ON THE GEOMETRY OF THE MODULI SPACES OF SEMI-STABLE SHEAVES SUPPORTED ON PLANE QUARTICS

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Abstract. We decompose each moduli space of semi-stable sheaves on the complex projective plane with support of dimension one and degree four into locally closed subvarieties, each subvariety being the good or geometric quotient of a set of morphisms of locally free sheaves modulo a reductive or a non-reductive group. We find locally free resolutions of length one for all these sheaves and describe them.

Contents

1. Introduction

Let V be a three-dimensional vector space over $\mathbb{C}$, and $\mathbb{P}^2 = \mathbb{P}(V)$ the projective plane of lines in V. Let $\mathrm{M}_{\mathbb{P}^2}(r,\chi)$ denote the moduli space of semi-stable sheaves $\mathcal{F}$ on $\mathbb{P}^2$ with Hilbert polynomial $P_{\mathcal{F}}(t) = rt + \chi$. The positive integer r is the multiplicity of $\mathcal{F}$ while χ is its Euler characteristic. The generic stable sheaves in this moduli space are the line bundles of Euler characteristic χ on smooth plane curves of degree r. The map sending $\mathcal{F}$ to the twisted sheaf $\mathcal{F}(1)$ gives an isomorphism between $\mathrm{M}_{\mathbb{P}^2}(r,\chi)$ and $\mathrm{M}_{\mathbb{P}^2}(r, r+\chi)$, so we can restrict our attention to the case $0 < \chi \le r$. It is known from [15] that the spaces $\mathrm{M}_{\mathbb{P}^2}(r,\chi)$ are projective, irreducible, locally factorial, of dimension r^2+1, and smooth at all points given by stable sheaves.

It is easy to see that $\mathrm{M}_{\mathbb{P}^2}(2,1)$ is isomorphic to the space of conic curves in $\mathbb{P}^2$ while $\mathrm{M}_{\mathbb{P}^2}(2,2)$ is the good quotient modulo the action by conjugation of the group $\mathrm{GL}(2,\mathbb{C}) \times \mathrm{GL}(2,\mathbb{C})$ on the space of 2×2-matrices with entries in V^* and non-zero determinant.

The case of multiplicity three is also well-understood. J. Le Potier showed in [15] that $\mathrm{M}_{\mathbb{P}^2}(3,2)$ and $\mathrm{M}_{\mathbb{P}^2}(3,1)$ are both isomorphic to the universal cubic in $\mathbb{P}(V) \times \mathbb{P}(S^3V^*)$.

It was first noticed in [16] that $\mathrm{M}_{\mathbb{P}^2}(3,2)$ is the geometric quotient of the set of injective morphisms $\varphi : \mathcal{O}(-2) \oplus \mathcal{O}(-1) \to 2\mathcal{O}$ for which φ_{12} and φ_{22} are linearly independent regarded as elements of V^*, modulo the action by conjugation of the non-reductive algebraic group $\mathrm{Aut}(\mathcal{O}(-2) \oplus \mathcal{O}(-1)) \times \mathrm{Aut}(2\mathcal{O})$. The corresponding result for $\mathrm{M}_{\mathbb{P}^2}(3,1)$ was established in [8] and [9].

According to [15] $\mathrm{M}_{\mathbb{P}^2}(3,3)$ contains an open dense subset which is a good quotient of the space of injective morphisms $3\mathcal{O}(-1) \to 3\mathcal{O}$ modulo the action by conjugation of $\mathrm{GL}(3,\mathbb{C}) \times \mathrm{GL}(3,\mathbb{C})$. The complement of this set is isomorphic to the space of cubic curves $\mathbb{P}(S^3V^*)$. We summarize in the following table the known facts about $\mathrm{M}_{\mathbb{P}^2}(r,\chi)$, $r = 1, 2, 3$:

$\mathrm{M}_{\mathbb{P}^2}(1,1)$	
$h^0(\mathcal{F}(-1)) = 0$ $h^1(\mathcal{F}) = 0$ $h^0(\mathcal{F} \otimes \Omega^1(1)) = 1$	$0 \longrightarrow \mathcal{O}(-1) \longrightarrow \mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0$

$\mathrm{M}_{\mathbb{P}^2}(2,1)$	
$h^0(\mathcal{F}(-1)) = 0$ $h^1(\mathcal{F}) = 0$ $h^0(\mathcal{F} \otimes \Omega^1(1)) = 0$	$0 \longrightarrow \mathcal{O}(-2) \longrightarrow \mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0$

$\mathrm{M}_{\mathbb{P}^2}(2,2)$	
$h^0(\mathcal{F}(-1)) = 0$ $h^1(\mathcal{F}) = 0$ $h^0(\mathcal{F} \otimes \Omega^1(1)) = 2$	$0 \longrightarrow 2\mathcal{O}(-1) \longrightarrow 2\mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0$

$\mathrm{M}_{\mathbb{P}^2}(3,1)$	
$h^0(\mathcal{F}(-1)) = 0$ $h^1(\mathcal{F}) = 0$ $h^0(\mathcal{F} \otimes \Omega^1(1)) = 0$	$0 \longrightarrow 2\mathcal{O}(-2) \longrightarrow \mathcal{O}(-1) \oplus \mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0$

$\mathrm{M}_{\mathbb{P}^2}(3,2)$	
$h^0(\mathcal{F}(-1)) = 0$ $h^1(\mathcal{F}) = 0$ $h^0(\mathcal{F} \otimes \Omega^1(1)) = 1$	$0 \longrightarrow \mathcal{O}(-2) \oplus \mathcal{O}(-1) \longrightarrow 2\mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0$

$\mathrm{M}_{\mathbb{P}^2}(3,3)$	
$h^0(\mathcal{F}(-1)) = 0$ $h^1(\mathcal{F}) = 0$ $h^0(\mathcal{F} \otimes \Omega^1(1)) = 3$	$0 \longrightarrow 3\mathcal{O}(-1) \longrightarrow 3\mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0$
$h^0(\mathcal{F}(-1)) = 1$ $h^1(\mathcal{F}) = 0$ $h^0(\mathcal{F} \otimes \Omega^1(1)) = 3$	$0 \longrightarrow \mathcal{O}(-2) \longrightarrow \mathcal{O}(1) \longrightarrow \mathcal{F} \longrightarrow 0$

For each moduli space (except $\mathrm{M}_{\mathbb{P}^2}(3,3)$) the left column indicates the cohomological conditions verified by the corresponding sheaves. These sheaves are isomorphic to cokernels of morphisms of locally free sheaves described in the right column. The moduli space is isomorphic to the good quotient, modulo the action of the appropriate group, of a certain open subset of the set of these morphisms. The moduli space $\mathrm{M}_{\mathbb{P}^2}(3,3)$ is the disjoint union of a dense open subset described in the first line, and of a closed subset described in the second line.

In this paper we will study the spaces $\mathrm{M}_{\mathbb{P}^2}(4,\chi)$ for $1 \leq \chi \leq 4$. We will decompose each moduli space into locally closed subvarieties (which we call *strata*) given by cohomological conditions and we will describe these strata as good or geometric quotients of spaces of morphisms. The work of finding resolutions for sheaves $\mathcal{F}$ in $\mathrm{M}_{\mathbb{P}^2}(4,\chi)$, apart from the case $\chi = 4$, $h^0(\mathcal{F}(-1)) = 1$, has already been carried out in [17] and is summarized in the next table. Each stratum in $\mathrm{M}_{\mathbb{P}^2}(4,\chi)$ described by the cohomological conditions from the left column of the table below is isomorphic to the good quotient, modulo the action of the appropriate group, of a certain open subset of the set of morphisms of locally free sheaves from the middle column. The right column gives the codimension of the stratum.

$\mathrm{M}_{\mathbb{P}^2}(4,1)$		
$h^0(\mathcal{F}(-1)) = 0$ $h^1(\mathcal{F}) = 0$ $h^0(\mathcal{F} \otimes \Omega^1(1)) = 0$	$0 \to 3\mathcal{O}(-2) \to 2\mathcal{O}(-1) \oplus \mathcal{O} \to \mathcal{F} \to 0$	0
$h^0(\mathcal{F}(-1)) = 0$ $h^1(\mathcal{F}) = 1$ $h^0(\mathcal{F} \otimes \Omega^1(1)) = 1$	$0 \to \mathcal{O}(-3) \oplus \mathcal{O}(-1) \to 2\mathcal{O} \to \mathcal{F} \to 0$	2

$\mathrm{M}_{\mathbb{P}^2}(4,2)$		
$h^0(\mathcal{F}(-1)) = 0$ $h^1(\mathcal{F}) = 0$ $h^0(\mathcal{F} \otimes \Omega^1(1)) = 0$	$0 \to 2\mathcal{O}(-2) \to 2\mathcal{O} \to \mathcal{F} \to 0$	0
$h^0(\mathcal{F}(-1)) = 0$ $h^1(\mathcal{F}) = 0$ $h^0(\mathcal{F} \otimes \Omega^1(1)) = 1$	$0 \to 2\mathcal{O}(-2) \oplus \mathcal{O}(-1) \to \mathcal{O}(-1) \oplus 2\mathcal{O} \to \mathcal{F} \to 0$	1
$h^0(\mathcal{F}(-1)) = 1$ $h^1(\mathcal{F}) = 1$ $h^0(\mathcal{F} \otimes \Omega^1(1)) = 3$	$0 \to \mathcal{O}(-3) \to \mathcal{O}(1) \to \mathcal{F} \to 0$	3

$\mathrm{M}_{\mathbb{P}^2}(4,3)$		
$h^0(\mathcal{F}(-1)) = 0$ $h^1(\mathcal{F}) = 0$ $h^0(\mathcal{F} \otimes \Omega^1(1)) = 2$	$0 \to \mathcal{O}(-2) \oplus 2\mathcal{O}(-1) \to 3\mathcal{O} \to \mathcal{F} \to 0$	0
$h^0(\mathcal{F}(-1)) = 1$ $h^1(\mathcal{F}) = 0$ $h^0(\mathcal{F} \otimes \Omega^1(1)) = 3$	$0 \to 2\mathcal{O}(-2) \to \mathcal{O}(-1) \oplus \mathcal{O}(1) \to \mathcal{F} \to 0$	2

$\mathrm{M}_{\mathbb{P}^2}(4,4)$		
$h^0(\mathcal{F}(-1))=0$ $h^1(\mathcal{F})=0$ $h^0(\mathcal{F}\otimes\Omega^1(1))=4$	$0\to 4\mathcal{O}(-1)\to 4\mathcal{O}\to\mathcal{F}\to 0$	0
$h^0(\mathcal{F}(-1))=1$ $h^1(\mathcal{F})=0$ $h^0(\mathcal{F}\otimes\Omega^1(1))=4$	$0\to\mathcal{O}(-2)\oplus\mathcal{O}(-1)\to\mathcal{O}\oplus\mathcal{O}(1)\to\mathcal{F}\to 0$	1

One of our difficulties will be to show that the natural maps from the spaces of morphisms of sheaves to the strata are good or geometric quotient maps. At 4.2.4 and 4.3.3 the difficulty is compounded by the fact that the group is non-reductive and we do not know a priori the existence of a good quotient. A similar situation was considered by the first author in [1], theorem D. Our method, exhibited in the proof of 3.1.6, is reminiscent of [1], in that we use an already established quotient modulo a reductive group, but different, in that we do not have a diagram as at 5.4 in [1], but only local morphisms. These local morphisms are constructed using the relative Beilinson spectral sequence.

The second section of this paper contains some tools that are used in the next sections to study the moduli spaces. The sections 3,4,5 are devoted to $\mathrm{M}_{\mathbb{P}^2}(4,1)$ and $\mathrm{M}_{\mathbb{P}^2}(4,3)$, $\mathrm{M}_{\mathbb{P}^2}(4,2)$, $\mathrm{M}_{\mathbb{P}^2}(4,4)$ respectively. Next we summarize the descriptions and properties of the moduli spaces $\mathrm{M}_{\mathbb{P}^2}(4,\chi)$ that are contained in this paper:

1.1. The moduli spaces $\mathrm{M}_{\mathbb{P}^2}(4,1)$ and $\mathrm{M}_{\mathbb{P}^2}(4,3)$

These moduli spaces are isomorphic: $\mathrm{M}_{\mathbb{P}^2}(4,1)\simeq\mathrm{M}_{\mathbb{P}^2}(4,-1)$ by duality (cf. 2.1) and $\mathrm{M}_{\mathbb{P}^2}(4,-1)\simeq\mathrm{M}_{\mathbb{P}^2}(4,3)$ (the isomorphism sending the point corresponding to the sheaf $\mathcal{F}$ to the point corresponding to $\mathcal{F}(1)$). We will treat them together in section 3. These varieties are smooth because in this case semi-stability is equivalent to stability. We show that, as in the preceding table, $\mathrm{M}_{\mathbb{P}^2}(4,3)$ is the disjoint union of two strata: $\mathrm{M}_{\mathbb{P}^2}(4,3)=X_0(4,3)\amalg X_1(4,3)$, where $X_0(4,3)$ is an open subset and $X_1(4,3)$ a closed smooth subvariety of codimension 2. These strata correspond to sheaves having the cohomological conditions given in the first column of the preceding table. For example, the open stratum $X_0(4,3)$ contains the points representing the sheaves $\mathcal{F}$ such that $h^0(\mathcal{F}(-1))=h^1(\mathcal{F})=0$ and $h^0(\mathcal{F}\otimes\Omega^1(1))=2$.

The sheaves in the open stratum $X_0(4,3)$ are isomorphic to cokernels of injective morphisms

$$\mathcal{O}(-2)\oplus 2\mathcal{O}(-1)\longrightarrow 3\mathcal{O}\quad .$$

Let $W=\mathrm{Hom}(\mathcal{O}(-2)\oplus 2\mathcal{O}(-1),3\mathcal{O})$, on which acts the non-reductive algebraic group

$$G=\big(\mathrm{Aut}(\mathcal{O}(-2)\oplus 2\mathcal{O}(-1))\times\mathrm{Aut}(3\mathcal{O})\big)/\mathbb{C}^*$$

in an obvious way. Following [7] we describe in 2.5 and 3.1 an open G–invariant subset $\mathbf{W}$ of W such that there exists a geometric quotient $\mathbf{W}/G$ which is a smooth projective variety. Then $X_0(4,3)$ is canonically isomorphic to the open subset of $\mathbf{W}/G$ corresponding to injective morphisms.

The sheaves in the closed stratum $X_1(4,3)$ are isomorphic to cokernels of injective morphisms

$$2\mathcal{O}(-2) \longrightarrow \mathcal{O}(-1) \oplus \mathcal{O}(1) \ ,$$

and $X_1(4,3)$ is isomorphic to a geometric quotient of a suitable open subset of the space $\mathrm{Hom}(2\mathcal{O}(-2), \mathcal{O}(-1) \oplus \mathcal{O}(1))$ by the non-reductive group

$$\big(\mathrm{Aut}(2\mathcal{O}(-2)) \times \mathrm{Aut}(\mathcal{O}(-1) \oplus \mathcal{O}(1))\big)/\mathbb{C}^* \ .$$

Similarly, $\mathrm{M}_{\mathbb{P}^2}(4,1)$ is the disjoint union of the open subset $X_0(4,1)$ and the closed smooth subvariety $X_1(4,1)$. Of course the canonical isomorphism $\mathrm{M}_{\mathbb{P}^2}(4,1) \simeq \mathrm{M}_{\mathbb{P}^2}(4,3)$ induces isomorphisms of the strata.

We can give a precise description of the sheaves that appear in the strata. We define a closed subvariety $\widetilde{Y}$ of $X_0(4,3)$ corresponding to sheaves $\mathcal{F}$ such that there is a non trivial extension

$$0 \longrightarrow \mathcal{O}_\ell(-1) \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_X(1) \longrightarrow 0$$

where ℓ is a line and X is a cubic curve. The open subset $X_0(4,3)\backslash\widetilde{Y}$ consists of kernels of surjective morphisms $\mathcal{O}_C(2) \to \mathcal{O}_Z$, where C is a quartic curve and Z a length 3 finite subscheme of $\mathbb{P}^2$ not contained in any line. The sheaves in $X_1(4,1)$ are the kernels of the surjective morphisms $\mathcal{O}_C(1) \to \mathcal{O}_P$, C being a quartic curve and P a closed point of C. The sheaves in the other strata can be described similarly using the isomorphism $\mathrm{M}_{\mathbb{P}^2}(4,1) \simeq \mathrm{M}_{\mathbb{P}^2}(4,3)$.

1.2. The moduli space $\mathrm{M}_{\mathbb{P}^2}(4,2)$

It is treated in section 4. Here we have 3 strata: an open subset X_0, a locally closed smooth subvariety X_1 of codimension 1, and a closed smooth subvariety X_2 of codimension 3 which is in the closure of X_1. These strata are defined by cohomological conditions on the corresponding sheaves, as indicated in the preceding table.

The smallest stratum X_2 is the set of sheaves of the form $\mathcal{O}_C(1)$, where C is a quartic curve. Hence X_2 is isomorphic to $\mathbb{P}(S^4V^*)$.

The stratum X_0 consists of the cokernels of the injective semi-stable morphisms

$$2\mathcal{O}(-2) \longrightarrow 2\mathcal{O} \ .$$

Let $N(6,2,2)$ denote the moduli space of semi-stable morphisms $2\mathcal{O}(-2) \to 2\mathcal{O}$ (cf. 2.4). The closed subset corresponding to non injective morphisms is naturally identified with $\mathbb{P}^2 \times \mathbb{P}^2$, hence X_0 is isomorphic to $N(6,2,2)\backslash(\mathbb{P}^2 \times \mathbb{P}^2)$.

The sheaves of the second stratum X_1 are the cokernels of injective morphisms

$$2\mathcal{O}(-2) \oplus \mathcal{O}(-1) \longrightarrow \mathcal{O}(-1) \oplus 2\mathcal{O}$$

corresponding to matrices $\varphi = \begin{pmatrix} X_1 & X_2 & 0 \\ \star & \star & Y_1 \\ \star & \star & Y_2 \end{pmatrix}$ where $X_1, X_2 \in V^*$ are linearly independent one-forms and the same for $Y_1, Y_2 \in V^*$. Consider the non-reductive group

$$G = \big(\mathrm{Aut}(2\mathcal{O}(-2) \oplus \mathcal{O}(-1)) \times \mathrm{Aut}(\mathcal{O}(-1) \oplus 2\mathcal{O})\big)/\mathbb{C}^* .$$

This non-reductive group acts naturally on the variety W_1 of the preceding matrices and we prove that there is a geometric quotient W_1/G which is isomorphic to X_1. Using this description we conclude that the generic sheaf in X_1 is of the form $\mathcal{O}_C(1)(P-Q)$, where C is a smooth quartic and P, Q are distinct points of C.

Let $X = X_0 \cup X_1$. We prove that the sheaves in X are precisely the cokernels of the injective morphisms

$$2\mathcal{O}(-2) \oplus \mathcal{O}(-1) \longrightarrow \mathcal{O}(-1) \oplus 2\mathcal{O}$$

which are G–semi-stable with respect to the polarization $(\frac{1-\mu}{2}, \mu, \mu, \frac{1-\mu}{2})$, where μ is a rational number such that $\frac{1}{3} < \mu < \frac{1}{2}$ (cf. [7], 2.5). Let W be the set of such injective G–semi-stable morphisms. We prove that there is a good quotient $W/\!\!/G$ which is isomorphic to X. The existence of this quotient cannot be obtained from the results of [7] and [5].

The inclusion $X_0 \subset N(6,2,2)$ can be extended to a morphism $\delta : X \to N(6,2,2)$. Let $\widetilde{\mathbf{N}}$ denote the blowing-up of $N(6,2,2)$ along $\mathbb{P}^2 \times \mathbb{P}^2$. We prove that X is naturally isomorphic to an open subset of $\widetilde{\mathbf{N}}$ and δ can be identified with the restriction to X of the natural projection $\widetilde{\mathbf{N}} \to N(6,2,2)$. The complement of X in $\widetilde{\mathbf{N}}$ is contained in the inverse image of the diagonal of $\mathbb{P}^2 \times \mathbb{P}^2 \subset N(6,2,2)$, and for every $P \in \mathbb{P}^2$ the inverse image of (P,P) contains exactly one point of $\widetilde{\mathbf{N}} \backslash X$. Hence $\widetilde{\mathbf{N}} \backslash X$ is isomorphic to $\mathbb{P}^2$, whereas $\mathrm{M}_{\mathbb{P}^2}(4,2) \backslash X$ is isomorphic to $\mathbb{P}(S^4V^*)$.

1.3. The moduli space $\mathrm{M}_{\mathbb{P}^2}(4,4)$

It has been completely described by J. Le Potier in [15]. Let $N(3,4,4)$ be the moduli space of semi-stable morphisms $4\mathcal{O}(-1) \to 4\mathcal{O}$ (cf. 2.4). Then $\mathrm{M}_{\mathbb{P}^2}(4,4)$ is isomorphic to the blowing-up of $N(3,4,4)$ along a subvariety isomorphic to $\mathbb{P}(V^*)$.

The open stratum X_0 of $\mathrm{M}_{\mathbb{P}^2}(4,4)$ contains the sheaves which are cokernels of injective morphisms $4\mathcal{O}(-1) \to 4\mathcal{O}$ (which are then semi-stable). Our contribution to the study of $\mathrm{M}_{\mathbb{P}^2}(4,4)$ is a description of the complement $X_1 = \mathrm{M}_{\mathbb{P}^2}(4,4)\backslash X_0$, i.e of the exceptional divisor in $\mathrm{M}_{\mathbb{P}^2}(4,4)$. We show that the sheaves in X_1 are precisely the cokernels of the injective morphisms

$$f : \mathcal{O}(-2) \oplus \mathcal{O}(-1) \longrightarrow \mathcal{O} \oplus \mathcal{O}(1)$$

such that $f_{12} \neq 0$. Let $W = \mathrm{Hom}(\mathcal{O}(-2) \oplus \mathcal{O}(-1), \mathcal{O} \oplus \mathcal{O}(1))$, on which acts the non-reductive group

$$G = \big(\mathrm{Aut}(\mathcal{O}(-2) \oplus \mathcal{O}(-1)) \times \mathrm{Aut}(\mathcal{O} \oplus \mathcal{O}(1))\big)/\mathbb{C}^* .$$

We denote by W_1 be the set of G-semi-stable points of W with respect to a polarization $(\lambda_1, \lambda_2, \mu_1, \mu_2)$ such that $\lambda_1 = \mu_2 < \frac{1}{4}$ (cf. 2.5). According to [7] there exists a good quotient $W_1/\!\!/G$ which is a projective variety. We prove that X_1 and $W_1/\!\!/G$ are canonically isomorphic.

1.4. Clifford's theorem

To show that there are no sheaves in $\mathrm{M}_{\mathbb{P}^2}(4, \chi)$ other than those from the table we prove at 3.2.3, 4.1.3 and 5.2.4 the following cohomology estimates:

For any sheaf $\mathcal{F}$ in $\mathrm{M}_{\mathbb{P}^2}(4, 0)$ or in $\mathrm{M}_{\mathbb{P}^2}(4, 1)$ we have $h^1(\mathcal{F}) \leq 1$.

For any sheaf $\mathcal{F}$ in $\mathrm{M}_{\mathbb{P}^2}(4, 2)$ we have $h^1(\mathcal{F}) = 0$, unless $\mathcal{F} \simeq \mathcal{O}_C(1)$ for a quartic curve $C \subset \mathbb{P}^2$.

For generic sheaves in $\mathrm{M}_{\mathbb{P}^2}(4, \chi)$ the above estimates already follow from Clifford's theorem. Indeed, a generic sheaf in $\mathrm{M}_{\mathbb{P}^2}(r, \chi)$ is a line bundle supported on a smooth curve of degree r. Clifford's theorem, cf. [11] p. 251, states that, if $\mathcal{L}$ is a line bundle on a compact Riemann surface S corresponding to an effective divisor and such that $h^1(\mathcal{L}) > 0$, then we have the inequality

$$h^0(\mathcal{L}) \leq 1 + \frac{\deg(\mathcal{L})}{2},$$

with equality only if $\mathcal{L} = \mathcal{O}_S$ or $\mathcal{L} = \omega_S$ or S is hyperelliptic.

If $\mathcal{L}$ has Hilbert polynomial $P(t) = rt + \chi$, the Riemann-Roch theorem and the genus formula give

$$\deg(\mathcal{L}) = g(S) - 1 + \chi = \frac{(r-1)(r-2)}{2} - 1 + \chi = \frac{r(r-3)}{2} + \chi$$

and the above inequality takes the form

$$h^0(\mathcal{L}) \leq 1 + \frac{\chi}{2} + \frac{r(r-3)}{4}.$$

Taking $r = 4$ and $0 \leq \chi < 4$ and noting that equality in Clifford's theorem is achieved for non-generic sheaves, we conclude that for a generic sheaf $\mathcal{F}$ in $\mathrm{M}_{\mathbb{P}^2}(4, \chi)$ we have the relation

$$h^0(\mathcal{F}) < 2 + \frac{\chi}{2}.$$

This yields 3.2.3, 4.1.3 and 5.2.4 for generic sheaves. What we prove in this paper is that, in fact, Clifford's theorem is true for all sheaves in $\mathrm{M}_{\mathbb{P}^2}(4, \chi)$, $0 \leq \chi < 4$. By inspecting the first table from above we see that Clifford's theorem is also true for all sheaves in $\mathrm{M}_{\mathbb{P}^2}(r, \chi)$, $r = 1, 2, 3$, $0 \leq \chi < r$. There is thus enough evidence to suggest that the following "generalized Clifford's theorem" be true:

Conjecture: *Let* $\mathcal{F}$ *be a semi-stable sheaf on* $\mathbb{P}^2$ *with Hilbert polynomial* $P(t) = rt + \chi$, *where* $r \geq 1$ *and* $0 \leq \chi < r$ *are integers. If* $h^1(\mathcal{F}) > 0$, *then we have the inequality*

$$h^0(\mathcal{F}) \;\leq\; 1 + \frac{\chi}{2} + \frac{r(r-3)}{4}$$

with equality only in the following two cases:

(i) $r = 3$, $\chi = 0$ *and* $\mathcal{F} = \mathcal{O}_C$ *for some cubic* C.
(ii) $r = 4$, $\chi = 2$ *and* $\mathcal{F} = \mathcal{O}_C(1)$ *for some quartic* C.

2. Preliminaries

2.1. Duality

For a sheaf $\mathcal{F}$ on $\mathbb{P}^2$ with support of dimension one we will consider the dual sheaf $\mathcal{F}^D = \mathcal{E}xt^1(\mathcal{F}, \omega_{\mathbb{P}^2})$. According to [18] we have the following duality result:

2.1.1. Theorem – *The map* $\mathcal{F} \longmapsto \mathcal{F}^D$ *gives an isomorphism* $\mathrm{M}_{\mathbb{P}^2}(r,\chi) \xrightarrow{\simeq} \mathrm{M}_{\mathbb{P}^2}(r,-\chi)$.

For a semi-stable sheaf $\mathcal{F}$ on $\mathbb{P}^2$ with support of dimension one Serre duality gives the relations

2.1.2. Proposition – *We have* $h^i(\mathcal{F} \otimes \Omega^j(j)) = h^{1-i}(\mathcal{F}^D \otimes \Omega^{2-j}(3-j))$, *in particular* $h^i(\mathcal{F}) = h^{1-i}(\mathcal{F}^D)$.

2.1.3. Proposition – *Let* $\mathcal{F}$ *be a semi-stable sheaf on* $\mathbb{P}^2$ *with Hilbert polynomial* $P_{\mathcal{F}}(t) = rt + \chi$. *Then we have* $h^0(\mathcal{F}(i)) = 0$ *for* $i < \dfrac{3-r}{2} - \dfrac{\chi}{r}$, *and* $h^1(\mathcal{F}(i)) = 0$ *for* $i > \dfrac{r-3}{2} - \dfrac{\chi}{r}$.

Proof: Assume that $h^0(\mathcal{F}(i)) > 0$. Then there is a non-zero morphism $\mathcal{O} \to \mathcal{F}(i)$. We claim that this morphism factors through an injective morphism $\mathcal{O}_C \to \mathcal{F}(i)$, for a curve $C \subset \mathbb{P}^2$ defined by a polynomial equation. This can be seen as follows: the morphism $\mathcal{O} \to \mathcal{F}(i)$ factors through a non-zero morphism $\sigma : \mathcal{O}_D \to \mathcal{F}(i)$ for some curve D defined by a homogeneous polynomial f. Let $\mathcal{I} = \mathcal{K}er(\sigma)$. According to 6.7 [17], there is a homogeneous polynomial g dividing f such that $\mathcal{I}$ is contained in the ideal sheaf $\mathcal{G}$ in $\mathcal{O}_D$ defined by g and such that $\mathcal{G}/\mathcal{I}$ is supported on finitely many points. Since $\mathcal{F}(i)$ has no zero-dimensional torsion, $\mathcal{G}/\mathcal{I}$ is mapped to zero in $\mathcal{F}(i)$, hence $\mathcal{G} = \mathcal{I}$ and we may take for C the support of $\mathcal{O}_D/\mathcal{G}$.

Let $d = \deg(C)$. We recall that the slope of every pure sheaf of codimension 1 on $\mathbb{P}^2$ is the ratio of the Euler characteristic and multiplicity. From the semi-stability of $\mathcal{F}(i)$ we know that the slope of $\mathcal{O}_C$ cannot exceed the slope of $\mathcal{F}(i)$, that is,

$$\frac{3-d}{2} \leq \frac{\chi}{r} + i. \quad \text{But } d \leq r, \text{ so we get } \quad \frac{3-r}{2} - \frac{\chi}{r} \leq i.$$

This proves the first part of the proposition. The second part follows from the first and 2.1.2, 2.1.1. □

2.2. Beilinson spectral sequence

For every coherent sheaf $\mathcal{F}$ on $\mathbb{P}^2$ there is a free monad, called the *Beilinson free monad*, with middle cohomology $\mathcal{F}$:

$$0 \longrightarrow \mathcal{C}^{-2} \longrightarrow \mathcal{C}^{-1} \longrightarrow \mathcal{C}^{0} \longrightarrow \mathcal{C}^{1} \longrightarrow \mathcal{C}^{2} \longrightarrow 0,$$

$$\mathcal{C}^i = \bigoplus_{0 \leq j \leq n} \mathrm{H}^{i+j}(\mathcal{F} \otimes \Omega^j(j)) \otimes \mathcal{O}(-j).$$

All maps $\mathrm{H}^{i+j}(\mathcal{F} \otimes \Omega^j(j)) \otimes \mathcal{O}(-j) \longrightarrow \mathrm{H}^{i+j+1}(\mathcal{F} \otimes \Omega^j(j)) \otimes \mathcal{O}(-j)$ in the monad are zero. For sheaves with support of dimension one the Beilinson free monad takes the form

$$0 \longrightarrow \mathcal{C}^{-2} \longrightarrow \mathcal{C}^{-1} \longrightarrow \mathcal{C}^{0} \longrightarrow \mathcal{C}^{1} \longrightarrow 0, \tag{2.2.1}$$

$$\begin{aligned}
\mathcal{C}^{-2} &= \mathrm{H}^0(\mathcal{F}(-1)) \otimes \mathcal{O}(-2),\\
\mathcal{C}^{-1} &= \big(\mathrm{H}^0(\mathcal{F} \otimes \Omega^1(1)) \otimes \mathcal{O}(-1)\big) \oplus \big(\mathrm{H}^1(\mathcal{F}(-1)) \otimes \mathcal{O}(-2)\big),\\
\mathcal{C}^{0} &= \big(\mathrm{H}^0(\mathcal{F}) \otimes \mathcal{O}\big) \oplus \big(\mathrm{H}^1(\mathcal{F} \otimes \Omega^1(1)) \otimes \mathcal{O}(-1)\big),\\
\mathcal{C}^{1} &= \mathrm{H}^1(\mathcal{F}) \otimes \mathcal{O}.
\end{aligned}$$

Dualizing (**2.2.1**) we get a free monad for $\mathcal{F}^D$:

$$0 \longrightarrow \mathcal{C}_D^{-2} \longrightarrow \mathcal{C}_D^{-1} \longrightarrow \mathcal{C}_D^{0} \longrightarrow \mathcal{C}_D^{1} \longrightarrow 0, \tag{2.2.2}$$

$$\mathcal{C}_D^i = \mathcal{H}om(\mathcal{C}^{-1-i}, \omega_{\mathbb{P}^2}).$$

For every coherent sheaf $\mathcal{F}$ on $\mathbb{P}^2$ there is a spectral sequence of sheaves, called the *Beilinson spectral sequence*, which converges to $\mathcal{F}$ in degree zero and to 0 in degree non-zero. Its first term, $\mathrm{E}^1(\mathcal{F})$, is given by

$$\mathrm{E}^1_{ij} = \mathrm{H}^j(\mathcal{F} \otimes \Omega^{-i}(-i)) \otimes \mathcal{O}(i).$$

If $\mathcal{F}$ is supported on a curve, the relevant part of $\mathrm{E}^1(\mathcal{F})$ is exhibited in the following tableau:

(2.2.3)

$$\begin{array}{ccccc}
\mathrm{H}^1(\mathcal{F}(-1)) \otimes \mathcal{O}(-2) & & \mathrm{H}^1(\mathcal{F} \otimes \Omega^1(1)) \otimes \mathcal{O}(-1) & & \mathrm{H}^1(\mathcal{F}) \otimes \mathcal{O} \\
\| & & \| & & \| \\
\mathrm{E}^1_{-2,1} & \xrightarrow{\varphi_1} & \mathrm{E}^1_{-1,1} & \xrightarrow{\varphi_2} & \mathrm{E}^1_{01} \\
\mathrm{E}^1_{-2,0} & \xrightarrow{\varphi_3} & \mathrm{E}^1_{-1,0} & \xrightarrow{\varphi_4} & \mathrm{E}^1_{00} \\
\| & & \| & & \| \\
\mathrm{H}^0(\mathcal{F}(-1)) \otimes \mathcal{O}(-2) & & \mathrm{H}^0(\mathcal{F} \otimes \Omega^1(1)) \otimes \mathcal{O}(-1) & & \mathrm{H}^0(\mathcal{F}) \otimes \mathcal{O}
\end{array}$$

All the other E^1_{ij} are zero. The relevant part of E^2 is

$$\begin{array}{lll}
\mathrm{E}^2_{-2,1} = \mathcal{K}er(\varphi_1) & \mathrm{E}^2_{-1,1} = \mathcal{K}er(\varphi_2)/\mathcal{I}m(\varphi_1) & \mathrm{E}^2_{01} = \mathcal{C}oker(\varphi_2)\ . \\
 & \searrow^{\varphi_5} & \\
\mathrm{E}^2_{-2,0} = \mathcal{K}er(\varphi_3) & \mathrm{E}^2_{-1,0} = \mathcal{K}er(\varphi_4)/\mathcal{I}m(\varphi_3) & \mathrm{E}^2_{00} = \mathcal{C}oker(\varphi_4)
\end{array}$$

All the other E^2_{ij} are zero. The relevant part of E^3 is

$$\begin{array}{lll}
\mathrm{E}^3_{-2,1} = \mathcal{K}er(\varphi_5) & \mathrm{E}^3_{-1,1} = \mathcal{K}er(\varphi_2)/\mathcal{I}m(\varphi_1) & \mathrm{E}^3_{01} = \mathcal{C}oker(\varphi_2)\ . \\
\mathrm{E}^3_{-2,0} = \mathcal{K}er(\varphi_3) & \mathrm{E}^3_{-1,0} = \mathcal{K}er(\varphi_4)/\mathcal{I}m(\varphi_3) & \mathrm{E}^3_{00} = \mathcal{C}oker(\varphi_5)
\end{array}$$

All the maps in $\mathrm{E}^3(\mathcal{F})$ are zero. This shows that $\mathrm{E}^3 = \mathrm{E}^\infty$, hence all the terms in E^3, except, possibly, E^3_{00} and $\mathrm{E}^3_{-1,1}$, are zero. Moreover, there is an exact sequence

$$0 \longrightarrow \mathcal{C}oker(\varphi_5) \longrightarrow \mathcal{F} \longrightarrow \mathcal{K}er(\varphi_2)/\mathcal{I}m(\varphi_1) \longrightarrow 0.$$

We conclude that φ_2 is surjective and that there are exact sequences

(2.2.4) $$0 \longrightarrow \mathrm{H}^0(\mathcal{F}(-1)) \otimes \mathcal{O}(-2) \xrightarrow{\varphi_3} \mathrm{H}^0(\mathcal{F} \otimes \Omega^1(1)) \otimes \mathcal{O}(-1) \xrightarrow{\varphi_4} \mathrm{H}^0(\mathcal{F}) \otimes \mathcal{O} \longrightarrow \mathcal{C}oker(\varphi_4) \longrightarrow 0,$$

(2.2.5) $$0 \longrightarrow \mathcal{K}er(\varphi_1) \xrightarrow{\varphi_5} \mathcal{C}oker(\varphi_4) \longrightarrow \mathcal{F} \longrightarrow \mathcal{K}er(\varphi_2)/\mathcal{I}m(\varphi_1) \longrightarrow 0.$$

Let S be a scheme over $\mathbb{C}$. For every coherent sheaf $\mathcal{F}$ on $\mathbb{P}^2 \times S$, flat on S, there is a spectral sequence of sheaves on $\mathbb{P}^2 \times S$, called the *relative Beilinson spectral sequence*, which converges to $\mathcal{F}$ in degree zero and to 0 in degree non-zero. Its E^1-term is given by

$$\mathrm{E}^1_{ij} = \mathrm{R}^j_{p_*}(\mathcal{F} \otimes \Omega^{-i}(-i)) \boxtimes \mathcal{O}_{\mathbb{P}^2}(i).$$

Here $p : \mathbb{P}^2 \times S \to S$ is the projection onto the second component.

2.2.6. Proposition – *Let s be a closed point of S. If all the base change homomorphisms*

$$\mathrm{R}^j_{p_*}(\mathcal{F} \otimes \Omega^{-i}(-i))_s \longrightarrow \mathrm{H}^j(\mathcal{F}_s \otimes \Omega^{-i}(-i))$$

are isomorphisms, then the restriction of the E^1*-term of the relative Beilinson spectral sequence for* $\mathcal{F}$ *to* $\mathbb{P}^2 \times \{s\}$ *is the* E^1*-term for the Beilinson spectral sequence for the restriction* $\mathcal{F}_s$ *of* $\mathcal{F}$ *to* $\mathbb{P}^2 \times \{s\}$.

Proof: Let $p_1, p_2 : \mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^2$ be the projections onto the first and second component. We consider the resolution of the diagonal $\Delta \subset \mathbb{P}^2 \times \mathbb{P}^2$ given on p. 242 in [21]:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-2) \boxtimes \Omega^2_{\mathbb{P}^2}(2) \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-1) \boxtimes \Omega^1_{\mathbb{P}^2}(1) \longrightarrow \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2} \longrightarrow \mathcal{O}_\Delta \longrightarrow 0.$$

The maps in $\mathrm{E}^1(\mathcal{F})$ are the induced maps

$$\begin{aligned} \mathrm{E}^1_{ij}(\mathcal{F}) = &\mathrm{R}^j_{(p_1 \times 1_S)_*}\big(\mathcal{O}_{\mathbb{P}^2}(i) \boxtimes (p_2 \times 1_S)^*(\mathcal{F} \otimes \Omega^{-i}_{\mathbb{P}^2}(-i))\big) \longrightarrow \\ &\mathrm{R}^j_{(p_1 \times 1_S)_*}\big(\mathcal{O}_{\mathbb{P}^2}(i+1) \boxtimes (p_2 \times 1_S)^*(\mathcal{F} \otimes \Omega^{-i-1}_{\mathbb{P}^2}(-i-1))\big) = \mathrm{E}^1_{i+1,j}(\mathcal{F}). \end{aligned}$$

Restricting to $\mathbb{P}^2 \times \{s\}$ we get the map

$$\mathcal{O}_{\mathbb{P}^2}(i) \otimes \mathrm{R}^j_{p_*}(\mathcal{F} \otimes \Omega^{-i}_{\mathbb{P}^2}(-i))_s \longrightarrow \mathcal{O}_{\mathbb{P}^2}(i+1) \otimes \mathrm{R}^j_{p_*}(\mathcal{F} \otimes \Omega^{-i-1}_{\mathbb{P}^2}(-i-1))_s.$$

From the naturality of the base-change homomorphism we see that the above is the induced map

$$\mathcal{O}_{\mathbb{P}^2}(i) \otimes \mathrm{H}^j(\mathcal{F}_s \otimes \Omega^{-i}_{\mathbb{P}^2}(-i)) \longrightarrow \mathcal{O}_{\mathbb{P}^2}(i+1) \otimes \mathrm{H}^j(\mathcal{F}_s \otimes \Omega^{-i-1}_{\mathbb{P}^2}(-i-1)).$$

But this is the map from $\mathrm{E}^1(\mathcal{F}_s)$, which finishes the proof of the claim. □

2.2.7. Proposition – *Let S be a noetherian integral scheme over* $\mathbb{C}$ *and let* $\mathcal{F}$ *be a coherent sheaf on* $\mathbb{P}^2 \times S$ *which is S-flat. For a closed point s in S we denote by* $\mathcal{F}_s$ *the restriction of* $\mathcal{F}$ *to* $\mathbb{P}^2 \times \{s\}$*. Assume that for all i and j,* $h^j(\mathcal{F}_s \times \Omega^{-i}(-i))$ *is independent of s. Then, for all closed points s in S, the restriction of* $\mathrm{E}^1(\mathcal{F})$ *to* $\mathbb{P}^2 \times \{s\}$ *is* $\mathrm{E}^1(\mathcal{F}_s)$.

Proof: According to III 12.9 from [12], all base change homomorphisms from 2.2.6 are isomorphisms, so 2.2.7 is a corollary of 2.2.6. □

2.3. Quotients by reductive groups and moduli spaces of sheaves

We first recall the definition of good and geometric quotients (cf. [19], [20]):

2.3.1. Definition – *Let an algebraic group G act on an algebraic variety X. Then a pair* (φ, Y) *of a variety and a morphism* $X \xrightarrow{\varphi} Y$ *is called a good quotient if*

(i) φ *is G–invariant (for the trivial action of G on Y),*

(ii) *φ is affine and surjective,*
(iii) *If U is an open subset of Y then φ^* induces an isomorphism $\mathcal{O}_Y(U) \simeq \mathcal{O}_X(\varphi^{-1}U)^G$, where the latter denotes the ring of G–invariant functions,*
(iv) *If F_1, F_2 are disjoint closed and G–invariant subvarieties of X then $\varphi(F_1)$, $\varphi(F_2)$ are closed and disjoint.*

If in addition the fibres of φ are the orbits of the action the quotient (φ, Y) is called a geometric quotient.

C. Simpson's construction of the moduli spaces $\mathrm{M}_{\mathbb{P}^2}(r,\chi)$ (cf. [15], [23]) is based on the following facts: there are a smooth variety R and a reductive group G (to be precise, G is a special linear group) acting algebraically on R, such that $\mathrm{M}_{\mathbb{P}^2}(r,\chi)$ is a good quotient of R by G. Moreover, the open subset $\mathrm{M}^{\mathrm{s}}_{\mathbb{P}^2}(r,\chi)$ of isomorphism classes of stable sheaves is the geometric quotient of an open subset $R_0 \subset R$ modulo G. There is a coherent R-flat sheaf $\widetilde{\mathcal{F}}$ on $\mathbb{P}^2 \times R$ whose restriction $\widetilde{\mathcal{F}}_s$ to every closed fiber $\mathbb{P}^2 \times \{s\}$ is a semi-stable sheaf with Hilbert polynomial $P(t) = rt + \chi$. The quotient morphism

$$\pi : R \longrightarrow \mathrm{M}_{\mathbb{P}^2}(r,\chi)$$

maps s to the stable-equivalence class of $\widetilde{\mathcal{F}}_s$, denoted $[\widetilde{\mathcal{F}}_s]$. The quotient morphism $R_0 \to \mathrm{M}^{\mathrm{s}}_{\mathbb{P}^2}(r,\chi)$ sends s to the isomorphism class of $\widetilde{\mathcal{F}}_s$.

2.3.2. Proposition – *Let X be an irreducible locally closed subvariety of $M_{\mathbb{P}^2}(r,\chi)$ and let $S' \subset R$ be the preimage of X equipped with the canonical reduced induced structure. Then*

(i) *The restriction of $\pi : S' \to X$ is a good quotient of S' by G.*
(ii) *There exists an irreducible component S of S' such that $\pi(S) = X$.*
(iii) *The restriction of $\pi : S \to X$ is a good quotient of S by G.*

Proof: It follows from [24], 3.2 (i), that $\pi^{-1}(X) \to X$ is a good quotient by G. We have $S' = \pi^{-1}(X)_{red}$. Since π is affine (i) is reduced to the following : let $Z = \mathrm{Spec}(A)$ be an affine scheme and suppose given an algebraic action of G on Z. Suppose that $Z/\!\!/G = \mathrm{Spec}(A^G)$ is reduced. Then the canonical morphism $\phi : A^G \to (A/\mathrm{rad}(A))^G$ is an isomorphism. We have $\mathcal{K}er(\phi) = A^G \cap \mathrm{rad}(A)$, and since A^G is reduced, ϕ is injective. We have a commutative square

$$\begin{array}{ccc} A & \xrightarrow{\ R\ } & A^G \\ \downarrow & & \downarrow \phi \\ A/\mathrm{rad}(A) & \xrightarrow{\ R\ } & (A/\mathrm{rad}(A))^G \end{array}$$

R being the Reynolds operators (cf. [19]), showing that ϕ is surjective. This proves (i).

Let S'' be an irreducible component of S. We have $S'' \subset G.S''$, and $G.S''$ is irreducible (because G and S'' are). It follows that $GS'' = S''$ and that $\pi(S'')$ is closed in X. Since

X is irreducible and is the union of the closed images of the irreducible components of S' there exists an irreducible component S of S' such that $\pi(S) = X$. This proves (ii).

Again in the case of (iii) the problem is local. So we can suppose that $S' = \mathrm{Spec}(A)$ and $X = \mathrm{Spec}(A^G)$. Let I be the G-invariant ideal of S. Then we have to prove that the canonical morphism $\phi : A^G \to (A/I)^G$ is an isomorphism. The surjectivity of $\pi : S \to X$ implies that the composition

$$A^G \xrightarrow{\phi} (A/I)^G \subset A/I$$

is injective, so ϕ is injective. The surjectivity of ϕ can be seen using the Reynolds operators as before. This proves (iii). □

2.3.3. Proposition – *With the above notations, let $\widetilde{\mathcal{F}}_S$ be the restriction of $\widetilde{\mathcal{F}}$ from $\mathbb{P}^2 \times R$ to $\mathbb{P}^2 \times S$. Assume that for all i and j, $h^j(\widetilde{\mathcal{F}}_s \otimes \Omega^{-i}(-i))$ is independent of the closed point s in S. Then, for all closed points s in S, the restriction of $\mathrm{E}^1(\widetilde{\mathcal{F}})$ to $\mathbb{P}^2 \times \{s\}$ is $\mathrm{E}^1(\widetilde{\mathcal{F}}_s)$.*

Proof: The hypotheses of 2.2.7 are satisfied because flatness is preserved under pulling back. □

2.4. Kronecker modules

Let L be a finite dimensional non-zero vector space over $\mathbb{C}$, and m, n positive integers. Let $W = \mathrm{Hom}(\mathbb{C}^m \otimes L, \mathbb{C}^n)\backslash\{0\}$. We have an algebraic action of

$$\Gamma = (\mathrm{GL}(m) \times \mathrm{GL}(n))/\mathbb{C}^*$$

on W given by

$$\begin{aligned} \Gamma \times W &\longrightarrow W \\ ((g_1, g_2), f) &\longmapsto g_2 \circ f \circ (g_1 \otimes I_L)^{-1} \end{aligned}$$

Let $\mathbb{P} = \mathbb{P}(\mathrm{Hom}(\mathbb{C}^m \otimes L, \mathbb{C}^n))$. The preceding action induces an action of the reductive group $G = \mathrm{SL}(m) \times \mathrm{SL}(n)$ on $\mathbb{P}$ with an obvious linearization. According to K. Hulek [13], the linear maps $\mathbb{C}^m \otimes L \to \mathbb{C}^n$ are called *L-Kronecker modules.* A Kronecker module will be called *semi-stable* (resp. *stable*) if it is non-zero and if the corresponding point in $\mathbb{P}$ is semi-stable (resp. stable) for the above action. We have

2.4.1. Proposition – *A L-Kronecker module $\tau : \mathbb{C}^m \otimes L \to \mathbb{C}^n$ is semi-stable (resp. stable) if and only for all linear subspaces $H \subset \mathbb{C}^m$, $K \subset \mathbb{C}^n$, with $H \neq \{0\}$, such that $\tau(H \otimes L) \subset K$ we have*

$$\frac{\dim(K)}{\dim(H)} \geq \frac{n}{m} \qquad \textit{(resp.} \quad > \textit{).}$$

(cf. [3], prop. 15, [14]).

Let $\mathbb{P}^{ss}$ (resp. $\mathbb{P}^s$) denote the G-invariant open subset of semi-stable (resp. stable) points of $\mathbb{P}$. Let

$$N(L,m,n) = \mathbb{P}^{ss}/\!\!/G, \quad N_s(L,m,n) = \mathbb{P}^s/G.$$

Of course these varieties depend only on m, n and $\dim(L)$. If $\dim(L) = q$ we will also use the notations $N(q,m,n)$ and $N_s(q,m,n)$.

The variety $N(q,m,n)$ is projective, irreducible, locally factorial, and $N_s(q,m,n)$ is a smooth open subset of $N(q,m,n)$.

Let x_q be the smallest solution of the equation $X^2 - qX + 1 = 0$. Then we have $\dim(N(q,m,n)) > 0$ if and only if $x_q < \frac{m}{n} < \frac{1}{x_q}$. In this case $N_s(q,m,n)$ is not empty and we have $\dim(N(q,m,n)) = qmn - m^2 - n^2 + 1$.

If m and n are relatively prime then $N(q,m,n) = N_s(q,m,n)$ hence $N(q,m,n)$ is a projective smooth variety. In this case there exists a *universal morphism* on $N(q,m,n)$: there are algebraic vector bundles E, F on $N(q,m,n)$ of ranks m, n respectively, and a morphism $\tau : E \otimes L \to F$ such that for every closed point x of $N(q,m,n)$, and isomorphisms $E_x \simeq \mathbb{C}^m$, $F_x \simeq \mathbb{C}^n$, the linear map $\tau_x : \mathbb{C}^m \otimes L \to \mathbb{C}^n$ belongs to the G-orbit represented by x.

The moduli spaces of Kronecker modules appear in the following context: suppose given two vector bundles U, V on $\mathbb{P}^2$. Then a morphism $U \otimes \mathbb{C}^m \to V \otimes \mathbb{C}^n$ is equivalent to a $\operatorname{Hom}(U,V)^*$-Kronecker module $\operatorname{Hom}(U,V)^* \otimes \mathbb{C}^m \to \mathbb{C}^n$.

2.5. Moduli spaces of morphisms

Let X be a projective algebraic variety, and r, s positive integers. For $1 \le i \le r$ (resp. $1 \le j \le s$) let m_i (resp. n_j) be a positive integer and $\mathcal{F}_i$ (resp. $\mathcal{E}_j$) a coherent sheaf on X. Let

$$\mathcal{E} = \bigoplus_{1 \le i \le r} \mathcal{E}_i \otimes \mathbb{C}^{m_i} \quad \text{and} \quad \mathcal{F} = \bigoplus_{1 \le j \le s} \mathcal{F}_j \otimes \mathbb{C}^{n_j}.$$

We suppose that the sheaves $\mathcal{E}_i$, $\mathcal{F}_j$ are simple and that

$$\operatorname{Hom}(\mathcal{E}_i, \mathcal{E}_k) = \operatorname{Hom}(\mathcal{F}_j, \mathcal{F}_l) = \{0\}$$

if $i > k$ and $j > l$.

Let $\mathbb{W} = \operatorname{Hom}(\mathcal{E}, \mathcal{F})$. The algebraic group $G = \operatorname{Aut}(\mathcal{E}) \times \operatorname{Aut}(\mathcal{F})$ acts on $\mathbb{W}$ in an obvious way. We can see the elements of $\operatorname{Aut}(\mathcal{E})$ as matrices

$$\begin{pmatrix} g_1 & 0 & \cdots & 0 \\ u_{21} & g_2 & & \vdots \\ \vdots & \ddots & \ddots & 0 \\ u_{r1} & \cdots & u_{r,r-1} & g_r \end{pmatrix}$$

where $g_i \in \mathrm{GL}(m_i)$ and $u_{ki} \in L(\mathbb{C}^{m_i}, \mathbb{C}^{m_k}) \otimes \mathrm{Hom}(\mathcal{E}_i, \mathcal{F}_k)$ (and similarly for $\mathrm{Aut}(\mathcal{F})$). Let $G_{red} = \prod \mathrm{GL}(m_i) \times \prod \mathrm{GL}(n_l)$, which is a reductive subgroup of G, and let H be the maximal normal unipotent subgroup of G, consisting of pairs of matrices with identities as diagonal terms.

The action of G_{red} is well known (cf. [14]). Let $\sigma = (\lambda_1, \ldots, \lambda_r, \mu_1, \ldots, \mu_s)$ be a sequence of positive rational numbers such that

$$\sum_{1 \le i \le r} \lambda_i m_i = \sum_{1 \le j \le s} \mu_j n_j = 1$$

(such a sequence is called a *polarization*). An element $f \in \mathbb{W}$ is called G_{red}*–semi-stable* (resp. G_{red}*–stable*) with respect to σ if for any choice of subspaces $M_i \subset \mathbb{C}^{m_i}$, $N_j \subset \mathbb{C}^{n_j}$ such that $N_j \neq \mathbb{C}^{n_j}$ for at least one j, and such that f maps $\oplus(\mathcal{E}_i \otimes M_i)$ into $\oplus(\mathcal{F}_j \otimes N_j)$, we have

$$\sum_{1 \le i \le r} \lambda_i \dim(M_i) \le \sum_{1 \le j \le s} \mu_j \dim(N_j) \quad (\text{resp. } <).$$

There exists a good quotient of the open subset of G_{red}–semi-stable points of $\mathbb{W}$ with respect to σ.

We consider now the action of the whole group G which is not reductive in general. An element $f \in \mathbb{W}$ is called G*–semi-stable* (resp. G*–stable*) with respect to σ if all the elements of $H.f$ are G_{red}–semi-stable (resp. G_{red}–stable) with respect to σ. Let $\mathbb{W}^{ss}(\sigma)$ (resp. $\mathbb{W}^{s}(\sigma)$) be the open G-invariant subset of G–semi-stable (resp. G–stable) points of $\mathbb{W}$ with respect to σ. If suitable numerical conditions are satisfied by σ then $\mathbb{W}^{ss}(\sigma)$ admits a good and projective quotient and $\mathbb{W}^{s}(\sigma)$ admits a geometric quotient, which is smooth (cf. [7]).

3. Euler Characteristics One and Three

3.1. The open strata

According to 4.2 in [17], the sheaves $\mathcal{G}$ giving a point in $\mathrm{M}_{\mathbb{P}^2}(4,3)$ and satisfying $h^0(\mathcal{G}(-1)) = 0$ are precisely the sheaves that have a resolution of the form

(3.1.1) $$0 \longrightarrow \mathcal{O}(-2) \oplus 2\mathcal{O}(-1) \xrightarrow{\varphi} 3\mathcal{O} \longrightarrow \mathcal{G} \longrightarrow 0$$

with φ_{12} having linearly independent maximal minors.

3.1.2. Moduli spaces of morphisms - It is easy to see, using the stability conditions of 2.4.1, that a morphism $f : 2\mathcal{O}(-1) \to 3\mathcal{O}$ is stable (as a V-Kronecker module) if and only if all its maximal minors are linearly independent. Moreover a stable morphism is injective (as a morphism of sheaves).

We consider now morphisms as in (3.1.1)

$$\varphi : \mathcal{O}(-2) \oplus 2\mathcal{O}(-1) \longrightarrow 3\mathcal{O}.$$

Let $W = \mathrm{Hom}(\mathcal{O}(-2) \oplus 2\mathcal{O}(-1), 3\mathcal{O})$. Then the linear algebraic group

$$G = \big(\mathrm{Aut}(\mathcal{O}(-2) \oplus 2\mathcal{O}(-1)) \times \mathrm{Aut}(3\mathcal{O})\big)/\mathbb{C}^*$$

acts on W in an obvious way. Good quotients of some G-invariant open subsets of W are given in [7], 9.3 (cf. 2.5). The quotient related to $\mathrm{M}_{\mathbb{P}^2}(4,3)$ is the obvious one, and we will describe it.

We begin with the following remark: let $f : 2\mathcal{O}(-1) \to 3\mathcal{O}$ be an injective morphism, and $\sigma \in \mathrm{H}^0(\mathcal{C}oker(f)(2))$. Then σ can be lifted to a morphism $\mathcal{O}(-2) \to 3\mathcal{O}$ and defines thus with f a morphism as in (3.1.1). All the morphisms constructed in this way are in the same G-orbit, and every morphism $\varphi : \mathcal{O}(-2) \oplus 2\mathcal{O}(-1) \to 3\mathcal{O}$ such that $\varphi_{12} = f$ comes from a section of $\mathcal{C}oker(f)(2)$.

Let $\tau : E \otimes V \to F$ be a universal morphism on $N(3,2,3)$ (cf. 2.4). Let p_1, p_2 be the projections $N(3,2,3) \times \mathbb{P}^2 \to N(3,2,3)$, $N(3,2,3) \times \mathbb{P}^2 \to \mathbb{P}^2$ respectively. From τ we get a morphism of sheaves on $N(3,2,3) \times \mathbb{P}^2$

$$\theta : p_1^*(E) \otimes p_2^*(\mathcal{O}(-1)) \longrightarrow p_1^*(F)\ .$$

For every $x \in N(3,2,3)$, this morphism is injective on the fiber $\{x\} \times \mathbb{P}^2$. Hence $\mathcal{C}oker(\theta)$ is flat on $N(3,2,3)$, and for every $x \in N(3,2,3)$ we have $\mathcal{C}oker(\theta)_x = \mathcal{C}oker(\theta_x)$. Let

$$U \ = \ p_{1*}(\mathcal{C}oker(\theta) \otimes p_2^*(\mathcal{O}(2))).$$

It is a rank 3 vector bundle on $N(3,2,3)$, and for every $x \in N(3,2,3)$ we have $U_x = \mathrm{H}^0(\mathcal{C}oker(\theta_x)(2))$.

Let $\mathbb{W} = \mathbb{P}(U)$. Let $\boldsymbol{W}$ be the open G-invariant subset of W consisting of morphisms φ such that φ_{12} is stable, and such that the section of $\mathcal{C}oker(\varphi_{12})$ defined by φ_{11} is non-zero. We have an obvious morphism $\boldsymbol{W} \to \mathbb{W}$ which is a geometric quotient, i.e. $\mathbb{W} = \boldsymbol{W}/G$. Hence $\boldsymbol{W}/G$ is a smooth projective variety.

3.1.3. The open stratum of $\mathbf{M}_{\mathbb{P}^2}(4,3)$ - Let $X_0(4,3)$ be the open subset of $\mathrm{M}_{\mathbb{P}^2}(4,3)$ corresponding to sheaves $\mathcal{G}$ such that $h^0(\mathcal{G}(-1)) = 0$. Let $W_0(4,3) \subset \boldsymbol{W}$ be the G-invariant open subset consisting of injective morphisms. We will see later that the morphism $\rho_{4,3} : W_0(4,3) \to X_0(4,3)$ sending a morphism to its cokernel is a geometric quotient by G, and hence $X_0(4,3)$ is isomorphic to the open subset of $\mathbb{W}$ corresponding to injective morphisms.

Now we will describe the open stratum of $\mathrm{M}_{\mathbb{P}^2}(4,1)$. As at 2.2.2, dualizing the exact sequence (3.1.1) we get a resolution for the sheaf $\mathcal{F} = \mathcal{G}^D(1)$. According to 2.1.1, the latter is in $\mathrm{M}_{\mathbb{P}^2}(4,1)$. Hence we obtain

3.1.4. Proposition – *The sheaves $\mathcal{F}$ in $\mathrm{M}_{\mathbb{P}^2}(4,1)$ satisfying $h^1(\mathcal{F})=0$ are precisely the sheaves with resolution of the form*

$$0 \longrightarrow 3\mathcal{O}(-2) \xrightarrow{\varphi} 2\mathcal{O}(-1)\oplus\mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0,$$

where φ_{11} has linearly independent maximal minors.

Let $W_0 = W_0(4,1)$ denote the set of injective morphisms

$$\varphi : 3\mathcal{O}(-2) \longrightarrow 2\mathcal{O}(-1)\oplus\mathcal{O},$$

such that φ_{11} is stable. The linear algebraic group

$$\big(\mathrm{Aut}(3\mathcal{O}(-2))\times\mathrm{Aut}(2\mathcal{O}(-1)\oplus\mathcal{O})\big)/\mathbb{C}^*$$

acts on $\mathrm{Hom}(3\mathcal{O}(-2), 2\mathcal{O}(-1)\oplus\mathcal{O})$ by conjugation. Of course this group is canonically isomorphic to G, and the isomorphism

$$\begin{array}{ccc} \mathrm{Hom}(\mathcal{O}(-2)\oplus 2\mathcal{O}(-1), 3\mathcal{O}) & \xrightarrow{\Lambda} & \mathrm{Hom}(3\mathcal{O}(-2), 2\mathcal{O}(-1)\oplus\mathcal{O}) \\ \varphi & \longmapsto & {}^t\varphi\otimes I_{\mathcal{O}(-2)} \end{array}$$

is G-invariant and sends $W_0(4,3)$ to $W_0(4,1)$.

3.1.5. The open stratum of $\mathbf{M}_{\mathbb{P}^2}(4,1)$ - Let $X_0 = X_0(4,1)$ be the open subset of $\mathrm{M}_{\mathbb{P}^2}(4,1)$ of sheaves $\mathcal{F}$ such that $h^1(\mathcal{F})=0$. Let

$$\begin{array}{ccc} \lambda : \mathrm{M}_{\mathbb{P}^2}(4,3) & \xrightarrow{\simeq} & \mathrm{M}_{\mathbb{P}^2}(4,1) \\ \mathcal{G} & \longmapsto & \mathcal{G}^D(1) \end{array}$$

be the isomorphism of 2.1.1.

The map $\rho : W_0 \to X_0$ (resp. $\rho_{4,3} : W_0(4,3) \to X_0(4,3)$) which sends φ to the isomorphism class of $\mathcal{C}oker(\varphi)$ is a surjective morphism whose fibers are G-orbits. We have a commutative diagram

$$\begin{array}{ccc} W_0(4,3) & \xrightarrow{\Lambda} & W_0(4,1) \\ \downarrow{\scriptstyle \rho_{4,3}} & & \downarrow{\scriptstyle \rho} \\ X_0(4,3) & \xrightarrow{\lambda} & X_0 \end{array}$$

We claim that ρ (and hence also $\rho_{4,3}$) is a geometric quotient map:

3.1.6. Theorem – *The geometric quotient W_0/G is isomorphic to X_0.*

Proof: We will show that ρ is a categorical quotient map and the isomorphism $W_0/G \simeq X_0$ will follow from the uniqueness of the categorical quotient. Given a G-invariant morphism of varieties $f : W_0 \to Y$, there is a unique map $g : X_0 \to Y$ such that $g \circ \rho = f$. We need to show that g is a morphism of varieties. To see this we consider the good quotient $\pi : S \to X_0$ of 2.3.2. We will show that S can be covered with open sets U for which there are morphisms $\varsigma_U : U \to W_0$ making the diagram commute:

$$\begin{array}{ccc} U & \xrightarrow{\varsigma_U} & W_0 \\ & \searrow^{\pi} \quad \swarrow_{\rho} & \\ & X_0 & \end{array}.$$

Now we note that $g \circ \pi : S \to Y$ is a morphism because its restriction to each open set U is $f \circ \varsigma_U$. Thus g is the unique morphism associated to $g \circ \pi$ by the categorical quotient property of π.

It remains to construct the morphisms ς_U. This can be achieved as follows: we notice that $\widetilde{\mathcal{F}}_S$ satisfies the hypotheses of 2.3.3. This is so because for every sheaf $\mathcal{F}$ giving a point in X_0 we have

$$h^1(\mathcal{F} \otimes \Omega^2(2)) = 3, \qquad h^1(\mathcal{F} \otimes \Omega^1(1)) = 2, \qquad h^1(\mathcal{F}) = 0,$$
$$h^0(\mathcal{F} \otimes \Omega^2(2)) = 0, \qquad h^0(\mathcal{F} \otimes \Omega^1(1)) = 0, \qquad h^0(\mathcal{F}) = 1.$$

Thus the higher direct images $\mathrm{R}^j_{p_*}(\widetilde{\mathcal{F}}_S \otimes \Omega^{-i}(-i))$ are locally free sheaves on S; we cover S with open subsets U on which they are trivial and we fix such trivializations. For an arbitrary closed point s in U we restrict $\mathrm{E}^1(\widetilde{\mathcal{F}}_S)$ to $\mathbb{P}^2 \times \{s\}$. In this manner we construct a morphism ζ_U from U to an open subset E of the space of spectral sequences with E^1-term

$$\begin{array}{ccc} \mathrm{E}^1_{-2,1} = 3\mathcal{O}(-2) \xrightarrow{\varphi_1} & \mathrm{E}^1_{-1,1} = 2\mathcal{O}(-1) \xrightarrow{\varphi_2} & \mathrm{E}^1_{01} = 0 \\ & & \\ \mathrm{E}^1_{-2,0} = 0 & \mathrm{E}^1_{-1,0} = 0 & \mathrm{E}^1_{00} = \mathcal{O} \end{array}.$$

All the other E^1_{ij} are zero. By 2.3.3 $\zeta_U(s)$ is isomorphic to $\mathrm{E}^1(\widetilde{\mathcal{F}}_s)$. It remains to construct a morphism $\xi : E \to W_0$ which maps $\zeta_U(s)$ to a point $\varphi \in W_0$ satisfying $\mathcal{C}oker(\varphi) \simeq \widetilde{\mathcal{F}}_s$; then we can put $\varsigma_U = \xi \circ \zeta_U$. In other words, we need to obtain a resolution of the form 3.1.4 for $\widetilde{\mathcal{F}}_s$ starting with $\mathrm{E}^1(\widetilde{\mathcal{F}}_s)$ and performing algebraic operations.

For the problem at hand it is easier to construct ξ for the dual sheaf $\mathcal{G} = \widetilde{\mathcal{F}}_s^D(1)$. In view of 2.1.1 and (2.2.2) we can dualize the problem: given $\mathcal{G}$ in $\mathrm{M}_{\mathbb{P}^2}(4,3)$ with $h^0(\mathcal{G}(-1)) = 0$, we would like to construct the dual to resolution 3.1.4 starting from $\mathrm{E}^1(\mathcal{G})$. Tableau (2.2.3) is

$$\begin{array}{ccc} \mathcal{O}(-2) & 0 & 0 \\ & & \\ 0 & 2\mathcal{O}(-1) \xrightarrow{\varphi_4} & 3\mathcal{O} \end{array}$$

and the exact sequence (2.2.5) takes the form

$$0 \longrightarrow \mathcal{O}(-2) \xrightarrow{\varphi_5} \mathcal{C}oker(\varphi_4) \longrightarrow \mathcal{G} \longrightarrow 0.$$

It is now clear that $\mathcal{G}$ is isomorphic to the cokernel of the morphism

$$\varphi = (\varphi_5, \varphi_4) : \mathcal{O}(-2) \oplus 2\mathcal{O}(-1) \longrightarrow 3\mathcal{O}.$$

This finishes the proof of 3.1.6. □

3.2. The closed strata

We will treat only the closed stratum in $\mathrm{M}_{\mathbb{P}^2}(4,1)$, the case of $\mathrm{M}_{\mathbb{P}^2}(4,3)$ can be deduced by duality as in 3.1. According to 2.1.3, we have $h^0(\mathcal{F}(-1)) = 0$ for all $\mathcal{F}$ in $\mathrm{M}_{\mathbb{P}^2}(4,1)$. From this and 5.3 in [17] we obtain:

3.2.1. Theorem – *The sheaves $\mathcal{F}$ in $\mathrm{M}_{\mathbb{P}^2}(4,1)$ satisfying $h^1(\mathcal{F}) = 1$ are precisely the sheaves with resolution of the form*

$$0 \longrightarrow \mathcal{O}(-3) \oplus \mathcal{O}(-1) \xrightarrow{\varphi} 2\mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0,$$

where φ_{12} and φ_{22} are linearly independent vectors in V^*.

The description of the closed strata is very similar to that of the open ones. Let $W_1 = W_1(4,1)$ denote the set of morphisms from 3.2.1 that is, the set of injective morphisms

$$\varphi : \mathcal{O}(-3) \oplus \mathcal{O}(-1) \longrightarrow 2\mathcal{O},$$

such that φ_{12} and φ_{22} are linearly independent. The linear algebraic group

$$G = \big(\mathrm{Aut}(\mathcal{O}(-3) \oplus \mathcal{O}(-1)) \times \mathrm{Aut}(2\mathcal{O})\big)/\mathbb{C}^*$$

acts on W_1 by conjugation. Note that a morphism $f : \mathcal{O}(-1) \to 2\mathcal{O}$ is stable (as a V-Kronecker module) if and only if f_{12} and f_{22} are linearly independent, and a stable morphism is injective. The variety $N(3,1,2)$ is canonically isomorphic to $\mathbb{P}^2$: to every point P of $\mathbb{P}^2$, represented by a line $D \subset V$, the corresponding stable morphism is the canonical one $\mathcal{O}(-1) \to \mathcal{O} \otimes (V/D)$. The cokernel of this morphism is isomorphic to $\mathcal{I}_P(1)$ ($\mathcal{I}_P$ being the ideal sheaf of P).

As in the case of the open strata, given a stable $f : \mathcal{O}(-1) \to 2\mathcal{O}$ with cokernel $\mathcal{I}_P(1)$, the morphisms $\mathcal{O}(-3) \oplus \mathcal{O}(-1) \to 2\mathcal{O}$ whose restriction to $\mathcal{O}(-1)$ is f correspond to sections of $\mathcal{I}_P(4)$. The morphisms corresponding to non-zero sections are injective. Let Q_4 denote the cokernel of the canonical morphism $\mathcal{O}(-4) \to \mathcal{O} \otimes S^4V$ (cf. [3]) which is a rank 14 vector bundle on $N(3,1,2) = \mathbb{P}^2$. We have a canonical isomorphism $Q_{4P}^* \simeq \mathrm{H}^0(\mathcal{I}_P(4))$, for every point P of $\mathbb{P}^2$. Let $\mathbb{W}' = \mathbb{P}(Q_4^*)$. Then it is easy to see that we obtain a geometric quotient $W_1 \to \mathbb{W}'$.

Let $X_1 = X_1(4,1)$ be the locally closed subset of $\mathrm{M}_{\mathbb{P}^2}(4,1)$ given by the condition $h^1(\mathcal{F}) = 1$. We equip X_1 with the canonical induced reduced structure.

3.2.2. Theorem – *The geometric quotient $\mathbb{W}'$ is isomorphic to X_1. In particular, X_1 is a smooth closed subvariety of codimension 2.*

Proof: As explained in the proof of 3.1.6, we have to construct the resolution of 3.2.1 starting from $\mathrm{E}^1(\mathcal{F})$. Tableau (2.2.3) is

$$3\mathcal{O}(-2) \xrightarrow{\varphi_1} 3\mathcal{O}(-1) \xrightarrow{\varphi_2} \mathcal{O} \; .$$

$$0 \qquad\qquad \mathcal{O}(-1) \xrightarrow{\varphi_4} 2\mathcal{O}$$

As φ_2 is surjective, we may assume that it is given by the matrix $\begin{pmatrix} X & Y & Z \end{pmatrix}$. Thus φ_1 has columns that are linear combinations of the columns of the matrix

$$\begin{pmatrix} -Y & -Z & 0 \\ X & 0 & -Z \\ 0 & X & Y \end{pmatrix}.$$

As $\mathcal{F}$ surjects onto $\mathcal{K}er(\varphi_2)/\mathcal{I}m(\varphi_1)$, the latter has rank zero, hence $\mathcal{I}m(\varphi_1)$ has rank two, hence the columns of φ_1 span a two or three dimensional vector space. In the former case we have the isomorphism $\mathcal{K}er(\varphi_2)/\mathcal{I}m(\varphi_1) \simeq \Omega^1/2\mathcal{O}(-2)$. This sheaf has Hilbert polynomial $P(t) = t - 2$, which contradicts the semistabilty of $\mathcal{F}$. We conclude that φ_1 has three linearly independent columns, hence $\mathcal{K}er(\varphi_2) = \mathcal{I}m(\varphi_1)$, $\mathcal{K}er(\varphi_1) \simeq \mathcal{O}(-3)$ and (2.2.5) takes the form

$$0 \longrightarrow \mathcal{O}(-3) \xrightarrow{\varphi_5} \mathcal{C}oker(\varphi_4) \longrightarrow \mathcal{F} \longrightarrow 0.$$

It follows that $\mathcal{F}$ is isomorphic to the cokernel of the morphism

$$\varphi = (\varphi_5, \varphi_4) : \mathcal{O}(-3) \oplus \mathcal{O}(-1) \longrightarrow 2\mathcal{O}.$$

This finishes the proof of 3.2.2. □

3.2.3. Theorem – *There are no sheaves $\mathcal{F}$ in $\mathrm{M}_{\mathbb{P}^2}(4,1)$ satisfying the relation $h^1(\mathcal{F}) \geq 2$.*

Proof: Let $\mathcal{F}$ be an arbitrary sheaf in $\mathrm{M}_{\mathbb{P}^2}(4,1)$. The Beilinson monad (2.2.1) for $\mathcal{F}(-1)$ has the form

$$0 \longrightarrow 7\mathcal{O}(-2) \longrightarrow 10\mathcal{O}(-1) \longrightarrow 3\mathcal{O} \longrightarrow 0.$$

Thus $\mathrm{M}_{\mathbb{P}^2}(4,1)$ is parametrized by an open subset M inside the space of monads of the form

$$0 \longrightarrow 7\mathcal{O}(-1) \xrightarrow{A} 10\mathcal{O} \xrightarrow{B} 3\mathcal{O}(1) \longrightarrow 0.$$

Let $\eta : M \to \mathrm{M}_{\mathbb{P}^2}(4,1)$ be the surjective morphism which sends a monad to the isomorphism class of its cohomology. The tangent space of M at an arbitrary point (A, B) is

$$\mathbb{T}_{(A,B)}M = \{(\alpha, \beta), \quad \beta \circ A + B \circ \alpha = 0\}.$$

Let $\Phi : M \to \mathrm{Hom}(10\mathcal{O}, 3\mathcal{O}(1))$ be the projection onto the second component. We claim that Φ has surjective differential at every point. Indeed, $\mathrm{d}\Phi_{(A,B)}(\alpha, \beta) = \beta$, so we need to show that, given β, there is α such that $\beta \circ A + B \circ \alpha = 0$. This follows from the surjectivity of the map

$$\mathrm{Hom}(7\mathcal{O}(-1), 10\mathcal{O}) \longrightarrow \mathrm{Hom}(7\mathcal{O}(-1), 3\mathcal{O}(1)), \qquad \alpha \longrightarrow B \circ \alpha.$$

To see this we apply the long $\mathrm{Ext}(7\mathcal{O}(-1), \_\_)$ sequence to the exact sequence

$$0 \longrightarrow \mathcal{K}er(B) \longrightarrow 10\mathcal{O} \xrightarrow{B} 3\mathcal{O}(1) \longrightarrow 0$$

and we use the vanishing $\mathrm{Ext}^1(7\mathcal{O}(-1), \mathcal{K}er(B)) = 0$. The latter follows from the exact sequence

$$0 \longrightarrow 7\mathcal{O}(-1) \longrightarrow \mathcal{K}er(B) \longrightarrow \mathcal{F} \longrightarrow 0$$

and the vanishing $\mathrm{H}^1(\mathcal{F}(1)) = 0$. The latter follows from 2.1.3.

Note that $h^0(\mathcal{F}) = 10 - \mathrm{rank}(B)$. The subset $N \subset M$ of monads with cohomology sheaf $\mathcal{F}$ satisfying $h^1(\mathcal{F}) \geq 2$ is the preimage under Φ of the set of morphisms of rank at most 7. Since any matrix of rank 7 is the limit of a sequence of matrices of rank 8, and since the derivative of Φ is surjective at every point, we deduce that N is included in $\overline{\eta^{-1}(X_1)} \setminus \eta^{-1}(X_1)$. But, according to 3.2.2, X_1 is closed. We conclude that N is empty, which proves 3.2.3. □

3.3. Description of the strata

We will describe the open stratum $X_0(4, 3)$ and the closed stratum $X_1(4, 1)$. One can obtain a description of the two other strata by duality. We consider first the open stratum $X_0(4, 3)$.

Let Y denote the closed subvariety of $N(3, 2, 3)$ corresponding to morphisms $f : 2\mathcal{O}(-1) \to 3\mathcal{O}$ such that $\mathcal{C}oker(f)$ is not torsion-free, and $\widetilde{Y}$ the closed subvariety of $X_0(4, 3)$ of points over Y.

From [4] prop. 4.5, we have

3.3.1. Proposition – *Let $f : 2\mathcal{O}(-1) \to 3\mathcal{O}$ be a stable morphism. Then $\mathcal{C}oker(f)$ is torsion-free if and only if it is isomorphic to $\mathcal{I}_Z(2)$, Z being a finite subscheme of length 3 of $\mathbb{P}^2$ not contained in any line, and $\mathcal{I}_Z$ its ideal sheaf.*

Let $f : 2\mathcal{O}(-1) \to 3\mathcal{O}$ be a morphism as in 3.3.1, i.e. such that $\mathcal{C}oker(f)$ is torsion-free. Let $\varphi : \mathcal{O}(-2) \oplus 2\mathcal{O}(-1) \to 3\mathcal{O}$ be a morphism in $W_0(4,3)$ such that $\varphi_{12} = f$, and $\mathcal{F} = \mathcal{C}oker(\varphi)$. Then φ is injective and we have an exact sequence

$$0 \longrightarrow \mathcal{O}(-2) \longrightarrow \mathcal{I}_Z(2) \longrightarrow \mathcal{F} \longrightarrow 0.$$

Let C be the quartic curve defined by the composition $\mathcal{O}(-2) \to \mathcal{I}_Z(2) \subset \mathcal{O}(2)$, this quartic contains Z. Then from the preceding exact sequence we get the following one

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_C(2) \longrightarrow \mathcal{O}_Z \longrightarrow 0.$$

It now follows easily that

3.3.2. Proposition – *The open subset $X_0(4,3)\backslash\widetilde{Y}$ consists of the kernels of the surjective morphisms $\mathcal{O}_C(2) \to \mathcal{O}_Z$, where C is a quartic curve and Z is a length 3 finite subscheme of $\mathbb{P}^2$ not contained in a line.*

The generic point of $X_0(4,3)$ is of the form $\mathcal{O}_C(2)(-P-Q-R)$, where C is a smooth quartic curve and P, Q, R are points of C not contained in the same line.

3.3.3. Non torsion-free cokernels - Let $f : 2\mathcal{O}(-1) \to 3\mathcal{O}$ be a stable morphism such that $\mathcal{C}oker(f)$ is not torsion-free. This is the case if and only if the maximal minors of f have a common divisor, which is a linear form. It follows easily that there exists a basis (z_0, z_1, z_2) of V^* such that f is given (up to the action of $\mathrm{GL}(2) \times \mathrm{GL}(3)$) by the matrix $\begin{pmatrix} z_1 & -z_0 \\ z_2 & 0 \\ 0 & z_2 \end{pmatrix}$. Here all the maximal minors are multiples of z_2. Let $D \subset V$ be the plane defined by z_2 and ℓ the corresponding line of $\mathbb{P}^2$. Then f is equivalent to the canonical morphism $\mathcal{O}(-1) \otimes D \to \mathcal{O} \otimes \Lambda^2 V$. Hence $\mathcal{C}oker(f)$ depends only on ℓ, and we can denote $E_\ell = \mathcal{C}oker(f)$. We have proved that Y is canonically isomorphic to $\mathbb{P}(V^*)$. Using the canonical complex

$$0 \longrightarrow \mathcal{O}(-2) \longrightarrow \mathcal{O}(-1) \otimes V \longrightarrow \mathcal{O} \otimes \Lambda^2 V \longrightarrow \mathcal{O}(1) \otimes \Lambda^3 V \longrightarrow 0$$

it is easy to see that we have an exact sequence

$$0 \longrightarrow \mathcal{O}_\ell(-1) \longrightarrow E_\ell \longrightarrow \mathcal{O}(1) \longrightarrow 0.$$

We have $\mathrm{Ext}^1(\mathcal{O}(1), \mathcal{O}_\ell(-1)) \simeq \mathbb{C}$ and the preceding extension is not trivial, because E_ℓ is simple.

A morphism $\mathcal{O}(-2) \to E_\ell$ is non injective if and only if its image is contained in $\mathcal{O}_\ell(-1)$. It follows that the fiber over ℓ of the projection $\widetilde{Y} \to Y$ is precisely $\mathbb{P}(\mathrm{H}^0(E_\ell(2))\backslash\mathrm{H}^0(\mathcal{O}_\ell(1))$, and that $(\boldsymbol{W}/G)\backslash X_0(4,3)$ is canonically isomorphic to the projective bundle $P(T_{\mathbb{P}(V^*)}(1))$ over $\mathbb{P}(V^*)$.

The closed subvariety $\widetilde{Y}$ can also be described precisely. It consists of the non trivial extensions

$$0 \longrightarrow \mathcal{O}_\ell(-1) \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_X(1) \longrightarrow 0$$

where ℓ is a line and X a cubic. We have $\dim(\mathrm{Ext}^1(\mathcal{O}_X(1), \mathcal{O}_\ell(-1)) = 3$, and $\widetilde{Y}$ is a projective bundle over $\mathbb{P}(V^*) \times \mathbb{P}(S^3V^*)$. The generic sheaves in $\widetilde{Y}$ are obtained as follows: take ℓ and X transverse, the sheaves $\mathcal{F}$ are obtained by glueing $\mathcal{O}_\ell(2)$ and $\mathcal{O}_X(1)$ at the intersection points of ℓ and X.

We will now describe the closed stratum $X_1(4,1)$. Using the description of $\mathbb{W}'$ in 3.2 we get easily

3.3.4. Proposition – *The sheaves of $X_1(4,1)$ are the kernels of the surjective morphisms $\mathcal{O}_C(1) \to \mathcal{O}_P$, C being a quartic curve in $\mathbb{P}^2$ and P a closed point of C.*

4. Euler Characteristic Two

4.1. Preliminaries

We quote 4.5 from [17]:

4.1.1. Theorem – *Let $\mathcal{F}$ be a sheaf in $\mathrm{M}_{\mathbb{P}^2}(4,2)$ satisfying $h^0(\mathcal{F}(-1)) = 0$ and $h^1(\mathcal{F}) = 0$. Then $h^0(\mathcal{F} \otimes \Omega^1(1))$ is zero or one. The sheaves of the first kind are precisely the sheaves with resolution of the form*

(i) $$0 \longrightarrow 2\mathcal{O}(-2) \longrightarrow 2\mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0.$$

The sheaves of the second kind are precisely the sheaves with resolution

(ii) $$0 \longrightarrow 2\mathcal{O}(-2) \oplus \mathcal{O}(-1) \xrightarrow{\varphi} \mathcal{O}(-1) \oplus 2\mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0,$$

$$\varphi = \begin{pmatrix} X_1 & X_2 & 0 \\ \star & \star & Y_1 \\ \star & \star & Y_2 \end{pmatrix}.$$

Here $X_1, X_2 \in V^*$ are linearly independent one-forms and the same for $Y_1, Y_2 \in V^*$.

4.1.2. Remark – Let $\mathcal{E}$ be a coherent sheaf on $\mathbb{P}^2$ with Hilbert polynomial $P_{\mathcal{E}}(t) = 4t+2$. Then there $\mathcal{E}$ is isomorphic to the cokernel on an injective morphism

$$f : 2\mathcal{O}(-2) \longrightarrow 2\mathcal{O}$$

if and only if $h^0(\mathcal{F} \otimes \Omega^1(1)) = 0$. We have then also $h^0(\mathcal{F}(-1)) = h^1(\mathcal{F}) = 0$. This can be seen with a *generalized Beilinson spectral sequence* using exceptional bundles on $\mathbb{P}^2$ (cf. [2], [10], [4] 5-). Here we use the triad $(\mathcal{O}(-2), \mathcal{O}, \Omega^1(2))$ (cf. [2]) instead of $(\mathcal{O}(-2), \mathcal{O}(-1), \mathcal{O})$. Then f is equivalent to the canonical morphism

$$\mathcal{O}(-2) \otimes \mathrm{H}^1(\mathcal{E}(-1)) \longrightarrow \mathcal{O} \otimes \mathrm{H}^1(\mathcal{E} \otimes Q_2(-1))$$

(where Q_2 is the exceptional bundle that is the cokernel of the canonical morphism $\mathcal{O}(-2) \to S^2V \otimes \mathcal{O}$).

4.1.3. Theorem – *There are no sheaves $\mathcal{F}$ in $\mathrm{M}_{\mathbb{P}^2}(4,2)$ satisfying the relations $h^0(\mathcal{F}(-1)) = 0$ and $h^1(\mathcal{F}) > 0$.*

Proof: Consider a semi-stable sheaf F with Hilbert polynomial $4t+2$. Let $p = h^0(\mathcal{F} \otimes \Omega^1(1))$ and suppose that $h^1(\mathcal{F}) > 0$ and $h^0(\mathcal{F}(-1)) = 0$. Then we can write $h^1(\mathcal{F}) = q+1$, with $q \geq 0$. The Beilinson diagram (2.2.3) of $\mathcal{F}$ is

$$\begin{array}{ccc} \mathcal{O}(-2) \otimes \mathbb{C}^2 & \mathcal{O}(-1) \otimes \mathbb{C}^p & \mathcal{O} \otimes \mathbb{C}^{q+1} . \\ & & \\ 0 & \mathcal{O}(-1) \otimes \mathbb{C}^p & \mathcal{O} \otimes \mathbb{C}^{q+3} \end{array}$$

The morphism $\varphi_2 : \mathcal{O}(-1) \otimes \mathbb{C}^p \to \mathcal{O} \otimes \mathbb{C}^{q+1}$ is surjective, hence we must have $p \geq q+3$. The morphism $\varphi_4 : \mathcal{O}(-1) \otimes \mathbb{C}^p \to \mathcal{O} \otimes \mathbb{C}^{q+3}$ is injective, hence we have $p \leq q+3$. Finally we get $p = q+3$.

The Beilinson complex of $\mathcal{F}$ is then

$$0 \longrightarrow 2\mathcal{O}(-2) \oplus (q+3)\mathcal{O}(-1) \longrightarrow (q+3)\mathcal{O}(-1) \oplus (q+3)\mathcal{O} \longrightarrow (q+1)\mathcal{O} \longrightarrow 0.$$

Let $\mathcal{E} = \mathcal{C}oker(\varphi_4)$. We have a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (q+3)\mathcal{O}(-1) & \xrightarrow{\varphi_4} & (q+3)\mathcal{O} & \longrightarrow & \mathcal{E} & \longrightarrow & 0 . \\ & & \downarrow & & \downarrow & & \downarrow \alpha & & \\ 0 & \longrightarrow & 2\mathcal{O}(-2) \oplus (q+3)\mathcal{O}(-1) & \longrightarrow & \mathcal{K}er(\varphi_2) \oplus (q+3)\mathcal{O} & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \end{array}$$

If follows that $\mathcal{K}er(\alpha) \subset 2\mathcal{O}(-2)$. But $\mathcal{E}$ is a torsion sheaf, hence α is injective. The Hilbert polynomial of $\mathcal{E}$ is $(q+3)(t+1)$, which contradicts the semi-stability of $\mathcal{F}$. $\Box$

4.2. The two sub-strata of the open stratum

Let X be the open subset of $\mathrm{M}_{\mathbb{P}^2}(4,2)$ corresponding to sheaves $\mathcal{F}$ satisfying the conditions

$$h^0(\mathcal{F}(-1)) = 0, \qquad h^1(\mathcal{F}) = 0, \qquad h^0(\mathcal{F} \otimes \Omega^1(1)) \leq 1.$$

It is the disjoint union of the open subset X_0 and the locally closed subset X_1, where for $i = 0, 1$, X_i corresponds to sheaves $\mathcal{F}$ such that $h^0(\mathcal{F} \otimes \Omega^1(1)) = i$. We will first describe the subsets X_0 and X_1.

4.2.1. The open subset X_0 **-** Let $W = \mathrm{Hom}(2\mathcal{O}(-2), 2\mathcal{O})$ on which acts the reductive group $G_0 = \mathrm{GL}(2) \times \mathrm{GL}(2)$. Let $W_0 \subset W$ be the set of morphisms from 4.1.1(i) that is, the set of injective morphisms

$$\varphi : 2\mathcal{O}(-2) \longrightarrow 2\mathcal{O}.$$

The corresponding S^2V-Kronecker modules $S^2V \otimes \mathbb{C}^2 \to \mathbb{C}^2$ are then semi-stable (cf. 2.4). Hence W_0 is an open G_0-invariant subset inside the set W^{ss} of semi-stable points and contains the set $W^{ss} \setminus W^s$ of properly semi-stable points. W^{ss} is the set of 2×2-matrices with entries in S^2V^* having linearly independent rows and columns. $W^{ss} \setminus W^s$ is the subset of matrices equivalent, modulo row and column operations, to a matrix having a zero entry. Incidentally, note that W_0 is a proper subset of W^{ss} (cf. 4.5). Thus $W_0/\!\!/G_0$ is a proper open subset of the projective variety $N(6, 2, 2)$.

The morphism $\rho_0 : W_0 \to X_0$ given by $\varphi \longmapsto [\mathcal{C}oker(\varphi)]$ is G_0-invariant. We claim that $\rho_0(\varphi_1) = \rho_0(\varphi_2)$ if and only if $\overline{G_0\varphi_1} \cap \overline{G_0\varphi_2} \neq \emptyset$. This is clear if $\mathcal{F} = \mathcal{C}oker(\varphi_1)$ is stable, in fact, in this case, we have $G_0\varphi_1 = G_0\varphi_2$. If $\mathcal{F}$ is properly semi-stable, then there is an extension

$$0 \longrightarrow \mathcal{O}_{C_1} \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_{C_2} \longrightarrow 0$$

for some conic curves $C_1 = \{f_1 = 0\}$ and $C_2 = \{f_2 = 0\}$. From the horseshoe lemma we get a resolution

$$0 \longrightarrow 2\mathcal{O}(-2) \xrightarrow{\psi} 2\mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0, \qquad \psi = \begin{pmatrix} f_2 & 0 \\ \star & f_1 \end{pmatrix}.$$

Thus $f_1 \oplus f_2$ is in the closure of the orbit $G_0\psi = G_0\varphi_1$. Analogously, $f_1 \oplus f_2$ is in the closure of $G_0\varphi_2$.

4.2.2. Theorem − *The good quotient $W_0/\!\!/G_0$ is isomorphic to X_0. The subvariety $(W^{ss} \setminus W^s)/\!\!/G_0$ of $W_0/\!\!/G_0$ given by properly semi-stable points is isomorphic to the subvariety of $\mathrm{M}_{\mathbb{P}^2}(4, 2)$ given by properly semi-stable sheaves and is isomorphic to the symmetric space $\big(\mathbb{P}(S^2V^*) \times \mathbb{P}(S^2V^*)\big)/\mathrm{S}_2$.*

Proof: We will show that ρ_0 is a categorical quotient map and the isomorphism $W_0/\!\!/G_0 \simeq X_0$ will follow from the uniqueness of the categorical quotient. Let $f : W_0 \to Y$ be a G_0-invariant morphism of varieties. On the closure of each G_0-orbit f is constant hence, by the claim above, f is constant on the fibers of ρ. Thus f factors through a map $g : X_0 \to Y$.

We continue the proof as at 3.1.6. We need to construct resolution 4.1.1(i) starting from the Beilinson spectral sequence for $\mathcal{F}$. Tableau (2.2.3) takes the form

$$\begin{array}{ccc} 2\mathcal{O}(-2) & 0 & 0 \\ & & \\ 0 & 0 & 2\mathcal{O} \end{array}$$

and (2.2.5) yields the resolution

$$0 \longrightarrow 2\mathcal{O}(-2) \xrightarrow{\varphi_5} 2\mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0.$$

This allows us to construct the morphism $\xi : E \to W_0$ as at 3.1.6 and proves the isomorphism $W_0/\!\!/G_0 \simeq X_0$.

The statement about properly semi-stable sheaves follows from the easily verifiable fact that $\rho_0(\varphi)$ is properly semi-stable if and only if φ is properly semi-stable. □

4.2.3. The locally closed subset X_1 **-** Let W_1 be the set of morphisms from 4.1.1(ii) that is, the set of injective morphisms

$$\varphi : 2\mathcal{O}(-2) \oplus \mathcal{O}(-1) \longrightarrow \mathcal{O}(-1) \oplus 2\mathcal{O}$$

satisfying $\varphi_{12} = 0$ and such that $\mathcal{C}oker(\varphi)$ is semi-stable. The linear algebraic group

$$G = \mathrm{Aut}(2\mathcal{O}(-2) \oplus \mathcal{O}(-1)) \times \mathrm{Aut}(\mathcal{O}(-1) \oplus 2\mathcal{O})$$

acts on W_1 in an obvious way. Let X_1 be the locally closed subset of $\mathrm{M}_{\mathbb{P}^2}(4,2)$ defined by the conditions

$$h^0(\mathcal{F}(-1)) = 0, \qquad h^1(\mathcal{F}) = 0, \qquad h^0(\mathcal{F} \otimes \Omega^1(1)) = 1.$$

We equip X_1 with the canonical induced reduced structure. The morphism

$$\begin{aligned} \rho_1 : W_1 &\longrightarrow X_1 \\ \varphi &\longmapsto [\mathcal{C}oker(\varphi)] \end{aligned}$$

is surjective and its fibers are G-orbits.

4.2.4. Theorem – *The morphism $\rho_1 : W_1 \to X_1$ is a geometric quotient by G.*

Proof: According to [19], prop. 0.2, since the fibers of ρ_1 are the G-orbits we need only to prove that ρ_1 is a categorical quotient. As in the proof of 3.1.6, we need to recover resolution 4.1.1(ii) of a sheaf $\mathcal{F}$ in X_1 starting from the Beilinson spectral sequence for $\mathcal{F}$. Tableau (2.2.3) is

$$\begin{matrix} 2\mathcal{O}(-2) \xrightarrow{\varphi_1} \mathcal{O}(-1) & & 0 \quad . \\ & & \\ 0 & & \mathcal{O}(-1) \xrightarrow{\varphi_4} 2\mathcal{O} \end{matrix}$$

$\mathcal{C}oker(\varphi_1)$ cannot be of the form $\mathcal{O}_L$ for a line $L \subset \mathbb{P}^2$ because, by semistability, $\mathcal{F}$ cannot surject onto such a sheaf. Thus $\mathcal{C}oker(\varphi_1)$ is supported on a point and $\mathcal{K}er(\varphi_1) \simeq \mathcal{O}(-3)$. Clearly φ_5 lifts to a morphism $\psi_5 : \mathcal{O}(-3) \to 2\mathcal{O}$. We have a resolution

$$0 \longrightarrow \mathcal{O}(-3) \oplus \mathcal{O}(-1) \xrightarrow{[\psi_5, \varphi_4]} 2\mathcal{O} \longrightarrow \mathcal{C}oker(\varphi_5) \longrightarrow 0 \ .$$

We now apply the horseshoe lemma to the extension

$$0 \longrightarrow \mathcal{C}oker(\varphi_5) \longrightarrow \mathcal{F} \longrightarrow \mathcal{C}oker(\varphi_1) \longrightarrow 0,$$

to the resolution of $\mathcal{C}oker(\varphi_5)$ given above and to the resolution

$$0 \longrightarrow \mathcal{O}(-3) \longrightarrow 2\mathcal{O}(-2) \longrightarrow \mathcal{O}(-1) \longrightarrow \mathcal{C}oker(\varphi_1) \longrightarrow 0.$$

We obtain the exact sequence

$$0 \longrightarrow \mathcal{O}(-3) \longrightarrow \mathcal{O}(-3) \oplus \mathcal{O}(-1) \oplus 2\mathcal{O}(-2) \longrightarrow 2\mathcal{O} \oplus \mathcal{O}(-1) \longrightarrow \mathcal{F} \longrightarrow 0.$$

From the fact that $h^1(\mathcal{F}) = 0$, we see that $\mathcal{O}(-3)$ can be cancelled to yield a resolution as in 4.1.1(ii). $\Box$

4.3. The open stratum

Let

$$\mathbb{W} = \mathrm{Hom}(2\mathcal{O}(-2) \oplus \mathcal{O}(-1), \mathcal{O}(-1) \oplus 2\mathcal{O}).$$

The elements of $\mathbb{W}$ are represented as matrices

$$\begin{pmatrix} U_1 & U_2 & \alpha \\ q_{11} & q_{12} & Y_1 \\ q_{21} & q_{22} & Y_2 \end{pmatrix},$$

where $\alpha \in \mathbb{C}$, U_1, U_2, Y_1, Y_2 are linear forms and q_{11},q_{12},q_{21},q_{22} quadratic forms on V. The linear algebraic group

$$G = \mathrm{Aut}(2\mathcal{O}(-2) \oplus \mathcal{O}(-1)) \times \mathrm{Aut}(\mathcal{O}(-1) \oplus 2\mathcal{O})$$

acts on $\mathbb{W}$ as in 4.2.3 and W_1 is a locally closed G-invariant subset of $\mathbb{W}$.

We are in the situation of 2.5, and we will use the polarization

$$\sigma = (\frac{1-\mu}{2}, \mu, \mu, \frac{1-\mu}{2}),$$

where μ is a rational number such that $\frac{1}{3} < \mu < \frac{1}{2}$. We denote by $\mathbb{W}_0^{ss}$ the set of injective morphisms which are semi-stable with respect to σ.

4.3.1. Lemma – *Let $f \in \mathbb{W}$ be an injective morphism. Then $\mathcal{C}oker(f)$ is semi-stable if and only if f is G–semi-stable with respect to σ.*

Proof: Let m_1, m_2, n_1, n_2 be integers such that $0 \leq m_1 \leq 2$, $0 \leq m_2 \leq 1$, $0 \leq n_1 \leq 1$, $0 \leq n_2 \leq 2$. Let $f \in \mathbb{W}$. We say that *"$(m_1, m_2) \to (n_1, n_2)$ is forbidden for f"* if for all linear subspaces $M_1 \subset \mathbb{C}^2$, $M_2 \subset \mathbb{C}$, $N_1 \subset \mathbb{C}$, $N_2 \subset \mathbb{C}^2$ such that $\dim(M_i) = m_i$ and $\dim(N_j) = n_j$ we don't have

$$f((\mathcal{O}(-2) \otimes M_1) \oplus (\mathcal{O}(-1) \otimes M_2)) \subset (\mathcal{O}(-1) \otimes N_1) \oplus (\mathcal{O} \otimes N_2).$$

Then f is G–semi-stable with respect to σ if and only if the following are forbidden for f:

$$(2,0) \to (1,0), (2,0) \to (0,1), (0,1) \to (0,1), (1,1) \to (0,2), (1,1) \to (1,0),$$

and the cases where $n_1 = n_2 = 0$ and $m_1 + m_2 \neq 0$. The results follows then easily from 4.2. $\Box$

The elements of the group G can be seen as pairs of matrices

$$(\nu_1, \nu_2) \ = \ \left(\begin{pmatrix} B & 0 \\ \psi & \beta \end{pmatrix}, \begin{pmatrix} \alpha & 0 \\ \phi & A \end{pmatrix} \right)$$

where $A, B \in \mathrm{GL}(2)$, $\alpha, \beta \in \mathbb{C}^*$, ϕ is a column vector $\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$ and $\psi = (\psi_1, \psi_2)$ a pair of linear forms on V. As in 4.2.1 we consider the space $W = \mathrm{Hom}(2\mathcal{O}(-2), 2\mathcal{O})$ on which acts the reductive group $G_0 = \mathrm{GL}(2) \times \mathrm{GL}(2)$. Let

$$\tau : G \longrightarrow G_0$$

(cf. 4.2.1) be the morphism of groups defined by $\tau(\nu_1, \nu_2) = (\beta B, \alpha A)$. An easy calculation shows that

4.3.2. Lemma – *The morphism*

$$\Delta : \mathbb{W} \longrightarrow W$$

defined by

$$\Delta\left(\begin{pmatrix} U_1 & U_2 & \alpha \\ q_{11} & q_{12} & Y_1 \\ q_{21} & q_{22} & Y_2 \end{pmatrix} \right) \ = \ \alpha \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix} - \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} (U_1, U_2)$$

is compatible with τ, i.e. for every $w \in \mathbb{W}$ and $g \in G$ we have $\Delta(gw) = \tau(g)\Delta(w)$.

Note that $\Delta(\mathbb{W}_0^{ss}) = W^{ss}$.

The locus of non injective morphisms in $N(6, 2, 2)$ is isomorphic to $\mathbb{P}^2 \times \mathbb{P}^2$. To a pair (P_0, P_1) of points of $\mathbb{P}^2$ corresponds the $\mathrm{GL}(2) \times \mathrm{GL}(2)$-orbit of the matrix $\begin{pmatrix} \alpha_0\alpha_1 & \alpha_0\beta_1 \\ \alpha_1\beta_0 & \beta_0\beta 1 \end{pmatrix}$, where for $i = 0, 1$, (α_i, β_i) is a pair of linear forms defining P_i. Thus we have an isomorphism

$$X_0 \ \simeq \ N(6, 2, 2) \backslash (\mathbb{P}^2 \times \mathbb{P}^2).$$

Let $\pi : W^{ss} \to N(6, 2, 2)$ be the quotient morphism. We have

$$\Delta(W_1) = \pi^{-1}(\mathbb{P}^2 \times \mathbb{P}^2) \subset W^s.$$

We have a surjective G–invariant morphism

$$\rho : \mathbb{W}_0^{ss} \longrightarrow X$$

$$\varphi \longmapsto [\mathcal{C}oker(\varphi)]$$

4.3.3. Theorem – *The morphism ρ is a good quotient by G.*

Proof: Let $\mathbb{W}_0 = \rho^{-1}(X^s)$. According to [19], prop. 0.2, the restriction $\mathbb{W}_0 \to X^s$ of ρ is a geometric quotient by G. Let

$$\mathbb{W}_2 = \Delta^{-1}\big(\pi^{-1}(N(6,2,2)\backslash(\mathbb{P}^2 \times \mathbb{P}^2))\big) = \rho^{-1}(X_0).$$

We will show that the restriction of ρ, $\mathbb{W}_2 \to X_0$ is a good quotient. Since $\mathbb{W} = \mathbb{W}_0 \cup \mathbb{W}_2$ it follows easily from definition 2.3.1 that ρ is a good quotient.

Let $H = \mathbb{C}^* \times (V^*)^4$. Let H_0 be the open subset of tuples $(\mu, U_1, U_2, Y_1, Y_2)$ for which $\{U_1, U_2\}$, respectively $\{Y_1, Y_2\}$ are linearly independent. We have an isomorphism

$$\theta : \pi^{-1}(N(6,2,2)\backslash(\mathbb{P}^2 \times \mathbb{P}^2)\big) \times H_0 \to \mathbb{W}_2$$

given by

$$\theta(q_0, \mu, U_1, U_2, Y_1, Y_2) = \begin{pmatrix} \vec{U} & \mu \\ Q & \vec{Y} \end{pmatrix} ,$$

with

$$\vec{U} = (U_1, U_2), \quad \vec{Y} = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}, \quad Q = \frac{1}{\mu}\big(q_0 + \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} (U_1, U_2)\big).$$

It is easy to verify that the G–orbits of $\mathbb{W}_2$ are of the form $Y \times H$, where Y is a G_0–orbit of $N(6,2,2)\backslash(\mathbb{P}^2 \times \mathbb{P}^2)$. The fact that ρ is a good quotient by G follows then immediately from the fact that the restriction of π,

$$\pi^{-1}(N(6,2,2)\backslash(\mathbb{P}^2 \times \mathbb{P}^2)\big) \longrightarrow N(6,2,2)\backslash(\mathbb{P}^2 \times \mathbb{P}^2)$$

is a good quotient by G_0. $\Box$

From Δ and 4.3.3 we get a surjective morphism

$$\delta : X \longrightarrow N(6,2,2)$$

which induces an isomorphism $X_0 \simeq N(6,2,2)\backslash(\mathbb{P}^2 \times \mathbb{P}^2)$.

4.3.4. The fibers of δ over points of $\mathbb{P}^2 \times \mathbb{P}^2$ - Let P_1, P_2 distinct points of $\mathbb{P}^2$. Let U_i, Z $(i = 1,2)$ be linearly independent linear forms on V vanishing at P_i. Then $\delta^{-1}(P_1, P_2)$ contains the sheaves which are cokernels of injective morphisms of type

$$\begin{pmatrix} U_1 & Z & 0 \\ q_{11} & q_{12} & Z \\ q_{21} & q_{22} & U_2 \end{pmatrix} .$$

Each G–orbit of such a morphism contains a matrix of the following type:

$$\begin{pmatrix} U_1 & Z & 0 \\ \alpha U_2^2 & q_{12} & Z \\ q_{21} & \beta U_1^2 & U_2 \end{pmatrix}$$

where the quadratic form q_{12} in U_1, U_2, Z has no term in Z^2, and all the matrices of this type in the G–orbit are obtained by replacing the submatrix $\begin{pmatrix} \alpha U_2^2 & q_{12} \\ q_{21} & \beta U_1^2 \end{pmatrix}$ by a non-zero multiple. The corresponding morphism is non injective if and only if the submatrix vanishes. It follows that $\delta^{-1}(P_1, P_2) \simeq \mathbb{P}^{12}$.

Let P_1 be a point of $\mathbb{P}^2$, and U_1, Z be linearly independant linear forms on V vanishing at P_1. Let U_2 be a linear form such that (U_1, U_2, Z) is a basis of V^*. Then $\delta^{-1}(P_1, P_1)$ contains the sheaves which are cokernels of injective morphisms of type

$$\begin{pmatrix} U_1 & Z & 0 \\ q_{11} & q_{12} & Z \\ q_{21} & q_{22} & U_1 \end{pmatrix}.$$

Each G–orbit of such a morphism contains a matrix of the following type:

$$\begin{pmatrix} U_1 & Z & 0 \\ \alpha U_2^2 & q_{12} & Z \\ q_{21} & \beta U_2^2 & U_1 \end{pmatrix}$$

where the quadratic form q_{12} in U_1, U_2, Z has no term in Z^2, and all the matrices of this type in the G–orbit are obtained by replacing the submatrix $\begin{pmatrix} \alpha U_2^2 & q_{12} \\ q_{21} & \beta U_2^2 \end{pmatrix}$ by a non-zero multiple. The corresponding morphism is non injective if and only if the submatrix vanishes or is a multiple of $\begin{pmatrix} U_2^2 & 0 \\ 0 & -U_2^2 \end{pmatrix}$. Hence we see that $\delta^{-1}(P_1, P_1)$ is isomorphic to the complement of a point in $\mathbb{P}^{12}$.

4.3.5. Theorem – *Let $\widetilde{\mathbf{N}}$ be the blowing-up of $N(6,2,2)$ along $\mathbb{P}^2 \times \mathbb{P}^2$. Then X is isomorphic to an open subset of $\widetilde{\mathbf{N}}$.*

PROOF: We have $\delta^{-1}(\mathbb{P}^2 \times \mathbb{P}^2) = X_1$, which is a smooth hypersurface of X. It follows from the universal property of the blowing-up that δ factors through $\widetilde{\mathbf{N}}$: we have a morphism $\widetilde{\delta} : X \to \widetilde{\mathbf{N}}$ such that $\delta = p \circ \widetilde{\delta}$, p being the projection $\widetilde{\mathbf{N}} \to N(6,2,2)$. We want to prove that $\widetilde{\delta}$ induces an isomorphism $X \simeq \widetilde{\delta}(X)$. For this it suffices to prove that $\widetilde{\delta}$ does not contract X_1 to a subvariety of codimension ≥ 2 (cf. [22], II,4, theorem 2).

The morphism $\widetilde{\delta}$ can be described precisely on X_1. Recall that $p^{-1}(\mathbb{P}^2 \times \mathbb{P}^2)$ is the projective bundle $\mathbb{P}(\mathcal{N}_{\mathbb{P}^2 \times \mathbb{P}^2})$, $\mathcal{N}_{\mathbb{P}^2 \times \mathbb{P}^2}$ being the normal bundle of $\mathbb{P}^2 \times \mathbb{P}^2$ in $N(6,2,2)$. Let $w \in X_1$ and C be a smooth curve through w in X, not tangent to X_1 at w. Then $\widetilde{\delta}(w)$ is the image in $\mathbb{P}(\mathcal{N}_{\mathbb{P}^2 \times \mathbb{P}^2, \delta(w)})$ of the tangent to C at w. Suppose first that $\delta(w)$ is a pair of distinct points (P_1, P_2) and that w is the cokernel of a morphism

$$\phi_0 = \begin{pmatrix} Y_1 & Z & 0 \\ q_{11} & q_{12} & Z \\ q_{21} & q_{22} & Y_2 \end{pmatrix}$$

(we use the notations of 4.3.4). Then $\overline{\phi_0} = \begin{pmatrix} ZY_1 & Z^2 \\ Y_1Y_2 & ZY_2 \end{pmatrix}$ is a point of W^s over $\delta(w)$ and $\mathcal{N}_{\mathbb{P}^2\times\mathbb{P}^2,\delta(w)}$ is isomorphic to $\mathcal{N}_{\Gamma,\overline{\phi_0}}$, where $\Gamma \subset W^s$ is the inverse image of $\mathbb{P}^2 \times \mathbb{P}^2$ and $\mathcal{N}_\Gamma$ is its normal bundle. The vector space $\mathcal{N}_{\Gamma,\overline{\phi_0}}$ is a quotient of W. Supppose that C is defined by the family (ϕ_t) (for t in a neighbourhood of 0 in $\mathbb{C}$), with

$$\phi_t = \begin{pmatrix} Y_1 & Z & t \\ q_{11} & q_{12} & Z \\ q_{21} & q_{22} & Y_2 \end{pmatrix} .$$

Then from the formula defining Δ we deduce that $\widetilde{\delta}(w)$ is the image of $\begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix}$ in $\mathcal{N}_{\Gamma,\overline{\phi_0}}$. It follows from 4.3.4 that $\mathbb{P}(\mathcal{N}_{\mathbb{P}^2\times\mathbb{P}^2,\delta(w)})$ is contained in the image of $\widetilde{\delta}$, and that $\widetilde{\delta}$ does not contract W_1 to a subvariety of codimension ≥ 2. $\Box$

4.3.6. Remark – Using 4.3.4 and the proof of 4.3.5 is is not difficult to prove that $\widetilde{\mathbf{N}}\backslash X \simeq \mathbb{P}^2$.

4.4. The closed stratum

4.4.1. Theorem – *A sheaf $\mathcal{F}$ giving a point in* $\mathrm{M}_{\mathbb{P}^2}(4,2)$ *satisfies* $h^0(\mathcal{F}(-1)) > 0$ *if and only if there is a quartic curve $C \subset \mathbb{P}^2$ such that $\mathcal{F} \simeq \mathcal{O}_C(1)$. The subvariety of such sheaves in* $\mathrm{M}_{\mathbb{P}^2}(4,2)$ *is isomorphic to* $\mathbb{P}(S^4V^*)$.

Proof: As explained in the comments before 2.1.3, a non-zero morphism $\mathcal{O} \to \mathcal{F}(-1)$ must factor through an injective morphism $\mathcal{O}_C \to \mathcal{F}(-1)$ for a curve $C \subset \mathbb{P}^2$. From the semistability of $\mathcal{F}$ we see that C must be a quartic, and the above morphism must be an isomorphism. $\Box$

4.5. Description of the strata

We will describe X_1.

Let $\varphi : 2\mathcal{O}(-2) \oplus \mathcal{O}(-1) \to \mathcal{O}(-1) \oplus 2\mathcal{O}$ be an injective morphism as in 4.1.1 (ii) and $\mathcal{F} = \mathcal{C}oker(\varphi)$. Let P (resp. Q) be the point of $\mathbb{P}^2$ defined by the linear forms X_1, X_2 (resp. Y_1, Y_2). We have a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}(-1) & \longrightarrow & 2\mathcal{O} & \longrightarrow & \mathcal{I}_Q(1) & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle\alpha} & & \\ 0 & \longrightarrow & 2\mathcal{O}(-2) \oplus \mathcal{O}(-1) & \xrightarrow{\varphi} & \mathcal{O}(-1) \oplus 2\mathcal{O} & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \end{array}$$

It follows that we have $\mathcal{K}er(\alpha) \simeq \mathcal{O}(-3)$ and $\mathcal{C}oker(\alpha) \simeq \mathcal{O}_P$ (the structural sheaf of P). So we have an exact sequence

$$0 \longrightarrow \mathcal{O}(-3) \xrightarrow{\beta} \mathcal{I}_Q(1) \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_P \longrightarrow 0.$$

Let C be the quartic curve given by the inclusion $\mathcal{O}(-3) \subset \mathcal{I}_Q(1) \subset \mathcal{O}(1)$. Up to a scalar multiple, there is only one surjective morphism $\mathcal{O}_C(1) \to \mathcal{O}_Q$. We denote by $\mathcal{O}_C(1)(-Q)$ its kernel. If C is not smooth at Q this sheaf can fail to be locally free. We have $\mathcal{C}oker(\beta) \simeq \mathcal{O}_C(1)(-Q)$, and an exact sequence

(4.5.1)
$$0 \longrightarrow \mathcal{O}_C(1)(-Q) \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_P \longrightarrow 0.$$

Hence we could denote $\mathcal{F}$ by "$\mathcal{O}_C(1)(P-Q)$". This notation is justified if $P \neq Q$ and if C is smooth at P. Thus we can already state

4.5.2. Proposition – *The generic sheaf in X_1 is of the form $\mathcal{O}_C(1)(P-Q)$, where C is a smooth quartic curve and P, Q are distinct points of C.*

We want to prove now that the notation "$\mathcal{O}_C(1)(P-Q)$" has a meaning if $P \neq Q$ even if C is not necessarily smooth. We have to compute $\mathrm{Ext}^1_{\mathcal{O}_C}\big(\mathcal{O}_P, \mathcal{O}_C(1)(-Q)\big)$. First we note that

$$\mathrm{Ext}^1_{\mathcal{O}_C}\big(\mathcal{O}_P, \mathcal{O}_C(1)(-Q)\big) \simeq \mathrm{Ext}^1_{\mathcal{O}_{\mathbb{P}_2}}\big(\mathcal{O}_P, \mathcal{O}_C(1)(-Q)\big)$$

(this can be seen using prop. 2.3.1 of [6]). We use now the exact sequence

$$0 \to \mathcal{O}(-3) \to \mathcal{I}_Q(1) \to \mathcal{O}_C(1)(-Q) \to 0$$

and get the exact sequence

$$0 \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_{\mathbb{P}_2}}\big(\mathcal{O}_P, \mathcal{I}_Q(1)\big) \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_{\mathbb{P}_2}}\big(\mathcal{O}_P, \mathcal{O}_C(1)(-Q)\big) \longrightarrow \mathrm{Ext}^2_{\mathcal{O}_{\mathbb{P}_2}}\big(\mathcal{O}_P, \mathcal{O}(-3)\big)$$
$$\longrightarrow \mathrm{Ext}^2_{\mathcal{O}_{\mathbb{P}_2}}(\mathcal{O}_P, \mathcal{I}_Q(1)) \longrightarrow \mathrm{Ext}^2_{\mathcal{O}_{\mathbb{P}_2}}(\mathcal{O}_P, \mathcal{O}_C(1)(-Q)) \longrightarrow 0.$$

Using the exact sequence $0 \to \mathcal{O}(-1) \to 2\mathcal{O} \to \mathcal{I}_Q(1) \to 0$, we find that

$$\mathrm{Ext}^1_{\mathcal{O}_{\mathbb{P}_2}}\big(\mathcal{O}_P, \mathcal{I}_Q(1)\big) = \{0\}\ , \quad \mathrm{Ext}^2_{\mathcal{O}_{\mathbb{P}_2}}\big(\mathcal{O}_P, \mathcal{I}_Q(1)\big) \simeq \mathbb{C}$$

if $P \neq Q$ and

$$\mathrm{Ext}^1_{\mathcal{O}_{\mathbb{P}_2}}\big(\mathcal{O}_P, \mathcal{I}_P(1)\big) \simeq \mathbb{C}\ , \quad \mathrm{Ext}^2_{\mathcal{O}_{\mathbb{P}_2}}\big(\mathcal{O}_P, \mathcal{I}_P(1)\big) \simeq \mathbb{C}^2.$$

It now follows easily that $\mathrm{Ext}^1_{\mathcal{O}_C}\big(\mathcal{O}_P, \mathcal{O}_C(1)(-Q)\big) \simeq \mathbb{C}$ if $P \neq Q$, hence there is only one non trivial extension (**4.5.1**) and the notation $\mathcal{O}_C(1)(P-Q)$ is justified in this case.

If $P = Q$ we first remark that C is never smooth at P, and this implies that the morphism

$$\mathrm{Ext}^2_{\mathcal{O}_{\mathbb{P}_2}}\big(\mathcal{O}_P, \mathcal{O}(-3)\big) \longrightarrow \mathrm{Ext}^2_{\mathcal{O}_{\mathbb{P}_2}}\big(\mathcal{O}_P, \mathcal{I}_Q(1)\big)$$

vanishes: by Serre duality it is the transpose of

$$\mathrm{Hom}(\mathcal{I}_Q(1), \mathcal{O}_P) \longrightarrow \mathrm{Hom}(\mathcal{O}(-3), \mathcal{O}_P)$$

which is just the multiplication by an equation of C. It follows that we have $\mathrm{Ext}^1_{\mathcal{O}_C}\big(\mathcal{O}_P, \mathcal{O}_C(1)(-P)\big) \simeq \mathbb{C}^2$ and an injective map

$$\lambda : \mathbb{C} = \mathrm{Ext}^1_{\mathcal{O}_{\mathbb{P}^2}}\big(\mathcal{O}_P, \mathcal{I}_P(1)\big) \hookrightarrow \mathrm{Ext}^1_{\mathcal{O}_C}\big(\mathcal{O}_P, \mathcal{O}_C(1)(-P)\big) \simeq \mathbb{C}^2.$$

Its image corresponds to the extension (**4.5.1**) given by $\mathcal{F} = \mathcal{O}_C(1)$ and the other extensions, which are in X_1 are defined by the other elements of $\mathrm{Ext}^1_{\mathcal{O}_C}\big(\mathcal{O}_P, \mathcal{O}_C(1)(-P)\big)$.

5. Euler Characteristic Four

5.1. The open stratum

Let X_0 be the open subset of $\mathrm{M}_{\mathbb{P}^2}(4,4)$ corresponding to sheaves $\mathcal{F}$ such that $h^0(\mathcal{F}(-1)) = 0$. The complement $\mathrm{M}_{\mathbb{P}^2}(4,4)\backslash X_0$ is the *theta divisor* (cf. [15]). Combining [15], 4.3 and [3], théorème 2, we obtain

5.1.1. Theorem – *1 - The sheaves on $\mathbb{P}^2$ with Hilbert polynomial $4t+4$ satisfying $h^0(\mathcal{F}(-1)) = 0$ are precisely the sheaves which are isomorphic to the cokernel of an injective morphism*

$$f : 4\mathcal{O}(-1) \longrightarrow 4\mathcal{O} .$$

Moreover, $\mathcal{F}$ is not stable if and only if φ is equivalent, modulo operations on rows and columns, to a morphism of the form

$$\begin{pmatrix} \varphi_{11} & 0 \\ \varphi_{21} & \varphi_{22} \end{pmatrix} \qquad \text{with} \quad \varphi_{22} : m\mathcal{O}(-1) \longrightarrow m\mathcal{O}, \quad m = 1,\, 2 \text{ or } 3.$$

2 - Let $\mathbf{N}_0$ denote the open subset of $N(3,4,4)$ corresponding to injective morphisms. By associating $\mathcal{F}$ to the $\big(GL(4)\times GL(4)\big)$-orbit of φ we get an isomorphism $\mathbf{N}_0 \simeq X_0$.

According to [15] the complement $N(3,4,4)\backslash X_0$ is isomorphic to $\mathbb{P}^2$ and the inclusion $X_0 \subset N(3,4,4)$ can be extended to a morphism $\mathrm{M}_{\mathbb{P}^2}(4,4) \to N(3,4,4)$ which is the blowing-up of $N(3,4,4)$ along $\mathbb{P}^2$.

5.2. The closed stratum

5.2.1. Proposition – *The sheaves $\mathcal{F}$ in $\mathrm{M}_{\mathbb{P}^2}(4,4)$ satisfying $h^0(\mathcal{F}(-1)) = 1$ are precisely the sheaves with resolution*

$$0 \longrightarrow \mathcal{O}(-2)\oplus\mathcal{O}(-1) \xrightarrow{\varphi} \mathcal{O}\oplus\mathcal{O}(1) \longrightarrow \mathcal{F} \longrightarrow 0, \qquad \varphi_{12} \neq 0.$$

Moreover, $\mathcal{F}$ is not stable if and only if φ_{12} divides φ_{11} or φ_{22}.

Proof: We assume that $\mathcal{F}$ has a resolution as above and we need to show that $\mathcal{F}$ is semi-stable. Assume that there is a destabilizing subsheaf $\mathcal{E}$ of $\mathcal{F}$. We may assume that $\mathcal{E}$ is semi-stable. As $\mathcal{F}$ is generated by global sections, we must have $h^0(\mathcal{E}) < h^0(\mathcal{F}) = 4$. Thus $\mathcal{E}$ is in $\mathrm{M}_{\mathbb{P}^2}(2,3)$, $\mathrm{M}_{\mathbb{P}^2}(1,3)$ or in $\mathrm{M}_{\mathbb{P}^2}(1,2)$. In other words, $\mathcal{E}$ must be isomorphic to $\mathcal{O}_C(1)$, $\mathcal{O}_L(2)$ or $\mathcal{O}_L(1)$ for a conic C or a line L in $\mathbb{P}^2$. In the first case we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}(-1) & \longrightarrow & \mathcal{O}(1) & \longrightarrow & \mathcal{E} & \longrightarrow & 0\,. \\ & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \alpha} & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{O}(-2)\oplus\mathcal{O}(-1) & \longrightarrow & \mathcal{O}\oplus\mathcal{O}(1) & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \end{array}$$

Notice that α is injective, because it is injective on global sections, hence β is injective too. We obtain $\varphi_{12} = 0$, which contradicts the hypothesis on φ. The other two cases similarly lead to contradictions: if $\mathcal{F} \simeq \mathcal{O}_L(2)$, then $\alpha = 0$; if $\mathcal{F} \simeq \mathcal{O}_L(1)$, then $\beta = 0$, which is impossible.

Conversely, we are given $\mathcal{F}$ in $\mathrm{M}_{\mathbb{P}^2}(4,4)$ satisfying the condition $h^0(\mathcal{F}(-1)) = 1$ and we need to construct a resolution as above. As explained in the comments above 2.1.3, there is an injective morphism $\mathcal{O}_C \to \mathcal{F}(-1)$ for a curve C in $\mathbb{P}^2$. From the semistability of $\mathcal{F}(-1)$ we see that C is a cubic or a quartic curve. Assume that C is a cubic curve. The quotient $\mathcal{F}/\mathcal{O}_C(1)$ has Hilbert polynomial $P(t) = t+1$ and has no zero-dimensional torsion. Indeed, if $\mathcal{F}/\mathcal{O}_C(1)$ had zero-dimensional torsion $\mathcal{T} \neq 0$, then the preimage of $\mathcal{T}$ in $\mathcal{F}$ would be a subsheaf which violates the semistability of $\mathcal{F}$. We conclude that $\mathcal{F}/\mathcal{O}_C(1) \simeq \mathcal{O}_L$ for a line $L \subset \mathbb{P}^2$. We apply the horseshoe lemma to the extension

$$0 \longrightarrow \mathcal{O}_C(1) \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_L \longrightarrow 0$$

and to the standard resolutions for $\mathcal{O}_C(1)$ and $\mathcal{O}_L$ to get the desired resolution for $\mathcal{F}$. This also proves the statement about properly semi-stable sheaves from the claim. Indeed, if $\mathcal{F}$ is properly semi-stable, then $\mathcal{F}$ has stable factors $\mathcal{O}_C(1)$ and $\mathcal{O}_L$ and we can apply the horseshoe lemma as above. For the rest of the proof we may assume that $\mathcal{F}$ is stable.

Next we examine the situation when C is a quartic curve. Notice that $\mathcal{F}/\mathcal{O}_C(1)$ has zero-dimensional support and Euler characteristic 2. There is a subsheaf $\mathcal{T} \subset \mathcal{F}/\mathcal{O}_C(1)$ satisfying $h^0(\mathcal{T}) = 1$. Let $\mathcal{E}$ be the preimage of $\mathcal{T}$ in $\mathcal{F}$ and $\mathcal{G} = \mathcal{F}/\mathcal{E}$. Notice that $\mathcal{E}$ is semi-stable because any subsheaf ruining the semistability of $\mathcal{E}$ must contradict the stability of $\mathcal{F}$. Also, $h^0(\mathcal{E}(-1)) > 0$. From our results on in section 3 we see that there is a resolution

$$0 \longrightarrow 2\mathcal{O}(-2) \longrightarrow \mathcal{O}(-1)\oplus\mathcal{O}(1) \longrightarrow \mathcal{E} \longrightarrow 0.$$

We apply the horseshoe lemma to the extension

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow 0,$$

to the resolution of $\mathcal{E}$ from above and to the resolution

$$0 \longrightarrow \mathcal{O}(-2) \longrightarrow 2\mathcal{O}(-1) \longrightarrow \mathcal{O} \longrightarrow \mathcal{G} \longrightarrow 0.$$

The morphism $\mathcal{O} \to \mathcal{G}$ lifts to a morphism $\mathcal{O} \to \mathcal{F}$ because $h^1(\mathcal{E}) = 0$. We obtain a resolution

$$0 \longrightarrow \mathcal{O}(-2) \longrightarrow 2\mathcal{O}(-1) \oplus 2\mathcal{O}(-2) \longrightarrow \mathcal{O} \oplus \mathcal{O}(-1) \oplus \mathcal{O}(1) \longrightarrow \mathcal{F} \longrightarrow 0.$$

From the condition $h^0(\mathcal{F}(-1)) = 1$ we see that the map $\mathcal{O}(-2) \to 2\mathcal{O}(-2)$ from the above sequence is non-zero. We may cancel $\mathcal{O}(-2)$ to get the exact sequence

$$0 \longrightarrow \mathcal{O}(-2) \oplus 2\mathcal{O}(-1) \longrightarrow \mathcal{O}(-1) \oplus \mathcal{O} \oplus \mathcal{O}(1) \longrightarrow \mathcal{F} \longrightarrow 0.$$

The morphism $2\mathcal{O}(-1) \to \mathcal{O}(-1)$ from the above sequence is non-zero, otherwise $\mathcal{F}$ would surject onto a sheaf of the form $\mathcal{O}_L(-1)$ for a line $L \subset \mathbb{P}^2$, in violation of the semistability of $\mathcal{F}$. Thus we may cancel $\mathcal{O}(-1)$ in the above sequence to get the resolution of $\mathcal{F}$ from the proposition. □

Let W_1 be the set of morphisms from 5.2.1 that is, the set of injective morphisms

$$\varphi : \mathcal{O}(-2) \oplus \mathcal{O}(-1) \longrightarrow \mathcal{O} \oplus \mathcal{O}(1),$$

for which $\varphi_{12} \neq 0$. The linear non-reductive algebraic group

$$G = \big(\mathrm{Aut}(\mathcal{O}(-2) \oplus \mathcal{O}(-1)) \times \mathrm{Aut}(\mathcal{O} \oplus \mathcal{O}(1))\big)/\mathbb{C}^*$$

acts on W_1 by conjugation. According to [7], there is a good and projective quotient $W_1/\!\!/G$ which contains a geometric quotient as a proper open subset. For a polarization $\sigma = (\lambda_1, \lambda_2, \mu_1, \mu_2)$ satisfying $\lambda_1 = \mu_2 < 1/4$ (cf. 2.5) the set of semi-stable morphisms is $W^{ss}(\sigma) = W_1$. The open subset of stable points $W^s(\sigma) \subset W^{ss}(\sigma)$ is given by the conditions $\varphi_{12} \nmid \varphi_{11}$ and $\varphi_{12} \nmid \varphi_{22}$. The geometric quotient $W^s(\sigma)/G$ is an open subset of $W_1/\!\!/G$.

Let X_1 be the locally closed subset of $\mathrm{M}_{\mathbb{P}^2}(4,4)$ given by the relation $h^0(\mathcal{F}(-1)) = 1$. We equip X_1 with the canonical induced reduced structure. The morphism $\rho : W_1 \to X_1$, $\rho(\varphi) = [\mathcal{C}oker(\varphi)]$, is surjective and G-invariant. Since W_1 is irreducible, X_1 is irreducible, too. From 5.2.1 we know that $\rho(\varphi)$ is the isomorphism class of a stable sheaf if and only if φ is in $W^s(\sigma)$. Arguments similar to those in the beginning of 4.2 show that $\rho(\varphi_1) = \rho(\varphi_2)$ if and only if $\overline{G\varphi_1} \cap \overline{G\varphi_2} \neq \emptyset$. If $\mathcal{F}$ is stable, then $\rho^{-1}([\mathcal{F}])$ is a G-orbit, so it has dimension equal to $\dim(G)$. Indeed, the stabilizer for any stable morphism consists only of the neutral element of G. We have

$$\dim(X_1) = \dim(W^s(\sigma)) - \dim(G) = 25 - 9 = 16.$$

5.2.2. Theorem – *The good quotient $W_1/\!\!/G$ is isomorphic to X_1. In particular, X_1 is a closed hypersurface of $\mathrm{M}_{\mathbb{P}^2}(4,4)$. The closed subvariety $(W_1 \setminus W^s(\sigma))/\!\!/G$ of X_1 is isomorphic to the subvariety of $\mathrm{M}_{\mathbb{P}^2}(4,4)$ given by non stable sheaves which have a factor of the form $\mathcal{O}_C(1)$ in their Jordan-Hölder filtration, for a cubic curve $C \subset \mathbb{P}^2$, and is isomorphic to $\mathbb{P}(S^3V^*) \times \mathbb{P}(V^*)$.*

Proof: As in the proof of 3.1.6, we need to construct resolution 5.2.1 starting from the Beilinson spectral sequence for $\mathcal{F}$. Tableau (2.2.3) takes the form

$$\begin{array}{ccc} \mathcal{O}(-2) & 0 & 0 \\ & & \\ \mathcal{O}(-2) & \xrightarrow{\varphi_3} 4\mathcal{O}(-1) & \xrightarrow{\varphi_4} 4\mathcal{O} \end{array} .$$

The exact sequence (2.2.5) becomes

$$0 \longrightarrow \mathcal{O}(-2) \xrightarrow{\varphi_5} \mathcal{C}oker(\varphi_4) \longrightarrow \mathcal{F} \longrightarrow 0.$$

Note that φ_5 lifts to a morphism $\mathcal{O}(-2) \to 4\mathcal{O}$ because $\mathrm{Ext}^1(\mathcal{O}(-2), \mathcal{C}oker(\varphi_3)) = \{0\}$. Combining with the exact sequence (2.2.4) we obtain the resolution

$$0 \longrightarrow \mathcal{O}(-2) \xrightarrow{\left(\begin{smallmatrix} 0 \\ \psi \end{smallmatrix}\right)} \mathcal{O}(-2) \oplus 4\mathcal{O}(-1) \xrightarrow{\varphi} 4\mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0.$$

The argument from remark 6.3 in [17] shows that, up to equivalence,

$$\psi^{\mathrm{T}} = \begin{pmatrix} 0 & X & Y & Z \end{pmatrix}$$

Now we distinguish two possibilities: firstly, up to equivalence,

$$\varphi = \begin{pmatrix} \varphi_{11} & 0 \\ \varphi_{21} & \varphi_{22} \end{pmatrix} \quad \text{for a morphism} \quad \varphi_{11} : \mathcal{O}(-2) \oplus \mathcal{O}(-1) \longrightarrow 2\mathcal{O}.$$

We see that $\mathcal{F}$ surjects onto $\mathcal{C}oker(\varphi_{11})$, hence this sheaf is supported on a proper closed subset of $\mathbb{P}^2$, hence φ_{11} is injective, hence $\mathcal{C}oker(\varphi_{11})$ has Hilbert polynomial $P(t) = 3t+2$. This contradicts the semistability of $\mathcal{F}$. The second possibility, in fact the only feasible one, is that, modulo equivalence,

$$\varphi = \begin{pmatrix} \varphi_{11} & 0 \\ \varphi_{21} & \varphi_{22} \end{pmatrix} \quad \text{with} \quad \varphi_{11} = \begin{pmatrix} q & \ell \end{pmatrix} \quad \text{and} \quad \varphi_{22} = \begin{pmatrix} -Y & X & 0 \\ -Z & 0 & X \\ 0 & -Z & Y \end{pmatrix},$$

$$\varphi_{11} : \mathcal{O}(-2) \oplus \mathcal{O}(-1) \longrightarrow \mathcal{O}, \qquad \varphi_{22} : 3\mathcal{O}(-1) \longrightarrow 3\mathcal{O}.$$

From the semistability of $\mathcal{F}$ we see that $\ell \neq 0$. It is also easy to see that $\mathcal{F}$ is properly semi-stable if and only if ℓ divides q. Let E denote the set parametrized by (ψ, φ). We can identify E with an open subset inside the affine space parametrized by the entries of φ_{11} and φ_{21}, so E is irreducible and smooth. The subset given by the condition that ℓ divide q has codimension 3, hence the morphism $\xi : E \to W_1$ can be defined on its complement and then extended algebraically to E. Thus, for the purpose of constructing ξ, we may assume in the sequel that ℓ does not divide q. The snake lemma gives the exact sequence

$$\begin{aligned} 0 \longrightarrow \mathcal{K}er(\varphi_{22}) \longrightarrow \mathcal{O}(-2) \longrightarrow \mathcal{K}er(\varphi_{11}) \longrightarrow \mathcal{C}oker(\varphi_{22}) \longrightarrow \mathcal{F} \longrightarrow \\ \longrightarrow \mathcal{C}oker(\varphi_{11}) \longrightarrow 0. \end{aligned}$$

As $\mathcal{K}er(\varphi_{22}) = \mathcal{O}(-2)$, $\mathcal{K}er(\varphi_{11}) = \mathcal{O}(-3)$, $\mathcal{C}oker(\varphi_{22}) = \mathcal{O}(1)$, we obtain the extension

$$0 \longrightarrow \mathcal{O}_C(1) \longrightarrow \mathcal{F} \longrightarrow \mathcal{C}oker(\varphi_{11}) \longrightarrow 0.$$

Here C is the quartic curve given as the zero-set of the polynomial

$$f = \begin{pmatrix} Z & -Y & X \end{pmatrix} \varphi_{21} \begin{pmatrix} -\ell \\ q \end{pmatrix}.$$

From here on we construct a morphism to W_1 in the same manner as in the proof of 5.2.1.
□

5.2.3. Remark – In the course of the above proof we have rediscovered lemma 4.10 from [15] which states that every stable sheaf $\mathcal{F}$ satisfying $h^0(\mathcal{F}(-1)) = 1$ occurs as an extension

$$0 \longrightarrow \mathcal{O}_C(1) \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_S \longrightarrow 0$$

with C a quartic curve and S a zero-dimensional scheme of length 2. The ideal of S is generated by q and ℓ.

5.2.4. Theorem – *There are no sheaves $\mathcal{F}$ in $\mathrm{M}_{\mathbb{P}^2}(4,4)$ satisfying the condition $h^0(\mathcal{F}(-1)) \geq 2$.*

Proof: Using the information in the Introduction on $h^0(\mathcal{E}(-1))$ for $\mathcal{E}$ in $\mathrm{M}_{\mathbb{P}^2}(r,r)$, $r = 1, 2, 3$, it is easy to see that, if $\mathcal{F}$ is a semi-stable but not stable sheaf with Hilbert polynomial $4t + 4$, then we have $h^0(\mathcal{F}(-1)) \leq 1$.

According to 5.1, $\mathrm{M}_{\mathbb{P}^2}(4,4)\backslash X_0$ is the exceptional divisor of the blowing-up $\mathrm{M}_{\mathbb{P}^2}(4,4) \to N(3,4,4)$ along $\mathbb{P}^2$. Hence it is an irreducible hypersurface. Since this hypersurface contains X_1 we have $\mathrm{M}_{\mathbb{P}^2}(4,4)\backslash X_0 = X_1$. The result follows immediately. □

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