Summer School: Large N limits
L'étang de Hasselfurth
August 2008

Finite and infinite dimensional spherical analysis

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Chapter I

GELFAND PAIRS

I.1 Gelfand pairs. — Let G be a locally compact group, K a compact subgroup, and let $\mathcal{C}_c(K\backslash G/K)$ denote the space of K-biinvariant continuous functions with compact support. It is a convution algebra. The pair (G,K) is said to be a *Gelfand pair* if the convolution algebra $\mathcal{C}_c(K\backslash G/K)$ is commutative.

Examples

a) Let V be a finite dimensional real vector space: $V \simeq \mathbb{R}^n$, and K a compact subgroup of $GL(V)$. Put $G = K \ltimes V$. Then (G,K) is a Gelfand pair. In fact, as convolutions algebras, $\mathcal{C}_c(K\backslash G/K) \simeq \mathcal{C}_c(K\backslash V)$, the space of K-invariant continuous functions on V with compact support.

b) Let U be a compact group, and put

$$G = U \times U, \quad K = \{(x,x) \mid x \in U\} \simeq U.$$

Then (G,K) is a Gelfand pair. In fact, as convolution algebras, $\mathcal{C}(K\backslash G/K) \simeq \mathcal{C}_{\rm central}(U)$, the space of central functions on U.

c) Let G be a locally compact group, K a compact subgroup. Assume that there is a continuous involutive automorphism θ such that

$$\forall x \in G,\ x^{-1} \in K\theta(x)K.$$

Then (G,K) is a Gelfand pair. For instance, if (G,K) is a Riemannian symmetric pair: G is a connected Lie group, K a compact group, and there is a continuous involutive automorphism θ such that

$$(G^\theta)_0 \subset K \subset G^\theta,$$

where G^θ denote the subgroup of G consisting of θ-fixed elements, and $(G^\theta)_0$ the identity component of G^θ. In fact G admits a Cartan decomposition: $G = KAK$, where A is a Cartan subgroup whose elements a satisty $\theta(a) = a^{-1}$.

I.2 Spherical functions. — Assume that (G, K) is a Gelfand pair. A *spherical functions* is a continuous function φ on G, K-biinvariant, with $\varphi(e) = 1$, which satisfies the following functional equation

$$\int_K \varphi(xky)\alpha(dk) = \varphi(x)\varphi(y),$$

where α denotes the normalized Haar measure on K.

PROPOSITION I.1. — *Let φ be a continuous function on G, K-biinvariant, with $\varphi(e) = 1$. Then φ is a spherical function if and only if the linear functional*

$$\chi(f) = \int_G f(x)\varphi(x)m(dx),$$

where m is a Haar measure of G, is a character of the convolution algebra $\mathcal{C}_c(K\backslash G/K)$.

One shows that, if (G, K) is a gelfand pair, then G is unimodular.

Proof. For a function $f \in \mathcal{C}_c(G)$, we put

$$f^\natural(x) = \int_{K\times K} f(k_1xk_2)\alpha(dk_1)\alpha(dk_2),$$

and

$$\Phi(f) = \int_G f(x)\varphi(x)m(dx).$$

For two functions f and g in $\mathcal{C}_c(G)$,

$$\begin{aligned}
&\Phi(f^\natural * g^\natural) - \Phi(f^\natural)\Phi(g^\natural) \\
&= \int_{G\times G} \big(\varphi(xy) - \varphi(x)\varphi(y)\big) f^\natural(x)g^\natural(y)m(dx)m(dy) \\
&= \int_{G\times G} \Big(\int_K \varphi(xky)\alpha(dk) - \varphi(x)\varphi(y)\Big) f(x)g(y)m(dx)m(dy).
\end{aligned}$$

The proposition follows from these equalities. ∎

The characters of the commutative Banach algebra $L^1(K\backslash G/K)$ are of the form

$$\chi(f) = \int_G f(x)\varphi(x)m(dx),$$

where φ is a bounded spherical function.

Examples

a) $G = K \ltimes V$, where V is a finite dimensional real vector space. A K-biinvariant function on G can be seen as a K-invariant function on V. Then the functional equation for the spherical functions becomes:

$$\int_K \varphi(x + k \cdot y)\alpha(dk) = \varphi(x)\varphi(y).$$

The bounded spherical functions are Fourier transforms of orbital measures. For a in V^*, the function φ, defined on V by

$$\varphi(x) = \int_K e^{i\langle x, k\cdot a\rangle}\alpha(dk),$$

is spherical, and every bounded spherical function is obtained in that way.

b) U is a compact group,

$$G = U \times U, \quad K = \{(x, x) \mid x \in U\}.$$

A K-biinvariant function on G can be seen as a central function on U. Then the functional equation for the spherical functions becomes

$$\int_U \varphi(xuyu^{-1})\alpha(du) = \varphi(x)\varphi(y),$$

where α is the normalized Haar measure on U.

Let $\hat{U}$ be the set of equivalence classes of irreducible representations of U. For $\lambda \in \hat{U}$, let $(\pi_\lambda, \mathcal{H}_\lambda)$ be a representation of U in the class λ. It character is the function χ_λ defined on U by

$$\chi_\lambda(u) = \mathrm{tr}\big(\pi_\lambda(u)\big).$$

Then $\chi_\lambda(e) = d_\lambda := \dim \mathcal{H}_\lambda$. It satisfies the following functional equation

$$\int_U \chi_\lambda(xuyu^{-1})\alpha(du) = \frac{1}{d_\lambda}\chi_\lambda(x)\chi_\lambda(y).$$

Hence, the normalized character

$$varphi(\lambda; u) = \frac{1}{d_\lambda}\chi_\lambda(u),$$

is a spherical function. One can show that all spherical functions are obtained in that way.

I.3 Gelfand-Naimark-Segal construction. — A function φ defined on a group G is said to be *of positive type* if

$$\sum_{i,j=1}^{N} \varphi(x_i^{-1}x_j)c_i\overline{c_j} \geq 0,$$

for $x_1, \ldots, x_N \in G$, $c_1, \cdots, c_N \in \mathbb{C}$. A function of positive type satisfies the Hermitian symmetry:

$$\varphi(x^{-1}) = \overline{\varphi(x)},$$

Furthermore

$$|\varphi(x)| \leq \varphi(e).$$

If π is a unitary representation of G on a Hilbert space $\mathcal{H}$, for $u \in \mathcal{H}$, the function

$$\varphi(x) = \big(u|\pi(x)u\big)$$

is of positive type. In fact

$$\sum_{i,j=1}^{N} \varphi(x_i^{-1}x_j)c_i\overline{c_j} = \sum_{i,j=1}^{N} \big(\pi(x_i)u|\pi(x_j)u\big)c_i\overline{c_j} = \|\sum_{i=1}^{N} c_i\pi(x_i)u\|^2 \geq 0.$$

Every function of positive type can be written in that way. This is the Gelfand-Naimark-Segal construction we will describe below.

If G is a topological group, we will always assume a unitary representation $(\pi, \mathcal{H})$ of G to be continuous: for every $v \in \mathcal{H}$, the map

$$x \mapsto \pi(x)v, \quad G \to \mathcal{H},$$

is continuous.

A vector $u \in \mathcal{H}$ is said to be *cyclic* if the subspace of $\mathcal{H}$ generated by the vectors $\pi(x)u$ for $x \in G$ is dense in $\mathcal{H}$.

In this section G will be a topological group, and K a closed subgroup.

Proposition I.2 (Gelfand-Naimark-Segal construction). — *a) Let φ be a continuous function on G of positive type, K-biinvariant.*

Then there exists a unitary representation $(\pi, \mathcal{H})$ *of* G *with a cyclic* K*-invariant vector* u *such that*

$$\varphi(x) = \big(u|\pi(x)u\big).$$

b) The triple $(\pi, \mathcal{H}, u)$ *is unique up to isomorphism. Precisely, if* $(\pi', \mathcal{H}', u')$ *is another triple with*

$$\varphi(x) = \big(u'|\pi'(x)u'\big),$$

u' *cyclic and* K*-invariant, then there is a unitary isomorphism*

$$A : \mathcal{H} \to \mathcal{H}',$$

such that

$$A\pi(x) = \pi'(x)A, \quad Au = u'.$$

Proof.

Let $\mathcal{H}_0^\varphi$ be the space of functions on G of the form

$$f(x) = \sum_{i=1}^{N} c_i \varphi(g_i^{-1}x),$$

with $g_1, \ldots, g_N \in G$, $c_1, \ldots, c_N \in \mathbb{C}$. Clearly, $\mathcal{H}_0^\phi$ is the subspace of $\mathcal{C}(G/K)$, the space of right K-invariant continuous functions on G.

The norm of such a function f is defined by

$$\|f\|^2 = \sum_{i,j=1}^{N} \varphi(g_i^{-1}g_j)c_i\overline{c_j}.$$

Since φ is of positive type, this number is ≥ 0. Writing

$$f = \mu * \varphi, \text{ with } \mu = \sum_{i=1}^{N} c_i\delta_{g_i},$$

we obtain

$$\|f\|^2 = \int_G (\mu * \phi)(x)\overline{\mu(dx)}.$$

By the Schwarz inequality, with $\nu = \sum_{i=1}^{N} d_i\delta_{g_i}$,

$$\left|\int_G f(x)\nu(dx)\right|^2 = \left|\sum_{i,j=1}^{N} \varphi(g_i^{-1}g_j)c_i\overline{d_j}\right|^2$$

$$\leq \sum_{i,j=1}^{N} \varphi(g_i^{-1}g_j)c_i\overline{c_j} \cdot \sum_{i,j=1}^{N} \varphi(g_i^{-1}g_j)d_i\overline{d_j},$$

Therefore, if $\|f\|^2 = 0$, then $f \equiv 0$. Observe that in general the representation

$$f(x) = \sum_{i=1}^{N} c_i \varphi(g_i^{-1} x)$$

of a function f is not unique. The above inequality shows that $\|f\|^2$ only depends on f, and not on the chosen representation. Hence $\|f\|$ is indeed a norm on $\mathcal{H}_0^\varphi$, and $\mathcal{H}_0^\varphi$ is a preHilbert space, with the inner product, for

$$f(x) = \sum_{i=1}^{N} c_i \varphi(g_i^{-1} x), \; f'(x) = \sum_{j=1}^{N'} c'_j \varphi\big((g'_j)^{-1} x\big),$$

defined by

$$(f|f') = \sum_{i=1}^{N} \sum_{j=1}^{N'} \varphi(g_i^{-1} g'_j) c_i \overline{c'_j}.$$

Define the representation π^φ of G on $\mathcal{H}_0^\varphi$ by the left action

$$(\pi^\varphi(g) f)(x) = f(g^{-1} x), \qquad f \in \mathcal{H}_0^\varphi, \; g, x \in G.$$

Then π^φ is unitary. In fact, if

$$f(x) = \sum_{i=1}^{N} c_i \varphi(g_i^{-1} x),$$

then

$$(\pi^\varphi(g) f)(x) = \sum_{i=1}^{N} c_i \phi((g g_i)^{-1} x),$$

and hence

$$\begin{aligned} \|\pi^\phi(g) f\|^2 &= \sum_{i,j=1}^{N} \varphi((g g_i)^{-1} (g g_j)) c_i \overline{c_j} \\ &= \sum_{i,j=1}^{N} \varphi(g_i^{-1} g_j) c_i \overline{c_j} = \|f\|^2. \end{aligned}$$

Let $\mathcal{H}^\varphi$ be the Hilbert completion of $\mathcal{H}_0^\varphi$. Since, for $f \in \mathcal{H}^\varphi$, $|f(x)| \leq \|f\|$ (by letting $\nu = \delta_x$), the Hilbert space $\mathcal{H}^\varphi$ can be realized as a subspace of $\mathcal{C}_b(G/K)$, the space of bounded continuous functions in $\mathcal{C}(G/K)$. By definition of $\mathcal{H}_0^\varphi$, the vector $\phi \in \mathcal{H}^\varphi$ is cyclic, K-invariant, and satisfies

$$\varphi(g) = (\varphi | \pi^\varphi(g) \varphi).$$

Let $(\pi^\varphi, \mathcal{H}^\varphi, u^\varphi)$ be the triple obtained via the Gelfand-Naimark-Segal construction, and $(\pi, \mathcal{H}, u)$ be a triple with a K-invariant and cyclic unit vector u in $\mathcal{H}$ such that

$$\varphi(g) = (u|\pi(g)u).$$

Let us define the map $A : \mathcal{H}^\varphi \to \mathcal{H}$, for

$$f(x) = \sum_{i=1}^N c_i \varphi(g_i^{-1}x).$$

by

$$Af = \sum_{i=1}^N c_i \pi(g_i)u.$$

Then

$$\begin{aligned} \|Af\|^2_{\mathcal{H}} &= \sum_{i,j=1}^N c_i \overline{c_j} \big(\pi(g_i)u|\pi(g_j)u\big) \\ &= \sum_{i,j=1}^N c_i \overline{c_j} \varphi(g_i^{-1}g_j) = \|f\|^2, \end{aligned}$$

Since u is cyclic, the range of A: $A(\mathcal{H}^\varphi)$ is dense in $\mathcal{H}$. It follows that A extends as an isometric isomorphism from $\mathcal{H}^\varphi$ onto $\mathcal{H}$. Furthermore

$$A\pi^\varphi(g) = \pi(g)A, \;\; Au^\varphi = u. \qquad \square$$

We will need below the following irreducibility criterium. If $(\pi, \mathcal{H})$ is a unitary representation of G, we will denote by $\mathcal{H}^K$ the subspace of K-fixed vectors in $\mathcal{H}$:

$$\mathcal{H}^K = \{u \in \mathcal{H} \mid \pi(k)u = u, (k \in K)\}.$$

PROPOSITION I.3. — *Let $(\pi, \mathcal{H})$ be a unitary representation of G with a cyclic K-invariant vector u. If* $\dim\mathcal{H}^K = 1$*, then the representation π is irreducible.*

Proof. Let $\mathcal{Y}$ be a closed invariant subspace in $\mathcal{H}$, and P the orthogonal projection onto $\mathcal{Y}$. The vector $v = Pu$ is K-invariant, hence $v = \lambda u$, with $\lambda \in \mathbb{C}$. If $\lambda = 0$, then $v = 0$, and u is orthogonal to $\mathcal{Y}$. Since $\mathcal{Y}$ is invariant, then, for all $x \in G$, $\pi(x)u$ is orthogonal to $\mathcal{Y}$. It follows that $\mathcal{Y} = \{0\}$, since u is cyclic. If $\lambda \neq 0$, then $\mathcal{Y} = \mathcal{H}$ since u is cyclic. $\square$

I.4 Extremality and irreducibility. — As in the previous section G is a topological group, and K a closed subgroup. Let $\mathcal{P}(K\backslash G/K)$ denote the convex cone of continuous functions of positive type on G, K-biinvariant, and $\mathcal{P}_1(K\backslash G/K)$ be the convex set of functions $\varphi \in \mathcal{P}(K\backslash G/K)$ with $\varphi(e) = 1$.

PROPOSITION I.4. — *For $\varphi \in \mathcal{P}_1(K\backslash G/K)$, let $(\pi, \mathcal{H})$ be the unitary representation obtained by the Gelfand-Naimark-Segal construction (Proposition I.2). The following properties are equivalent.*

(i) φ is extremal in the convex set $\mathcal{P}_1(K\backslash G/K)$.

(ii) The unitary representation $(\pi, \mathcal{H})$ is irreducible.

Proof.

(i) $\Rightarrow$ (ii). Assume φ extremal and let $u \in \mathcal{H}$ be a K-invariant cyclic unit vector. Suppose that $\mathcal{H}$ decomposes as the sum $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ of two orthogonal closed invariant subspaces. The vector u decomposes as $u = u_1' + u_2'$, $(u_i' \in \mathcal{H}_i)$. Put $\alpha = \|u_1'\|^2$. Then $0 \leq \alpha \leq 1$ since

$$1 = \|u\|^2 = \|u_1'\|^2 + \|u_2'\|^2.$$

If either $\alpha = 0$ or $\alpha = 1$, then we have either $u_1' = u$ or $u_2' = u$. Since u is cyclic, either $\mathcal{H} = \mathcal{H}_1$ or $\mathcal{H} = \mathcal{H}_2$, which means that $\mathcal{H}$ is irreducible.

Assume now that $0 < \alpha < 1$, and put

$$u_1 = \frac{u_1'}{\sqrt{\alpha}}, \qquad u_2 = \frac{u_2'}{\sqrt{1-\alpha}},$$
$$\varphi_1(g) = (u_1|\pi(g)u_1), \qquad \varphi_2(g) = (u_2|\pi(g)u_2).$$

Then $\varphi = \alpha\varphi_1 + (1-\alpha)\varphi_2$. Since φ is extremal, $\varphi = \varphi_1 = \varphi_2$. Observing that

$$\big(u_i|\pi(g)u_i\big) = \big(u_i|\pi(g)u\big) \quad (i = 1, 2),$$

we get

$$\big(u_1|\pi(g)u\big) = \big(u_2|\pi(g)u\big),$$

and, since u is cyclic, $u_1 = u_2$: a contradiction. We have proven that π is irreducible.

(ii) $\Rightarrow$ (i). Assume π irreducible and that φ is expressed as $\varphi = \alpha\varphi_1 + (1-\alpha)\varphi_2$ for some $\varphi_1, \varphi_2 \in \mathcal{P}_1(K\backslash G/K)$, and $0 < \alpha < 1$. For $f \in \mathcal{H}_0$, expressed as

$$f(x) = \sum_{i=1}^{N} c_i \varphi(g_i^{-1}x),$$

put

$$H(f) = \alpha \sum_{i,j=1}^{N} \varphi_1(g_j^{-1} g_i) c_i \overline{c_j}.$$

This defines an invariant Hermitian form on $\mathcal{H}_0$. Furthermore, since

$$\alpha \sum_{i,j=1}^{N} \varphi_1(g_i^{-1} g_j) c_i \overline{c_j} \leq \sum_{i,j=1}^{N} \varphi(g_i^{-1} g_j) c_i \overline{c_j},$$

we get

$$0 \leq H(f) \leq \|f\|^2,$$

hence H extends as a continuous invariant Hermitian form on $\mathcal{H}$. This form can be written $H(f) = (Af|f)$, where A is a selfadjoint operator on $\mathcal{H}$, $0 \leq A \leq I$, which commutes with the representation π: $A\pi(g) = \pi(g)A$. By Schur's Lemma, $A = \lambda I$, with $0 \leq \lambda \leq 1$. It follows that $\alpha\varphi_1 = \lambda\varphi$. Since $\varphi(e) = \varphi_1(e) = 1$, we get $\lambda = \alpha$, and $\varphi_1 = \varphi$. This means that φ is extremal. $\square$

I.5 Spherical functions and irreducibility. — We assume in this section that (G, K) is a Gelfand pair.

PROPOSITION I.5. — *If the unitary representation* $(\pi, \mathcal{H})$ *is irreducible, then*

$$\dim \mathcal{H}^K \leq 1.$$

Proof. Let P be the orthogonal projection onto $\mathcal{H}^K$. Observe that, for $v \in \mathcal{H}$,

$$Pv = \int_K \pi(k) v \alpha(dk)$$

(α is the normalized Haar measure of K), or $P = \pi(\alpha)$.

Since (G, K) is a Gelfand pair, the algebra $L^1(K\backslash G/K)$ is commutative, and the algebra $M^b(K\backslash G/K)$ of bounded K-biinvariant measures on G is commutative as well. In particular, for $x, y \in G$,

$$\alpha * \delta_x * \alpha * \delta_y * \alpha = \alpha * \delta_y * \alpha * \delta_x * \alpha,$$

and, since $P = \pi(\alpha)$,

$$P\pi(x)P\pi(y)P = P\pi(y)P\pi(x)P.$$

Let $\mathcal{A}$ denote the closed algebra (for the operator norm) generated the the operators $P\pi(x)P$, for $x \in G$. By the equality above, the algebra $\mathcal{A}$ is commutative. The space $\mathcal{H}^K$ is invariant under $\mathcal{A}$. Since an irreducible representation of a commutative Banach algebra is commutative, it is sufficient to prove that $\mathcal{H}^K$ is irreducible under $\mathcal{A}$.

Assume that $\mathcal{H}^K = \mathcal{H}_1 \oplus \mathcal{H}_2$, where $\mathcal{H}_1$ and $\mathcal{H}_2$ are two $\mathcal{A}$-invariant orthogonal subspaces of $\mathcal{H}^K$. Let $u_1 \in \mathcal{H}_1$ $(u_1 \neq 0)$. For any $u_2 \in \mathcal{H}_2$ and $x \in G$, $\big(P\pi(x)Pu_1|u_2\big) = 0$. Since $Pu_1 = u_1$, $Pu_2 = u_2$, this means that $\big(\pi(x)u_1|u_2\big) = 0$. We use now the fact that the representation π is irreducible, and hence that any non zero vector is cyclic. In particular u_1 is cyclic. This implies $u_2 = 0$, and $\mathcal{H}_2 = \{0\}$. ∎

Proposition I.6. — *For $\varphi \in \mathcal{P}_1(K\backslash G/K)$ the following properties are equivalent:*

(i) *The function φ is spherical.*

(ii) *The unitary representation $(\pi, \mathcal{H})$ associated to φ via the Gelfand-Naimark-Segal construction is irreducible.*

Proof. Recall that $\varphi(x) = \big(u \mid \pi(x)u\big)$, where u is a cyclic vector in $\mathcal{H}^K$, and $\|u\| = 1$ since $\varphi(e) = 1$.

(i) $\Rightarrow$ (ii). Assume the function φ to be spherical: for $x, y \in G$,

$$\int_K \big(u|\pi(xky)u\big)\alpha(dk) = \varphi(x)\varphi(y).$$

This can be written

$$\big(\pi(x^{-1})u|P\pi(y)u\big) = \varphi(y)\big(\pi(x^{-1})u|u\big),$$

or, since u is cyclic,

$$P\pi(y)u = \varphi(y)u.$$

It follows that $\mathcal{H}^K = \mathbb{C}u$, and, by Proposition I.3 that the representation $(\pi, \mathcal{H})$ is irreducible.

(ii) $\Rightarrow$ (i). Assume the representation $(\pi, \mathcal{H})$ to be irreducible. By Proposition I.5, $\dim \mathcal{H}^K = 1$, hence $\mathcal{H}^K = \mathbb{C}u$, and, for $v \in \mathcal{H}$,

$$Pv = (v|u)u.$$

Therefore, for $y \in G$,

$$\big(v|P\pi(y)Pu\big) = (v|u)\big(u|\pi(y)u\big) = \varphi(y)(v|u),$$

and, since u is cyclic,

$$P\pi(y)Pu = \varphi(y)u.$$

Taking the inner product of both sides with $\pi(x^{-1})u$, one obtains

$$\big(u|\pi(x)P\pi(y)u\big) = \varphi(x)\varphi(y),$$

or

$$\int_K \varphi(xky)\alpha(dk) = \varphi(x)\varphi(y). \qquad \square$$

I.6 Multiplicity free representations. — Let $(\pi, \mathcal{H})$ be a unitary representation of a compact group U. It decomposes as

$$\pi = \bigoplus_{\lambda \in \hat{U}} m_\lambda \pi_\lambda,$$
$$\mathcal{H} \simeq \bigoplus_{\lambda \in \hat{U}} \mathcal{H}_\lambda^{m_\lambda}.$$

The numbers $m_\lambda \in \mathbb{N}$ are the multiplicities. Let $\mathcal{A}$ denote the commutant of the representation:

$$\mathcal{A} = \{A \in \mathcal{L}(\mathcal{H}) \mid \forall x \in G,\ A\pi(x) = \pi(x)A\}.$$

Then

$$\mathcal{A} \simeq \bigoplus_{\lambda \in \hat{U}} M(m_\lambda, \mathbb{C}).$$

Since, for $m \geq 2$, the algebra $M(m, \mathbb{C})$ is not commutative, the commutant $\mathcal{A}$ is a commutative algebra if and only if, for all $\lambda \in \hat{U}$, $m_\lambda = 0$ or 1. In such a case, one says that the representation π is multiplicity free.

In the general case, this property is taken as a definition: one says that a representation π is *multiplicity free* if the commutant $\mathcal{A}$ of π is commmutative.

Theorem I.7. — *Assume that (G, K) is a Gelfand pair. Let $(\pi, \mathcal{H})$ be a unitary representation of G, with a cyclic K-invariant vector. Then the representation $(\pi, \mathcal{H})$ is multiplicity free.*

Proof.

Let u be a unit cyclic K-invariant vector.

a) We define first a conjugation on $\mathcal{H}^K$. One starts from the observation that, for $f \in L^1(K\backslash G/K)$,

$$\|\pi(f)u\| = \|\pi(f^*)u\|,$$

with

$$f^*(x) = \overline{f(x^{-1})}.$$

In fact, since $\pi(f)$ and $\pi(f^*)$ commute,

$$\|\pi(f)u\|^2 = \big(\pi(f^*)\pi(f)u|u\big) = \big(\pi(f)\pi(f^*)u|u\big) = \|\pi(f^*)u\|^2.$$

Since the set $\{\pi(f)u \mid f \in L^1(K\backslash G/K)\}$ is dense in $\mathcal{H}^K$, the map

$$\pi(f)u \mapsto \pi(f^*)u)$$

extends as an antilinear isometry $v \mapsto \bar{v}$ of $\mathcal{H}^K$. This map is characterized by

$$\forall f \in L^1(K\backslash G/K),\ \big(\bar{v}|\pi(f^*)u\big) = \big(\pi(f)u|v\big).$$

In fact, for $v = \pi(h)u$, $\bar{v} = \pi(h^*)u$,

$$\big(\pi(h^*)u|\pi(f^*)u\big) = \big(\pi(f)\pi(h^*)u|u\big) = \big(\pi(h^*)\pi(f)u|u\big) = \big(\pi(f)u|\pi(h)v\big).$$

b) Let A be a bounded operator on $\mathcal{H}^K$, commuting with the operators $\pi(f)$, for $f \in L^1(K\backslash G/K)$. Let us show that

$$\overline{Au} = A^*u.$$

In fact

$$\big(A^*u|\pi(f^*)u\big) = \big(\pi(f)A^*u|u\big) = \big(A^*\pi(f)u|u\big) = \big(\pi(f)u|Au\big).$$

It follows that, if

$$Au = \lim_{n\to\infty} \pi(f_n)u,$$

for a sequence $f_n \in L^1(K\backslash G/K)$, then

$$A^*u = \lim_{n\to\infty} \pi(f_n^*)u.$$

c) Let $\mathcal{A}_0$ denote the commutant of the representation of $L^1(K\backslash G/K)$ on $\mathcal{H}^K$:

$$\mathcal{A}_0 = \{A \in \mathcal{L}(\mathcal{H}^K) \mid \forall f \in L^1(K\backslash G/K),\ A\pi(f) = \pi(f)A\}.$$

The algebra $\mathcal{A}_0$ is selfadjoint. We will prove that $\mathcal{A}_0$ is commutative. Let $A, B \in \mathcal{A}_0$. Observe that, for $f, h \in L^1(K\backslash G/K)$,

$$\big(AB\pi(f)u|\pi(h)u\big) = \big(B\pi(f)u|A^*\pi(h)u\big).$$

Therefore, to prove the commutativity of $\mathcal{A}_0$ amounts to showing:

$$\big(\pi(f)Bu|\pi(h)A^*u\big) = \big(\pi(f)Au|\pi(h)B^*u\big).$$

There are sequences $\{\phi_p\}$ and $\{\psi_q\}$ in $L^1(K\backslash G/K)$ such that

$$Au = \lim_{p\to\infty} \pi(\phi_p)u,$$

$$Bu = \lim_{q\to\infty} \pi(\psi_q)u,$$

and, by b),

$$A^*u = \lim_{p\to\infty} \pi(\phi_p)u,$$

$$B^*u = \lim_{q\to\infty} \pi(\psi_q)u.cr$$

Hence it is enough to prove that, for $f, h, \phi, \psi \in L^1(K\backslash G/K)$,

$$\big(\pi(f)\pi(\psi)u|\pi(h)\pi(\phi^*)u\big) = \big(\pi(f)\pi(\phi)u|\pi(h)\pi(\psi^*)u\big),$$

which can be checked.

d) If A commutes with the representation π of G, then A commutes with the orthogonal projection $P = \pi(\alpha)$ onto $\mathcal{H}^K$, and hence $A(\mathcal{H}^K) \subset \mathcal{H}^K$. Let us consider the map

$$A \mapsto A_0,\ \mathcal{A} \to \mathcal{A}_0,$$

where A_0 is the restriction of A to $\mathcal{H}^K$. This is an algebra morphism. It is injective. In fact, if $Au = 0$, then, for all $x \in G$,

$$A\pi(x)u = \pi(x)Au = 0,$$

and this implies that $A = 0$ since u is cyclic. This proves that $\mathcal{A}$ is commutative. ∎

Proposition I.8. — *Let G be a locally compact group, and K a compact subgroup. If the quasiregular representation of G on $L^2(G/K)$ is multiplicity free, then (G, K) is a Gelfand pair.*

Proof.

To $f \in L^1(K\backslash G/K)$, one associates the bounded orpertor $R(f)$ on $L^2(G/K)$ by

$$R(f)h = h * f.$$

The operator $R(f)$ commutes withs the quasiregular representation of G on $L^2(G/K)$: $R(f)$ belongs to the commutant $\mathcal{A}$ of the quasiregular representation, and the map $f \mapsto R(f)$ is an injective algebra morphism. Therefore $L^1(K\backslash G/K)$ is commutative: (G, K) is a Gelfand pair. ∎

Conclusion

Let (G, K) be a Gelfand pair. For a function $\varphi \in \mathcal{P}(K\backslash G/K)$, we have seen that the following three properties are equivalent:

- φ is spherical,
- φ is extremal in the convex set $\mathcal{P}_1(K\backslash G/K)$,

- the representation associated to φ by the Gelfand-Naimark-Segal construction is irreducible. (One says that φ is *pure*.)

An irreducible representation $(\pi, \mathcal{H})$ of G with $\dim \mathcal{H}^K = 1$ is said to be *spherical*, and the set Ω of equivalence classes of spherical representations will be called the *spherical dual* of the Gelfand pair (G, K). Equivalently
- Ω is the set of spherical functions of positive type,
- Ω is the set of extremal points in the convex set $\mathcal{P}_1(K\backslash G/K)$.

In numerous cases a set of parameters for Ω is known. That is why we will write $\varphi(\omega; x)$ for a spherical function of positive type, with $x \in G$, $\omega \in \Omega$, thought of as a parameter.

Basic questions in harmonic analysis for a Gelfand pair (G, K) are:
- Determine the spherical dual Ω of (G, K).
- For a spherical function of positive type φ describe a realization $(\pi, \mathcal{H})$ of the spherical representation associated to φ.

Chapter II

BOCHNER-GODEMENT THEOREM

Theorem II.1 (Bochner-Godement). — *Let (G, K) be a Gelfand pair, and Ω its spherical dual. For $\varphi \in \mathcal{P}(K\backslash G/K)$, there exists a unique bounded positive measure on Ω such that*

$$\varphi(x) = \int_\Omega \varphi(\omega; x)\mu(d\omega).$$

We will give a proof of this theorem by using integral representation theory. The idea to use integral representation theory for proving Bochner Theorem can be found in a paper of Cartan and Godement:

H. Cartan, R. Godement (1947). Théorie de la dualité et analyse harmonique dans les groupes abéliens localement compacts, *Ann. Sci. E.N.S.*, **64**, 79–99.

This method is used by van Dijk for Bochner-Godement Theorem:

G. van Dijk (1969). Spherical functions on the p-adic group $PGL(2)$, *Indagationes Math.*, **31**, 213–241.

I.1 Basic theorems about integral representation theory. — Let Q be a convex subset in a locally convex toppological vector space V. A point $p \in Q$ is said to be *extremal* when the folowing holds: if

$$p = \alpha p_1 + (1 - \alpha)p_2,$$

with $p_1, p_2 \in Q$, $0 < \alpha < 1$, then $p = p_1 = p_2$. Let $\mathcal{E}(Q)$ denote the set of extremal points in Q.

Theorem II.2 (Krein-Milman,1940). — *Assume Q compact. Then Q is the convex hull of $\mathcal{E}(Q)$: the finite convex combinations of extremal points are dense in Q.*

It follows that, if furthermore $\mathcal{E}(Q)$ is closed (hence compact), then, for every $p \in Q$, there is a probability measure μ on $\mathcal{E}(Q)$ such that

$$p = \int_{\mathcal{E}(Q)} q\mu(dq).$$

The point p is said to be the *barycentre* of the measure μ. It means that, for every continuous linear form ℓ on V,

$$\ell(p) = \int_{\mathcal{E}(Q)} \ell(q)\mu(dq).$$

Theorem II.3 (Choquet, 1960). — *If Q is compact and metrizable, then $\mathcal{E}(Q)$ is a G_δ (countable intersection of open sets), hence a Borel set. Every $p \in Q$ is the barycentre of a probability measure μ which is concentrated on $\mathcal{E}(Q)$.*

This means that μ is a probability measure on Q such that $\mu\big(Q \setminus \mathcal{E}(Q)\big) = 0$.

Theorem II.4 (Choquet-Meyer, 1963). — *The measure μ is unique if and only if Q is a simplex.*

Define

$$\tilde{Q} = \{(p, 1) \mid p \in Q\} \subset V \times R,$$

and let Γ be the convex cone generated by $\tilde{Q}$. Then Q is a *simplex* ifand only if the ordered set Γ is a lattice, *i.e.*, for any pair $p, q \in \Gamma$, there is a least upper bound $p \vee q$.

II.2 Proof of Bochner-Godement Theorem. — We apply the theorems of Section 1 to the convex set

$$Q = \mathcal{P}_{\leq 1}(K\backslash G/K) = \{\varphi \in \mathcal{P}(K\backslash G/K) \mid \varphi(e) \leq 1\}.$$

We will assume that G is separable. It follows that $L^1(G)$ is separable. By Propositions I.4 and I.6,

$$\mathcal{E}(Q) = \Omega \cup \{0\}.$$

1) We will consider on Q a topology for which Q is compact and metrizable. We will consider Q as a subset of the unit ball in $L^\infty(G)$, and see that it is closed for the $*$-weak topology $\sigma(L^\infty, L^1)$, and hence compact since, for that topology, the unit ball in $L^\infty(G)$ is compact.

To a function $\varphi \in \mathcal{P}(K\backslash G/K)$, we associate the continuous linear form Φ on $L^1(G)$ given by

$$\Phi(f) = \int_G f(x)\varphi(x^{-1})m(dx).$$

Then

$$\Phi(f^* * f) = \int_{G\times G} \varphi(x^{-1}y)f(x)\overline{f(y)}m(dx)m(dy) \geq 0.$$

We will say that a linear form L on $L^1(G)$ is positive if

$$\forall f \in L^1(G),\ L(f^* * f) \geq 0.$$

THEOREM II.5 (GELFAND-RAIKOV). — *Let L be a continuous positive linear form on $L^1(G)$, and K-biinvariant:*

$$\forall f \in L^1(G),\ L(f) = L(f^\natural).$$

Then there is a function $\varphi \in \mathcal{P}(K\backslash G/K)$ such that

$$L(f) = \int f(x)\varphi(x^{-1})m(dx).$$

Furthermore

$$\|L\| = \varphi(e).$$

Proof.

Let B be the positive sesquibilinear form defined on $L^1(G)$ by

$$B(f,g) = L(g^* * f).$$

By the Schwarz inequality,

$$|B(f,g)|^2 \le B(f,f)B(g,g),$$

and furthermore

$$B(f,f) \le \|L\|\|f\|_1^2.$$

We get a preHilbert structure on the quotient $L^1(G)/\mathcal{N}$, with

$$\mathcal{N} = \{f \in L^1(G) \mid B(f,f) = 0\},$$

and, by completion, a Hilbert space $\mathcal{H}$. We get also a unitary representation by letting

$$\big(\pi(x)f\big)(y) = f(x^{-1}y).$$

For every neighborhood of the identy e of G, let h_U be a positive continuous function with integral equal to one, and with support in U. The system $\{h_U\}$ is an identity approximation:

$$\forall f \in L^1(G),\ \lim_{\mathcal{U}} h_U * f = \lim_{\mathcal{U}} f * h_U = f,$$

where $\mathcal{U}$ is filter of the neiborhoods of e. By the Schwarz inequality

$$|L(h_U^* * f)|^2 \le B(f,f)B(h_U,h_U),$$

and, since

$$B(h_U,h_U) \le \|L\|\|h_U\|_1^2 = \|L\|,$$
$$\lim_{\mathcal{U}} L(h_U^* * f) = L(f),$$

it follows that

$$|L(f)|^2 \leq \|L\| B(f,f).$$

By Riesz representation Theorem, there exists $u \in \mathcal{H}$ such that

$$L(f) = (\dot{f}|u),$$

where $\dot{f}$ is the image of f by the map $L^1(G) \to L^1(G)/\mathcal{N} \subset \mathcal{H}$. The vector u is unique and K-invariant. From the equalities

$$\dot{g}|\dot{f}) = B(g,f) = L(f^* * g) = \big(\pi(f^*)\dot{g}|u\big) = \big(\dot{g}|\pi(f)u\big),$$

it follows that $\pi(f)u = \dot{f}$, and this shows that the vector u is cyclic. The function φ defined on G by

$$\varphi(x) = \big(u|\pi(x)u\big)$$

is continous, K-biinvariant, of positive type, and

$$L(f) = \big(\pi(f)u|u\big) = \int_G f(x)\varphi(x^{-1})m(dx).$$

Since $|\varphi(x)| \leq \varphi(e)$,

$$|L(f)| \leq \varphi(e)\|f\|_1.$$

Furthermore

$$\lim_{\mathcal{U}} L(h_U) = \varphi(e),$$

therefore : $|L\| = \varphi(e)$. ∎

Corollary II.6. — *The set $\mathcal{P}_{\leq 1}(K\backslash G/K)$, seen as a part of $L^\infty(G)$, topological dual of $L^1(G)$, is compact for the $*$-weak topology $\sigma(L^\infty, L^1)$.*

Proof. Since the unit ball of $L^\infty(G)$ is compact for the $*$-weak topology, it amounts to showing that $\mathcal{P}_{\leq 1}(K\backslash G/K)$ is closed. Let $\{\varphi_n\}$ be a sequence in $\mathcal{P}_{\leq 1}(K\backslash G/K)$ such that, for every $f \in L^1(G)$, the limit

$$L(f) = \lim_{n\to\infty} \int_G f(x)\varphi_n(x^{-1})m(dx)$$

exists. Then L is a positive continuous linear form on $L^1(G)$ with $\|L\| \leq 1$, and, by Theorem II.5, there is a function $\varphi \in \mathcal{P}(K\backslash G/K)$ such that, for $f \in L^1(G)$,

$$L(f) = \int_G f(x)\varphi(x^{-1})m(dx).$$

This means that φ_n converges to φ for the $*$-weak topology. ∎

Since $L^1(G)$ is separable, the closed unit ball in $L^\infty(G)$ is metrizable for the $*$-weak topology. Hence $\mathcal{P}_{\leq 1}(K\backslash G/K)$ is metrizable.

2) For the uniqueness we will give two proofs. The first one is very simple. The second one, which is more elaborated, has the advantage to rely the uniqueness to the multiplicity freedom, and will be used in a more general setting.

We define first the *spherical Fourier transform* of a function $f \in L^1(K\backslash G/K)$. It is the function $\hat{f}$ defined on Ω by

$$\hat{f}(\omega) = \int_G f(x)\varphi(\omega; x^{-1})m(dx).$$

As the classical Fourier transformation does, the spherical Fourier transformation carries the convolution product onto the ordinary one: for $f, g \in L^1(K\backslash G/K)$,

$$\widehat{f * g}(\omega) = \hat{f}(\omega)\hat{g}(\omega).$$

Observe also that

$$f * \varphi_\omega = \hat{f}(\omega)\varphi_\omega,$$

with $\varphi_\omega(x) = \varphi(\omega; x)$.

a) *1st proof.* We show that, if the measure μ exists, it is unique.

For $f \in L^1(K\backslash G/K)$, by Fubini theorem

$$\int_G f(x)\varphi(x^{-1})m(dx) = \int_\Omega \hat{f}(\omega)\mu(\omega).$$

Uniqueness follows since the space $\{\hat{f} \mid f \in L^1(K\backslash G/K)\}$ is dense in $\mathcal{C}_0(\Omega)$ by Stone-Weierstrass Theorem.

b) *2nd proof.* By Theorem II.3 it amounts to showing that the cone $\mathcal{P}(K\backslash G/K)$ is a lattice.

Let G be a topological group, and K a closed subgroup. For $\varphi \in \mathcal{P}(K\backslash G/K)$, define the face

$$\mathcal{P}_\varphi(K\backslash G/K) = \{\psi \in \mathcal{P}(K\backslash G/K) \mid \exists\lambda > 0,\ \psi \ll \lambda\varphi\}.$$

($\psi \ll \lambda\varphi$ means that $\lambda\varphi - \psi$ is of positive type.)

Theorem II.7. — *The face $\mathcal{P}_\varphi(K\backslash G/K)$ is a lattice if and only if the representation associated to φ by the Gelfand-Naimark-Segal construction is multiplicity free.*

Proof.

We will give only one way of the proof. We will prove that, if the representation $(\pi, \mathcal{H})$ is multiplicity free, then the face $\mathcal{P}_\varphi(K\backslash G/K)$ is a lattice. One observes first that the face $\mathcal{P}_\varphi(K\backslash G/K)$ is linearly isomorphic to the cone

$$\mathcal{A}^+ = \{A \in \mathcal{A} \mid \forall v \in \mathcal{H},\ (Av|v) \geq 0\}.$$

See the second part of the proof of Proposition I.4.

The algebra $\mathcal{A}$ is a commutative C^*- algebra, hence isomorphic to the space $\mathcal{C}(S)$ of continuous functions on the spectrum S of $\mathcal{A}$, which is compact: $\mathcal{A} \simeq \mathcal{C}(S)$. Furthermore the cone $\mathcal{A}^+$ is linearly isomorphic to the cone $\mathcal{C}(S)^+$ of positive functions in $\mathcal{C}(S)$, and $\mathcal{C}(S)^+$ is a lattice. □

II.4 Plancherel-Godement Theorem. — By the Bochner-Godement theorem, for $\varphi \in \mathcal{P}(K\backslash G/K)$, there is a unique measure μ_φ on the spherical dual Ω such that

$$\varphi(x) = \int_\Omega \varphi(\omega; x)\mu_\varphi(d\omega).$$

For $f \in L^1(K\backslash G/K)$,

$$f * \varphi(x) = \int_\Omega \hat{f}(\omega)\varphi(\omega; x)\mu_\varphi(d\omega).$$

Hence, if $\varphi, \psi \in \mathcal{P}(K\backslash G/K) \cap L^1(G)$, then

$$\hat{\varphi}(\omega)\mu_\psi(d\omega) = \hat{\psi}(\omega)\mu_\varphi(d\omega).$$

It follows that there is a measure σ on Ω such that, for $\varphi \in \mathcal{P}(K\backslash G/K) \cap L^1(G)$,

$$\mu_\varphi(d\omega) = \hat{\varphi}(\omega)\sigma(d\omega).$$

The measure σ is called the Plancherel measure.

THEOREM II.8 (PLANCHEREL-GODEMENT). — (i) *If $f \in \mathcal{P}(K\backslash G/K)$ is integrable, then $\hat{f}$ is integrable with respect to the Plancherel measure σ, and*

$$f(x) = \int_\Omega \hat{f}(\omega)\varphi(\omega; x)\sigma(d\omega).$$

(ii) *If $f \in L^1 \cap L^2(K\backslash G/K)$, then $\hat{f}$ is square integrable with respect to the Plancherel measure σ, and*

$$\int_G |f(x)|^2 m(dx) = \int_\Omega |\hat{f}(\omega)|^2 \sigma(d\omega).$$

(iii) *The spherical Fourier transformation, as a map from $L^1 \cap L^2(K\backslash G/K)$ into $L^2(\Omega, \sigma)$, extends as an isometric isomorphism from $L^2(K\backslash G/K)$ onto $L^2(\Omega, \sigma)$.*

Proof.

Statement (i) follows directly from the definition of the Plancherel measure σ.

For $f \in L^1 \cap L^2(K\backslash G/K)$, put $h = f * f^*$:

$$h(x) = \int_G f(xy)\overline{f(y)}m(dy),$$
$$h(e) = \int_G |f(y)|^2 m(dy).$$

The function h is continuous, of positive type, and $\hat{h}(\omega) = |\hat{f}(\omega)|^2$. By (i), with $x = e$,

$$h(e) = \int_\Omega |\hat{f}(\omega)|^2 \sigma(d\omega).$$

This gives (ii).

For (iii) it is sufficient to prove that the space

$$\{\hat{f} \mid f \in L^1 \cap L^2(K\backslash G/K)\}$$

is dense in $L^2(\Omega, \sigma)$. Let F be a continuous function on Ω with compact support. Since the space $\{\hat{f} \mid \in L^1(K\backslash G/K)\}$ is dense in $\mathcal{C}_0(\Omega)$, there exists a continuous function g on G, with compact support and K-biinvariant such that $\hat{g}(\omega) > 0$ on the support of F, and a continuous function F' on Ω with compact support such that

$$\forall \omega \in \Omega,\ F(\omega) = \hat{g}(\omega)F'(\omega).$$

For $\varepsilon > 0$, there exists a continuous function h on G, with compact support and K-biinvariant, such that for all $\omega \in \Omega$

$$|F'(\omega) - \hat{h}(\omega)| < \varepsilon,$$

and

$$|F(\omega) - \widehat{g * h}(\omega)| = |F(\omega) - \hat{g}(\omega)\hat{h}(\omega)| = |\hat{g}(\omega)|\,|F'(\omega) - \hat{h}(\omega)| \leq \varepsilon|\hat{g}(\omega)|.$$

Therefore

$$\int_\Omega |F(\omega) - \widehat{g * h}(\omega)|^2 \sigma(d\omega) \leq \varepsilon^2 \int_\Omega |\hat{g}(\omega)|^2 \sigma(d\omega).$$ ∎

Corollary II.9 (Inversion formula). — *If, for $f \in \mathcal{C}_c(K\backslash G/K)$,*

$$\int_\omega |\hat{f}(\omega)|\sigma(d\omega) < \infty,$$

then, for $x \in G$,

$$f(x) = \int_\Omega \hat{f}(\omega)\varphi(\omega; x)\sigma(d\omega).$$

Proof.

For $h \in \mathcal{C}_c(K\backslash G/K)$, by Theorem II.8,

$$f * h(x) = \int_\Omega \hat{f}(\omega)\hat{h}(\omega)\varphi(\omega; x)\sigma(d\omega).$$

If (h_V) is the approximation of the identity we considered in the proof of Theorem II.5,

$$\lim_{\mathcal{V}} f * h_V(x) = f(x).$$

Furthermore

$$|\widehat{h_V}(\omega)| \leq 1, \quad \lim_{\mathcal{V}} \widehat{h_V}(\omega) = 1.$$

Therefore, by the dominated convergence theorem,

$$\lim_{\mathcal{V}} \int_\Omega \hat{f}(\omega)\widehat{h_V}(\omega)\varphi(\omega; x)\sigma(d\omega) = \int_\Omega \hat{f}(\omega)\varphi(\omega; x)\sigma(d\omega).$$ ∎

II.5 Decomposition of representations into irreducible ones. In this section (G, K) is a Gelfand pair. We will describe the decomposition of a unitary representation $(\pi, \mathcal{H})$ with a K-invariant cyclic vector. as a direct integral of spherical representations.

Let u be a K-invariant cyclic vector, and define:

$$\varphi(x) = \big(u|\pi(x)u\big).$$

Tthe function φ belongs tp $\mathcal{P}(K\backslash G/K)$. By Theorem II.1 φ admits an integral representation:

$$\varphi(x) = \int_\Omega \varphi(\omega; x)\mu(d\omega),$$

where μ is a bounded positive measure on the spherical dual Ω.

Theorem II.10. — *The unitary representation $(\pi, \mathcal{H})$ associated to φ by the Gelfand-Naimark-Segal construction decomposes as*

$$\pi = \int_\Omega^\oplus \pi_\omega \mu(d\omega),$$
$$\mathcal{H} = \int_\Omega^\oplus \mathcal{H}_\omega \mu(d\omega).$$

One can say that μ is the *spectral measure* of the representation $(\pi, \mathcal{H})$.

Proof. We will denote by $\|\cdot\|$ the norm in $\mathcal{H}$, and $\|\cdot\|_\omega$ the norm in $\mathcal{H}_\omega$. For $f \in L^1(G)$, $f * \varphi$ belongs to $\mathcal{H}$, and

$$\|f * \varphi\|^2 = \int_G \varphi(xy) f(x) \overline{f(y)} m(dx) m(dy).$$

Similarly, $f * \varphi_\omega \in \mathcal{H}_\omega$ (with $\varphi_\omega(x) = \varphi(\omega; x)$), and

$$\|f * \varphi_\omega\|_\omega = \int_G \varphi(\omega; xy) f(x) \overline{f(y)} m(dx) m(dy).$$

By using Fubini Theorem, we obtain

$$\|f * \varphi\|^2 = \int_\Omega \|f * \varphi_\omega\|_\omega^2 \mu(d\omega).$$

It follows that the map T_0

$$f * \varphi \mapsto \int_\Omega f * \varphi_\omega \mu(d\omega),$$
$$L^1(G) * \varphi \to \int_\Omega^\oplus \mathcal{H}_\omega \mu(d\omega),$$

extends as an isometry T

$$\mathcal{H} \to \int_\Omega^\oplus \mathcal{H}_\omega \mu(d\omega).$$

It remains to show that the isometry T is surjective, or that the image of the map T_0 is dense. Let (ξ_ω) be a vector field such that, for every $f \in L^1(G)$,

$$\int_\Omega \big(f * \varphi_\omega | \xi_\omega\big) \mu(d\omega) = 0.$$

Replace f by $f * h$ with $h \in L^1(K \backslash G / K)$. Then

$$f * h * \varphi_\omega = \hat{h}(\omega) f * \varphi_\omega,$$

hence

$$\int_\Omega \hat{h}(\omega) \big(f * \varphi_\omega | \xi_\omega\big) \mu(d\omega) = 0.$$

By the density of the space $\{\hat{h} \mid h \in L^1(K\backslash/K)\}$ in $\mathcal{C}_0(\Omega)$, it follows that

$$\big(f * \varphi_\omega | \xi_\omega\big) = 0 \quad \mu \text{ a.e..}$$

Since φ_ω is cyclic in $\mathcal{H}_\omega$, and that $L^1(G)$ is separable, it follows that $\xi_\omega = 0$ μ a.e. ∎

One proves also that an operator A in the commutant of the representation $(\pi, \mathcal{H})$ is diagonal: if $\xi = (\xi_\omega)$ is a vector field, then $A\xi = (\psi(\omega)\xi_\omega)$, with $\psi \in L^\infty(\Omega, \mu)$: $\mathcal{A} \simeq L^\infty(\Omega, \mu)$.

A basic question in harmonic analysis for Gelfand pairs is, given a representation $(\pi, \mathcal{H})$ with a K-invariant cyclic vector, to determine the spectral measure μ on the spherical dual Ω.

II.5 Example. — Let $V = Herm(n, \mathbb{F})$ ($\mathcal{H} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$), $K = U(n, \mathbb{F})$ acting on V by the transformations $x \mapsto kxk^*$, and $G = K \ltimes V$. The spherical dual Ω is the set of K-orbits in V: $\Omega = K\backslash V$. By the classical spectral theorem, every $x \in V$ has real eigenvalues, and can be diagonalized in an orthonormal basis. It follows that $K\backslash V$ can be parametrized by real diagonal matrices

$$\Omega \simeq \mathfrak{S}_n \backslash \mathbb{R}^n.$$

If a is a real diagonal matrix, the corresponding spherical function is the Fourier transform of the normalized invariant orbital measure supported by the orbit kak^*:

$$\varphi(a; x) = \int_K e^{i\,\mathrm{tr}(xkak^*)} \alpha(dk),$$

where α is the normalized Haar measure on K. It is a *generalized Bessel function.*

Let M be a K-invariant bounded positive measure on V, and $\mathcal{H} = L^2(V, M)$. The group G acts on $\mathcal{H}$ by, if $g^{-1}.y = kyk^* + v$,

$$\pi(g)f(y) = e^{i\,\mathrm{tr}(yv)} f(kyk^*) \quad (k \in K,\ v \in V).$$

We obtain a unitary representation. The vector $f_0 \equiv 1$ is K-invariant and cyclic. The associated spherical function φ is given by

$$\varphi(g) = \big(f_0 | \pi(g) f_0\big) = \int_V e^{\mathrm{tr}(xv)} M(dy).$$

It only depends on v: $\varphi(g) = \varphi(v)$, and is the Fourier transform of M.

In the present case the Bochner-Godement theorem says that there is a bounded positive measure μ on $\Omega \simeq \mathfrak{S}_n \backslash \mathbb{R}^n$ such that

$$\varphi(x) = \int_{\mathbb{R}^n} \varphi(a;x)\mu(da).$$

In particular, consider the case

$$M(dy) = q(y)m(dy),$$

where q is a K-invariant positive integrable function on v. Then $q(kak^*) = Q(a)$, where Q is a symmetric function on $\mathbb{R}^n$. By the Weil integration formula

$$\mu(da) = \frac{1}{Z_n} Q(a)|V(a)|^\beta da_1 \dots da_n,$$

where V is the Vandermonde polynomial

$$V(a) = \prod_{i<j}(a_j - a_i),$$

$\beta = \dim_{\mathbb{R}} \mathbb{F} = 1, 2$ or 4, and

$$Z_n = \int_{\mathbb{R}^n} Q(a)|V(a)|^\beta da_1 \dots da_n.$$

The formula for φ can also be written as a desintegration of the measure M. Let M_a be the orbital measure associated to the diagonal measure a:

$$\int_V f(y)M_a(dy) = \int_K f(kak^*)\alpha(dk).$$

Then

$$\int_V f(y)M(dy) = \int_{\mathbb{R}^n} \Big(\int_V f(y)M_a(dy) \Big) \mu(da),$$

Define $\mathcal{H}_a = L^2(V, M_a)$. The unitary representation of G on $\mathcal{H}_a$ is irreducible. It is the spherical representation associated to $a \in \Omega$. In fact by the Gelfand-Naimark-Segal construction, the representation π_a associated to $\varphi(a;\cdot)$ is realized on $\mathcal{F}\big(L^2(V, M_a)\big)$, and the one associated to φ on $\mathcal{F}\big(L^2(V, M)\big)$. By Theorem II.7,

$$\mathcal{H} = \int_{\mathbb{R}^n}^{\oplus} \mathcal{H}_a \; \mu(da).$$

Chapter III

OLSHANSKI SPHERICAL PAIRS

III.1. Definitions. — Let $(G(n),K(n))_{n\geq 1}$ be an increasing sequence of Gelfand pairs: $G(n)$ is a closed subgroup of $G(n+1)$, $K(n)$ of $K(n+1)$, and $K(n)=G(n)\cap K(n+1)$. Define

$$G=\bigcup_{n=1}^{\infty} G(n), \qquad K=\bigcup_{n=1}^{\infty} K(n).$$

We consider on G the inductive limit topology. Then K is a closed subgroup of G. But in general G is not locally compact, and K is not compact. We say that the pair (G,K) is an *Olshanski spherical pair.*

Let us give a simple example of such a sequence $(G(n),K(n))$.

Example

Let $K(n)=O(n)$ be the orthogonal group and let $G(n)=O(n)\ltimes \mathbb{R}^n$ the motion group. The product in $G(n)$ is given by:

$$(g_1,\xi_1)\cdot(g_2,\xi_2)=(g_1g_2,\xi_1+g_1\xi_2),$$

$(g_1,g_2\in O(n),\ \xi_1,\xi_2\in\mathbb{R}^n)$. Then

$$K=O(\infty)=\bigcup_{n=1}^{\infty} O(n),$$

the *infinite dimensional orthogonal group.* An element $k=(k_{ij})_{i,j\geq 1}$ in $O(\infty)$ satisfies $k_{ij}=\delta_{ij}$ for i and j large enough. Define $\mathbb{R}^{(\infty)}$ by

$$\mathbb{R}^{(\infty)}=\bigcup_{n=1}^{\infty}\mathbb{R}^n.$$

A vector $\xi\in\mathbb{R}^{(\infty)}$ is a sequence $\xi=(\xi_1,\xi_2,\ldots)$ of real numbers with $\xi_i=0$ for i large enough.

The group $O(\infty)$ naturally acts on $\mathbb{R}^{(\infty)}$, and $G=O(\infty)\ltimes\mathbb{R}^{(\infty)}$.

Let (G,K) be an Olshanski spherical pair, inductive limit of an increasing sequence of Gelfand pairs $\big(G(n),K(n)\big)$. A K-biinvariant continuous function φ on G is said to be *spherical* if, for $x,y\in G$,

$$\lim_{n\to\infty}\int_{K(n)}\varphi(xky)\alpha_n(dk)=\varphi(x)\varphi(y),$$

where α_n is the normalized Haar measure on the compact group $K(n)$.

III.2 Spherical functions and irreducibility. — Let G be a topological group, and $(K(n))_{n\geq 1}$ an increasing sequence of compact subgroups of G. Put $K=\bigcup_{n=1}^{\infty} K(n)$. For a unitary representation $(\pi,\mathcal{H})$ of G, the orthogonal projection P_n onto the space $\mathcal{H}^{K(n)}$ of $K(n)$-invariant vectors is given by

$$P_n v=\int_{K(n)} \pi(k)v\alpha_n(dk) \quad (v\in\mathcal{H}),$$

where α_n is the normalized Haar measure of $K(n)$. The sequence of the subspaces $\mathcal{H}^{K(n)}$ is decreasing, and the projections P_n strongly converge to the projection P onto

$$\mathcal{H}^K=\bigcap_{n=1}^{\infty}\mathcal{H}^{K(n)}.$$

It follows that, if $\mathcal{Y}\subset\mathcal{H}$ is an invariant closed subspace, then $P(\mathcal{Y})\subset\mathcal{Y}$.

Proposition III.1. — *Let $(\pi,\mathcal{H})$ be a unitary representation of G with a K-invariant cyclic vector $u\in\mathcal{H}$. If $\dim\mathcal{H}^K=1$, then π is irreducible.*

Proof.
Let $\mathcal{Y}$ be a closed G-invariant subspace of $\mathcal{H}$. We will show that either $\mathcal{Y}=\{0\}$ or $\mathcal{Y}=\mathcal{H}$. If $P(\mathcal{Y})=\{0\}$, then $\mathcal{Y}$ is orthogonal to $u\in\mathcal{H}^K$. Since u is cyclic, it follows that $\mathcal{Y}=\{0\}$. If $P(\mathcal{Y})\neq\{0\}$, then $\mathcal{H}^K\subset\mathcal{Y}$, and $\mathcal{Y}=\mathcal{H}$ since $u\in\mathcal{H}^K$ is cyclic. Thus, we have proven that the representation π is irreducible. ∎

We assume now that (G,K) is an Olshanski spherical pair, inductive limit of an increasing sequence $\big(G(n),K(n)\big)$ of Gelfand pairs.

Proposition III.2. — *Let (G,K) be an Olshanski spherical pair. For any irreducible unitary representation $(\pi,\mathcal{H})$ of G,*

$$\dim\mathcal{H}^K\leq 1.$$

Proof
Assume $\mathcal{H}^K\neq\{0\}$. Since $\big(G(n),K(n)\big)$ is a Gelfand pair, the convolution algebra $L^1\big(K(n)\backslash G(n)/K(n)\big)$ is commutative, and the algebra $M^b\big(K(n)\backslash G(n)/K(n)\big)$ of K-biinvariant bounded measures is commutative as well. In particular, for $x,y\in G(n)$,

$$\alpha_n*\delta_x*\alpha_n*\delta_y*\alpha_n=\alpha_n*\delta_y*\alpha_n*\delta_x*\alpha_n,$$

and, since $P_n = \pi(\alpha_n)$,

$$P_n\pi(x)P_n\pi(y)P_n = P_n\pi(y)P_n\pi(x)P_n.$$

Observing that $P_{n+1} = P_nP_{n+1} = P_{n+1}P_n$, we obtain, for $m, m' \geq 0$,

$$P_{n+m}\pi(x)P\pi(y)P_{n+m'} = P_{n+m}\pi(y)P\pi(x)P_{n+m'}.$$

As $m, m' \to \infty$, and then $n \to \infty$, we obtain

$$P\pi(x)P\pi(y)P = P\pi(y)P\pi(x)P,$$

since P_n strongly converges to P.

Let $\mathcal{A}$ be the closed algebra (for the operator norm) generated by the operators $P\pi(x)P$, for $x \in G$. As proven above, the algebra $\mathcal{A}$ is commutative. The space $\mathcal{H}^K$ is invariant under $\mathcal{A}$. Since an irreducible representation of a commutative Banach algebra is one dimensional, it is sufficient to prove that $\mathcal{H}^K$ is irreducible under $\mathcal{A}$.

Assume that $\mathcal{H}^K = \mathcal{H}_1 \oplus \mathcal{H}_2$, where $\mathcal{H}_1$ and $\mathcal{H}_2$ are two $\mathcal{A}$-invariant orthogonal subspaces of $\mathcal{H}^K$. Let $u_1 \in \mathcal{H}_1$ ($u_1 \neq 0$). For any $u_2 \in \mathcal{H}_2$ and $x \in G$, $\big(P\pi(x)Pu_1|u_2\big) = 0$. Since $Pu_1 = u_1$, $Pu_2 = u_2$, this means that $(\pi(x)u_1|u_2) = 0$. We use now the fact that the representation π is irreducible, and hence that any non zero vector is cyclic, in particular u_1 is cyclic. This implies $u_2 = 0$, and $\mathcal{H}_2 = \{0\}$. ∎

Theorem III.3. — *Let (G, K) be an Olshanski spherical pair. For $\varphi \in \mathcal{P}_1(K\backslash G/K)$, the following properties are equivalent:*

(i) *φ is spherical.*

(ii) *The representation $(\pi, \mathcal{H})$ associated to φ by the Gelfand-Naimark-Segal construction is irreducible.*

Proof.

Recall that $\varphi(g) = \big(u|\pi(g)u\big)$, where u is a cyclic unit vector in $\mathcal{H}^K$.

(ii) $\Rightarrow$ (i). Assume the representation $(\pi, \mathcal{H})$ irreducible. By Proposition III.2 we know that $\dim \mathcal{H}^K = 1$. Therefore the orthogonal projection P onto $\mathcal{H}^K$ can be written

$$Pv = (v|u)u.$$

For $y \in G$, and any $v \in \mathcal{H}$,

$$\big(v|P\pi(y)Pu\big) = \big(Pv|\pi(y)u\big) = (v|u)\big(u|\pi(y)u\big) = \varphi(y)(v|u).$$

Therefore $P\pi(y)Pu = \phi(y)u$. Hence, for $x \in G$,

$$P\pi(x)P\pi(y)Pu = \varphi(y)P\pi(x)Pu = \varphi(x)\varphi(y)u,$$

and

$$\big(u|\pi(x)P\pi(y)u\big) = \varphi(x)\varphi(y).$$

Since the projections P_n strongly converge to P, we get

$$\begin{aligned}\varphi(x)\varphi(y) &= \lim_{n\to\infty}\big(u|\pi(x)P_n\pi(y)u\big)\\ &= \lim_{n\to\infty}\int_{K(n)}(u|\pi(xky)u)\alpha_n(dk)\\ &= \lim_{n\to\infty}\int_{K(n)}\varphi(xky)\alpha_n(dk),\end{aligned}$$

which means that φ is spherical.

(i) $\Rightarrow$ (ii). Assume φ spherical. We will show that, for $g \in G$, $P\pi(g)u = \varphi(g)u$. If this holds, then the subspace $\mathcal{H}^K$ is one dimensional: $\mathcal{H}^K = \mathbb{C}u$. Therefore, by Proposition III.1, the representation π is irreducible. By assumption, for $x, y \in G$,

$$\begin{aligned}\varphi(x)\varphi(y) &= \lim_{n\to\infty}\int_{K(n)}\varphi(xky)\alpha_n(dk)\\ &= \lim_{n\to\infty}\big(u|\pi(x)P_n\pi(y)u\big) = \big(u|\pi(x)P\pi(y)u\big).\end{aligned}$$

This can be written as

$$\big(\pi(x^{-1})u|P\pi(y)u\big) = \varphi(y)\big(\pi(x^{-1})u|u\big).$$

Since u is cyclic, we obtain $P\pi(y)u = \varphi(y)u$. $\square$

III.3 Examples. — Let us give two simple examples of Olshanski spherical pairs.

Example 1

We come back to the example of Section 1.2. Let $G(n) = O(n) \ltimes \mathbb{R}^n$ and $K(n) = O(n)$. Then $G = O(\infty) \ltimes \mathbb{R}^{(\infty)}$ and $K = O(\infty)$. For an element $x = (g, \xi) \in G$, we denote by $\|x\|$ the radius of ξ: $\|x\| = \|\xi\| = \sqrt{\xi_1^2 + \xi_2^2 + \cdots}$ for $\xi = (\xi_1, \xi_2, \ldots, 0, 0, \ldots) \in \mathbb{R}^{(\infty)}$. Let φ be a K-biinvariant function of G. Then, for any $g_1, g_2, g \in O(\infty)$ and $\xi \in \mathbb{R}^{(\infty)}$,

$$\varphi(g, \xi) = \varphi((g_1, 0)\cdot(g, \xi)\cdot(g_2, 0)) = \varphi(g_1gg_2, g_1\xi).$$

Therefore $\phi(g, \xi)$ only depends on the radius $\|\xi\|$, i.e., there exists a function Φ on $\mathbb{R}_{\geq 0}$ such that

$$\varphi(x) = \Phi(\|x\|^2).$$

Assume ϕ spherical:

$$\lim_{n\to\infty} \int_{K(n)} \varphi(xky)\alpha_n(dk) = \varphi(x)\varphi(y), \qquad x, y \in G.$$

By classical harmonic analysis,

$$\int_{K(n)} \varphi(xky)\alpha_n(dk) = c_n \int_0^{\pi} \Phi(a^2 + b^2 + 2abcos\theta) \sin^{n-1}\theta d\theta.$$

where $a = \|x\|$ and $b = \|y\|$. Note that the constant c_n is given by

$$c_n = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{n}{2}\right)}.$$

One shows easily that, if f is a continuous function on $[0, \pi]$,

$$\lim_{n\to\infty} c_n \int_0^{\pi} f(\theta) \sin^{n-1}\theta d\theta = f\left(\frac{\pi}{2}\right).$$

Therefore we obtain the following functional equation

$$\Phi(a^2 + b^2) = \Phi(a^2)\Phi(b^2).$$

This equation implies that $\Phi(a) = e^{-\lambda a}$ for some $\lambda \in \mathbb{C}$. Hence the spherical functions ϕ for the Olshanski spherical pair (G, K) are given by

$$\phi(x) = e^{-\lambda\|x\|^2}, \qquad \lambda \in \mathbb{C}.$$

Furthermore φ is of positive type if and only if $\lambda \geq 0$. Thus, the spherical dual Ω can be identifed with $[0, \infty[$. Observe that a spherical function, which is essentially a function on $\mathbb{R}^{(\infty)}$, extends as a continuous function on

$$\ell^2(\mathbb{N}) = \left\{ (\xi_1, \xi_2, \ldots) \;\middle|\; \xi_k \in \mathbb{R}, \sum_{k\geq 1} \xi_k^2 < \infty \right\}.$$

Example 2

Let $G(n) = O(n+1)$ and $K(n) = O(n)$. Here $K(n)$ is seen as a subgroup of $G(n)$ as follows:

$$O(n) \ni u \mapsto \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \in O(n+1).$$

Let $\{e_0, e_1, \ldots, e_{n+1}\}$ be the canonical basis of $\mathbb{R}^{n+1}$. A K-biinvariant continuous function φ on G can be written as

$$\varphi(g) = \Phi((ge_0|e_0)),$$

where Φ is a continuous function on $[-1,1]$. We get

$$\int_{K(n)} \varphi(xky)\alpha_n(dk) = c_n \int_0^\pi \Phi(\cos a \cos b + \sin a \sin b \cos\theta) \sin^{n-1}\theta d\theta$$

where $\cos a = (xe_0|e_0)$ and $\cos b = (ye_0|e_0)$ (c_n is the same constant as in Example 1). If φ is spherical, then

$$\Phi(\cos a \cos b) = \Phi(\cos a)\Phi(\cos b).$$

Finally, the spherical functions φ of the spherical pair (G,K) are the following:

$$\varphi(g) = (ge_0|e_0)^m, \qquad (m \in \mathbb{N}).$$

They are of positive type Thus the spherical dual Ω can be identified with $\mathbb{N}$.

III.4 Harmonic analysis on Olshanski spherical pairs, more or less recent results. — Given an Olshanski spherical apir (G,K), one of the first tasks is to determine the spherical dual Ω.

For

$$G = U(\infty) \ltimes Herm(\infty, \mathbb{C}), \quad K = U(\infty),$$

the spherical dual has been determined by Pickrell, Olshanski and Vershik. It is remarkable that the problem is related to the classical notion of total positivity.

D. Pickrel (1991). Mackey analysis of infinite classical motion groups, *Pacific J. Math.*, **150**, 139–166.

G. Olshanski, A. Vershik (1996). Ergodic unitarily invariant measures on the space of infinite Hermitian matrices, in *Comtemporary Mathematical Physics* (eds. R.L. Dobroshin, R.A. Minlos, M.A. Shubin, A.M. Vershik), *Amer. Math. Soc. Translations*, **175**, 137–175.

For the infinite symmetric group,

$$G = U(\infty) \times U(\infty), \quad K = U(\infty),$$

the spherical dual has been determined by Voiculescu, completed by Boyer, Kerov and Vershik. In that case also, the problem is related to total positivity.

D. Voiculescu (1976). Représentations factorielles de type II_1 de $U(\infty)$, *J. Math. Pures Appl.*, **55**, 1–20.

A. Vershik, S. Kerov (1982). Characters and factor representations of the infinite unitary group, *Soviet Math. Dokl.*, **26**, 570–574.

R.P. Boyer (1983). Infinite traces of AF-algebras and characters of $U(\infty)$, *J. Operator Theory*, **9**, 205-236.

For

$$G = \mathfrak{S}_\infty \times \mathfrak{S}_\infty, \quad K = \mathfrak{S}_\infty,$$

see

E. Thoma (1964). Die unzerlegbaren, positiv-definiten Klassenfunktionen der abzählbaar unendlichen, symmetrischen Gruppe, *Math. Z.*, **85**, 40–61.

A. Vershik, S. Kerov (1981). Asymptotic theory of characters of a symmetric group, *Funct. Anal. Appl.*, **15**, 246–255.

S. Kerov (2003). Asymptotic Representation Theory of the Symmetric Group and its Applications in Analysis. *Translations of Mathematical Monographs, A.M.S.*.

See also

T. Hirai, E. Hirai (2005). Positive definite class functions on a topological group and characters of factor representations, *J. Math. Kyoto Univ.*, **45**, 355–379.

T. Hirai, E. Hirai (2005). Characters of wreath products of finite groups with the infinite symmetric group, *J. Math. Kyoto Univ.*, **45**, 547–597.

A natural question arises: is it possible to obtain the spherical functions for the Olshanski spherical pair (G, K) as limits of spherical functions of $\big(G(n), K(n)\big)$? So far I know, there is no general answer. Important special cases have been investigated by Olshanski and Vershik, by Okunkov and Olshanski. We will present their results in next talks.

G. Olshanski, A. Vershik (1996). Ergodic unitarily invariant measures on the space of infinite Hermitian matrices, in *Comtemporary Mathematical Physics* (eds. R.L. Dobroshin, R.A. Minlos, M.A. Shubin, A.M. Vershik), *Amer. Math. Soc. Translations*, **175**, 137–175.

A. Okunkov, G. Olshanski (1998). Asymptotics of Jack polynomials as the number of variables goes to infinity, *Internat. Math. Res. Notices*, **13**, 641–682.

A. Okunkov, G. Olshanski (2006). Limits of BC-type orthogonal polynomials as the number of variables goes to infinity, *Contemporary Mathematics*, **417**, 281–318.

See also

Bouali, M. (2007). Application des théorèmes de Minlos et Poincaré à l'étude asymptotique d'une intégrale orbitale, *Ann. Fac. Sci. Toulouse, Math (6)*, **16**, 49-70.

Consider now, for an Olshanski spherical pair (G, K), a unitary representation $(\pi, \mathcal{H})$ with a cyclic K-invariant vector. The problem is to determine the spectrral measure μ. This problem has been investigated by Borodin and Olshanski.

In the case of

$$G = U(\infty) \ltimes Herm(\infty, \mathbb{C}), \quad K = U(\infty),$$

such a representation can be realized in the space $L^2(H_\infty, M)$, where M is a bounded positive measure on the space H_∞ of infinite dimensional Hermitian matrices. Borodin and Olshanski solved the problem in the case of the Hua-Pickrell measure. It amounts to describing a probability measure on the space of point configurations on the real line.

Their work is closely related to Random Matrix Theory, where one uses the analysis of orthogonal polynomials.

A. Borodin, G. Olshanski (2001). Infinite random matrices and ergodic measures, *Comm. Math. Phys.*, **223**, 87–123.

In the case of

$$G = U(\infty) \times U(\infty), \quad K = U(\infty),$$

Borodin and Olshanski consider isometric linear maps

$$L^2\big(G(n)/K(n)\big) \to L^2\big(G(n+1)/K(n+1)\big),$$

and the inductive limit

$$\mathcal{H} = \overline{L^2(G/K)} = \lim_{\to} L^2\big(G(n)/K(n)\big).$$

Here also their work is related to the methods used in Random Matrix Theory.

G. Olshanski (2003). The problem of harmonic analysis on the infinite-dimensional unitary group, *J. Funct. Anal.*, **205**, 464–524.

A. Borodin, G. Olshanski (2005). Harmonic analysis on the infinite-dimensional unitary group and determinantal point processes, *Annals of Math.*, **161**, 1319–1422.

A different construction of $L^2(G/K)$ as an inductive limit has been investigated by J.W. Wolf for inductive limits of compact Gelfand pairs.

Wolf, J.A. (2007). Infinite dimensional multiplicity free spaces I: Limits of compact commutative spaces. *To appear.*

An analogue of the Bochner-Godement theorem has been established in several special cases:
- by Thoma, for the infinite symmetric group $\mathfrak{S}_\infty$,
- by Voiculescu, for $U(\infty)$,
- by Olshanski, for the space $Herm(\infty, \mathbb{C})$.

Recently such a theorem has been established by Rabaoui in the case of a general Olshanski spherical pair.

Rabaoui, M. (2008). A Bochner type theorem for inductive limits of Gelfand pairs, *Ann. Inst. Fourier*, **58**, –.

Finally, I mention three introductory papers:

J. Faraut (2006). Infinite dimensional harmonic analysis and probability, in *Probability measures on groups: recent directions and trends,* (eds. S.G. Dani and P. Graczyk), *Tata Institute of Fundamental Research, Narosa,New Dehli*, –, 179–254.

J. Faraut (2008). Infinite dimensional spherical analysis. *COE Lecture notes, Kyushu University.*

J. Faraut (2008). Asymptotics of spherical functions for large rank, an introduction. *Preprint.*