4}$ | $\frac{1}{2}$ | $-\frac{n-2}{4}$ |
| (2) | $\frac{p}{2}$ | $-\frac{p-2}{2}$ | $0$ | $-\frac{p-2}{2}$ |
| (3) | $1+3\frac{d}{2}$ | $-3\frac{d}{2}$ | $-\frac{d}{2}$ | $-2\frac{d}{2}$ |

**Theorem 6.1.** (i) *The inner product of $\mathcal{H}$ is $\mathbf{g}_{\mathbb{R}}$-invariant if*

$$c_m = \frac{(\eta+1)_m}{(\eta+\alpha_2)_m(\eta+\alpha_3)_m} \frac{1}{m!}.$$

(ii) *The reproducing kernel of $\mathcal{H}$ is given by*

$$\mathcal{K}(\xi,\xi') = {}_1F_2(\eta+1; \eta+\alpha_2, \eta+\alpha_3; H(z,z')w\overline{w'}),$$

*for $\xi = (w,z)$, $\xi' = (w',z')$.*

(This corresponds to Theorems 6.6 and 8.1 in [Brylinski,1998].)

*Proof.* (i) Recall that

$$\mathbf{p}_{\mathbb{R}} = \{p \in \mathbf{p} \mid \beta(p) = p\},$$

where $\beta$ is the conjugation of $\mathbf{p}$, we introduced at the end of Section 4. Recall also that

$$\beta(\kappa(g)p) = \kappa\big(\alpha(g)\big)\beta(p).$$

The inner product of $\mathcal{H}$ is $\mathbf{g}_{\mathbb{R}}$-invariant if and only if, for every $p \in \mathbf{p}$,

$$\rho(p)^* = -\rho\big(\beta(p)\big).$$

But this is equivalent to the single condition

$$\rho(E)^* = -\rho(F).$$

In fact, assume that this condition is satisfied. Then, for $p = \kappa(g)E$, $(g \in K)$,

$$\rho(p) = \pi(g)\rho(E)\pi(g^{-1}), \quad \rho(p)^* = -\pi(g^{-1})^*\rho(F)\pi(g)^*.$$

Since $\pi(g)^* = \pi\big(\alpha(g)\big)^{-1}$, we get

$$\begin{aligned}\rho(p)^* &= -\pi\big(\alpha(g)\big)\rho(F)\pi\big(\alpha(g^{-1})\big) = -\rho\big(\kappa(\alpha(g))F\big) \\ &= -\rho\big(\kappa(\alpha(g))\beta(E)\big) = -\rho\big(\beta(\kappa(g)E)\big) = -\rho\big(\beta(p)\big).\end{aligned}$$

Finally observe that the vector $E$ is cyclic in $\mathbf{p}$ for the $K$-action.

The condition $\rho(E)^* = -\rho(F)$ is equivalent to: for $m \geq 0$, $\phi \in \mathcal{O}_{m+1}(\Xi)$, $\phi' \in \mathcal{O}_m(\Xi)$,

$$\frac{1}{c_{m+1}}(\phi \mid \mathcal{M}^\sigma \phi')_{m+1} = \frac{1}{c_m}\delta_m(\mathcal{D}\phi \mid \phi')_m.$$

Recall that $m_0(dz) = H_0(z)m(dz)$ with

$$H_0(z) = H(z)^{-2\eta},$$

and the norm of $\tilde{\mathcal{O}}_m(V)$ can be written

$$\|\psi\|_m^2 = \frac{1}{a_m}\int_V |\psi(z)|^2 H(z)^{-m-2\eta} m(dz).$$

Then, the required condition of invariance becomes

$$\begin{aligned}&\frac{1}{c_{m+1}a_{m+1}}\int_V \psi(z)\overline{Q(z)\psi'(z)}H(z)^{-(m+1)-2\eta}m(dz) \\ &= \frac{\delta_m}{c_m a_m}\int_V \Big(Q\Big(\frac{\partial}{\partial z}\Big)\psi\Big)(z)\overline{\psi'(z)}H(z)^{-m-2\eta}m(dz).\end{aligned}$$

By integrating by parts:

$$\begin{aligned}&\int_V \Big(Q\Big(\frac{\partial}{\partial z}\Big)\psi\Big)(z)\overline{\psi'(z)}H(z)^{-m-2\eta}m(dz) \\ &= \int_V \psi(z)\overline{\psi'(z)}\bigg(Q\Big(\frac{\partial}{\partial z}\Big)H(z)^{-m-2\eta}\bigg)m(dz),\end{aligned}$$

and, by the relation

$$Q\Big(\frac{\partial}{\partial z}\Big)H(z)^{-m-2\eta} = B(-m-2\eta)\overline{Q(z)}H(z)^{-(m+1)-2\eta},$$

the condition can be written

$$\frac{1}{c_{m+1}} = \frac{a_{m+1}}{a_m}\delta_m B(-m-2\eta)\frac{1}{c_m}.$$

From Proposition 3.2 it follows that

$$\frac{a_{m+1}}{a_m} = \frac{B(-m-\eta)}{B(-m-2\eta)}.$$

We obtain finally

$$\frac{c_{m+1}}{c_m} = \frac{m+\eta+1}{(m+\eta+\alpha_2)(m+\eta+\alpha_3)(m+1)},$$

and, since $c_0 = 1$,

$$c_m = \frac{(\eta+1)_m}{(\eta+\alpha_2)_m(\eta+\alpha_3)_m}\frac{1}{m!}.$$

(ii) By Theorem 2.5 the reproducing kernel of $\mathcal{H}$ is given by

$$\begin{aligned}\mathcal{K}(\xi,\xi') &= \sum_{m=0}^{\infty} c_m H(z,z')^m w^m \overline{w'}^m \\ &= {}_1F_2\big(\eta+1;\eta+\alpha_2,\eta+\alpha_3; H(z,z')w\overline{w'}\big),\end{aligned}$$

with $\xi = (w,z)$, $\xi' = (w',z')$. □

We will see that the Hilbert space $\mathcal{H}$ is a pseudo-weighted Bergman space. By this we mean that the norm is given by an integral of $|\phi|^2$ with respect to a weight taking both positive and negative values. The weight involves a Meijer $G$-function:

$$G(u) = \frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty} \frac{\Gamma(\beta_1+s)\Gamma(\beta_2+s)\Gamma(\beta_3+s)}{\Gamma(\alpha+s)} u^{-s} ds,$$

where $\alpha,\beta_1,\beta_2,\beta_3$ are real numbers, and $c > \sigma = -\inf\{\beta_1,\beta_2,\beta_3\}$. This function is denoted by

$$G(u) = G^{3,0}_{1,3}\left(x \middle| \begin{matrix} \alpha & & \\ \beta_1 & \beta_2 & \beta_3 \end{matrix}\right)$$

(see for instance [Mathai,1993]). By the inversion formula for the Mellin transform

$$\int_0^\infty G(u)u^{s-1}du = \frac{\Gamma(\beta_1+s)\Gamma(\beta_2+s)\Gamma(\beta_3+s)}{\Gamma(\alpha+s)},$$

for $\operatorname{Re} s > \sigma$, and the integral is absolutely convergent. If the numbers $\beta_1, \beta_2, \beta_3$ are distinct, then

$$G(u) = \varphi_1(u)u^{\beta_1} + \varphi_2(u)u^{\beta_2} + \varphi_3(u)u^{\beta_3},$$

where $\varphi_1, \varphi_2, \varphi_3$ are holomorphic near 0. ($\varphi_1, \varphi_2, \varphi_3$ are ${}_1F_2$ hypergeometric functions.)

The function $G$ may be not positive on $]0, \infty[$, but is positive for $u$ large enough. In fact

$$G(u) \sim \sqrt{\pi} u^{\theta} e^{-2\sqrt{u}} \quad (u \to \infty),$$

where

$$\theta = \beta_1 + \beta_2 + \beta_3 - \alpha - \frac{1}{2}.$$

([Paris-Wood,1986], Theorem 3, p.32.)

Now take

$$\alpha = \eta - 1,\ \beta_1 = 2\eta - 1,\ \beta_2 = 2\eta + a - 1,\ \beta_3 = 2\eta + b - 1.$$

| | $\alpha$ | $\beta_1$ | $\beta_2$ | $\beta_3$ |
|---|---|---|---|---|
| (1) | $\frac{n}{4} - 1$ | $\frac{n-2}{2}$ | $\frac{n-1}{2}$ | $\frac{n-2}{4}$ |
| (2) | $\frac{p}{2} - 1$ | $p - 1$ | $p - 1$ | $\frac{p}{2}$ |
| (3) | $3\frac{d}{2}$ | $3d + 1$ | $5\frac{d}{2} + 1$ | $2d + 1$ |

The Mellin transform of $G$ vanishes at $-\alpha$, with changing sign. One can check that $-\alpha > \sigma$ in all cases. Therefore there are real values $s > \sigma$ for which the integral

$$\int_0^\infty G(u)u^{s-1}du < 0.$$

This implies that the function $G$ takes negative values on $]0, \infty[$.

**Theorem 6.2.** *For $\phi \in \mathcal{H}$,*

$$\|\phi\|^2 = \int_{\mathbb{C}\times V} |\phi(w,z)|^2 p(z,w) m(dw) m_0(dz),$$

*with*

$$p(w,z) = CG\big(|w|^2 H(z)\big)H(z).$$

*The integral is absolutely convergent.*

*Proof.* We will follow the proof of Theorem 5.7 in [Brylinski,1997].

a) From the proof of Theorem 6.1 it follows that

$$\begin{aligned}\frac{1}{a_m c_m} &= \frac{(2\eta)_m(2\eta+\alpha_2)_m(2\eta+\alpha_3)_m}{(\eta)_m} \\ &= C\frac{\Gamma(2\eta+m)\Gamma(2\eta+\alpha_2+m)\Gamma(2\eta+\alpha_3+m)}{\Gamma(\eta+m)} \\ &= C\int_0^\infty G(u)u^m du.\end{aligned}$$

(One checks that $\sigma < 1$, *i.e.* $G$ is integrable.) By the computation we did for the proof of Theorem 2.6, we obtain, for $\phi(w,z) = w^m\psi(z) \in \mathcal{O}_m$,

$$\int_{\mathbb{C}\times V} |\phi(w,z)|^2 p(z,w)m(dw)m_0(dz) = \|\phi\|^2.$$

Furthermore, if $\phi \in \mathcal{O}_m$, $\phi' \in \mathcal{O}_{m'}$, with $m \neq m'$,

$$\int_{\mathbb{C}\times V} \phi(w,z)\overline{\phi'(w,z)}m(dw)m_0(dz) = 0.$$

It follows that, for $\phi \in \mathcal{O}_{\text{fin}}$,

$$\int_{\mathbb{C}\times V} |\phi(w,z)|^2 p(z,w)m(dw)m_0(dz) = \|\phi\|^2.$$

The computation is justified by the fact that, for $s > \sigma$,

$$\int_0^\infty |G(u)|u^{s-1}du < \infty.$$

b) Let us consider the weighted Bergman space $\mathcal{H}^1$ whose norm is given by

$$\|\phi\|_1^2 = \int_{\mathbb{C}\times V} |\phi(w,z)|^2|p(w,z)|m(dw)m_0(dz).$$

By Theorem 2.6,

$$\|\phi\|_1^2 = \sum_{m=0}^\infty \frac{1}{c_m^1}\|\psi_m\|_m^2,$$

with

$$\frac{1}{a_m c_m^1} = C\int_0^\infty |G(u)|u^m du.$$

Obviously $c^1_m \leq c_m$, therefore $\mathcal{H}^1 \subset \mathcal{H}$. We will show that $\mathcal{H} \subset \mathcal{H}^1$. For that we will prove that there is a constant $A$ such that

$$c_m \leq A c^1_m.$$

As observed above there is $u_0 \geq 0$ such that $G(u) \geq 0$, for $u \geq u_0$, and then

$$\int_0^\infty |G(u)| u^m \leq \int_0^\infty G(u) u^m du + 2 \int_0^{u_0} |G(u)| u^m du.$$

Hence

$$\frac{1}{c^1_m} \leq \frac{1}{c_m} + 2 a_m u_0^m \int_0^{u_0} |G(u)| du.$$

By the formula we gave at the beginning of a), the sequence $a_m c_m u_0^m$ is bounded. Therefore there is a constant $A$ such that

$$\frac{1}{c^1_m} \leq A \frac{1}{c_m},$$

and this implies that $\mathcal{H} \subset \mathcal{H}_1$. □

Let $\tilde{G}_{\mathbb{R}}$ be the connected and simply connected Lie group with Lie algebra $\mathbf{g}_{\mathbb{R}}$ and denote by $\tilde{K}_{\mathbb{R}}$ the subgroup of $\tilde{G}_{\mathbb{R}}$ with Lie algebra $\mathbf{k}_{\mathbb{R}}$. It is a covering of $K_{\mathbb{R}}$. We denote by $s : \tilde{K}_{\mathbb{R}} \to K_{\mathbb{R}}, g \mapsto s(g)$ the canonical surjection.

**Theorem 6.3.** (i) *There is a unique unitary irreducible representation $\tilde{\pi}$ of $\tilde{G}_{\mathbb{R}}$ on $\mathcal{H}$ such that $d\tilde{\pi} = \rho$. And, for all $k \in \tilde{K}_{\mathbb{R}}$, $\tilde{\pi}(k) = \pi(s(k))$.*

(ii) *The representation $\tilde{\pi}$ is spherical.*

*Proof.* (i) Notice that if the operators $\rho(E+F)$ and $\rho(i(E-F))$ are skew-symmetric, then for each $p \in \mathbf{p}_{\mathbb{R}}$, the operator $\rho(p)$ is skew-symmetric. In fact, since the $\mathbf{sl}_2$-triple $(E, F, H)$ is strictly normal (see [Sekiguchi,1987]), which means that $H \in i\mathbf{k}_{\mathbb{R}}, E+F \in \mathbf{p}_{\mathbb{R}}, i(E-F) \in \mathbf{p}_{\mathbb{R}}$, and since $\mathbf{p} = \mathcal{U}(\mathbf{k})E$, hence $\mathbf{p}_{\mathbb{R}} = \mathcal{U}(\mathbf{k}_{\mathbb{R}})(E+F) + \mathcal{U}(\mathbf{k}_{\mathbb{R}})(i(E-F))$, and the assertion follows.

Now, by Nelson's criterion, it is enough to prove that the operator $\rho(\mathcal{L})$ is essentially self-adjoint where $\mathcal{L}$ is the Laplacian of $\mathbf{g}_{\mathbb{R}}$. Let's consider a basis $\{X_1, \ldots, X_k\}$ of $\mathbf{k}_{\mathbb{R}}$ and a basis $\{p_1, \ldots, p_l\}$ of $\mathbf{p}_{\mathbb{R}}$, orthogonal with respect to the Killing form. As $\mathbf{g}_{\mathbb{R}} = \mathbf{k}_{\mathbb{R}} + \mathbf{p}_{\mathbb{R}}$ is the Cartan decomposition of $\mathbf{g}_{\mathbb{R}}$, then the Laplacian and the Casimir operators of $\mathbf{g}_{\mathbb{R}}$ are given by

$$\mathcal{L} = X_1^2 + \ldots + X_k^2 + p_1^2 + \ldots + p_l^2,$$

$$\mathcal{C} \;=\; X_1^2 + \ldots + X_k^2 - p_1^2 - \ldots - p_l^2.$$

It follows that $\mathcal{L} = 2(X_1^2+\ldots+X_k^2)-\mathcal{C}$ and $\rho(\mathcal{L}) = 2\rho(X_1^2+\ldots+X_k^2)-\rho(\mathcal{C})$. Since $\rho(X_1^2+\ldots+X_k^2) = d\pi(X_1^2+\ldots+X_k^2)$ and as $\pi$ is a unitary representation of $K_{\mathbb{R}}$, hence the image $\rho(X_1^2+\ldots+X_k^2)$ of the Laplacian of $\mathbf{k}_{\mathbb{R}}$ is essentially self-adjoint. Moreover, since the dimension of $\mathcal{O}(\Xi)_{\text{fin}}$ is countable, then the commutant of $\rho$, which is a division algebra over $\mathbb{C}$, has a countable dimension too, and is equal to $\mathbb{C}$ (see [Cartier,1979], p.118). It follows that $\rho(\mathcal{C})$ is scalar. We deduce that $\rho(\mathcal{L})$ is essentially self-adjoint and that the irreducible representation $\rho$ of $\mathbf{g}_{\mathbb{R}}$ integrates to an irreducible unitary representation of $\tilde{G}_{\mathbb{R}}$, on the Hilbert space $\mathcal{H}$.

(ii) The space $\mathcal{O}_0(\Xi)$ reduces to the constant functions which are the $K$-fixed vectors. ∎

We don't know whether the representation $\tilde{\pi}$ goes down to a representation of a real Lie group $G_{\mathbb{R}}$ with $K_{\mathbb{R}}$ as a maximal compact subgroup.

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Institut de Mathématiques de Jussieu
Université Pierre et Marie Curie
4 place Jussieu, case 247, 75252 Paris cedex 05
achab@math.jussieu.fr, faraut@math.jussieu.fr