Diffusion rate in the wind-tree model

V. Delecroix
(joint work with P. Hubert, S. Lelièvre)

We study periodic versions of the wind-tree model introduced by P. & T. Ehrenfest in 1912 [3]. A point moves in the plane $\mathbb{R}^2$ and bounces elastically off rectangular scatterers following the usual law of reflection. The scatterers are translates of the rectangle $[0,a] \times [0,b]$, $0 < a < 1$ and $0 < b < 1$, one centered at each point of $\mathbb{Z}^2$. We denote the complement of obstacles in the plane by $T(a,b)$ and refer to it as *the wind-tree model*. Our aim is to understand its dynamical properties following the general scheme.

- Does there exists a typical behavior for trajectories ? If so describe it ?
- Quantify the set of non-typical behavior.

Typical behavior can be thought in topological or measurable sense and with respect to different dynamical properties: recurrence/divergence, diffusion rate, ergodicity, ...

The first study of the periodic wind-tree model is due to J. Hardy and J. Weber [7]. They proved that the rate of diffusion is $\log(t)\log\log(t)$ for very specific directions (generalized diagonals). Their result was recently completed by J.P. Conze and E. Gutkin [2] who build the ergodic decomposition of the billiard flow for those specific directions. In another direction, P. Hubert, S. Lelièvre and S. Troubetzkoy [6] proved that for a dense set of parameters a, b, for almost every direction θ, the flow in the direction θ is recurrent. In this paper, we compute the polynomial rate of diffusion.

Let $0 < a, b < 1$ be a fixed size for the scatterer. If we consider an initial condition with angle θ then, as the barriers are either horizontal or vertical, the ball will only takes direction θ, $-\theta$, $\pi-\theta$ and $\pi+\theta$. We call *flow in direction* θ and denote by ϕ_t^θ the billiard flow associated to the quadruple of directions $\{\theta, -\theta, \pi-\theta, \pi+\theta\}$. The phase space for the billiard flow is $T(a,b) \times (++,+-,-+,--)$ where $++,+-,-+,--$ referes to the four possible directions.

theorem 1. *Let ϕ_t^θ be the billiard flow in direction θ in the table $T(a,b)$ and $d(.,.)$ be the euclidean distance on $\mathbb{R}^2$.*

(1) *If (a,b) are rational numbers, then for almost every θ and for every point x in $T(a,b)$ (with an infinite trajectory), we have*

$$\limsup_{t\to\infty} \frac{\log(d(x,\phi_t^\theta(x)))}{\log(t)} = \frac{2}{3}.$$

(2) *If $(a,b) \in \mathbb{Q}[\sqrt{D}]$ are quadratic numbers with the additional condition that: $1/(1-a) = x + z\sqrt{D}$ and $1/(1-b) = (1-x) + z\sqrt{D}$ then for almost every θ and for every point x in $T(a,b)$ (with an infinite trajectory), we have*

$$\limsup_{t\to\infty} \frac{\log(d(x,\phi_t^\theta(x)))}{\log(t)} = \frac{2}{3}.$$

(3) *For almost all* $(a,b) \in (0,1)^2$*, for almost every* θ *and for every point* x *in* $T(a,b)$ *(with an infinite trajectory), we have*

$$\limsup_{t\to\infty} \frac{\log(d(x,\phi_t^\theta(x)))}{\log(t)} = \frac{2}{3}.$$

The conclusion of the first and second statement holds for specific parameters while the third one is the answer in the generic case. We do not know if the latter result holds for *every* parameters $(a,b) \in (0,1)^2$.

By the $\mathbb{Z}^2$ periodicity of the billiard table $T(a,b)$, our problem reduces to estimations of a $\mathbb{Z}^2$ cocycle over the billiard in a fundamental domain. On the other hand, a standard construction consisting of unfolding the trajectories [12], the billiard flow can be replaced by a linear flow on a (non compact) translation surface that we denote $\mathrm{X}_\infty(a,b)$. The surface $\mathrm{X}_\infty(a,b)$ keeps the $\mathbb{Z}^2$-periodicity of the billiard table $T(a,b)$. We denote $\mathrm{X}(a,b)$ the quotient of $X_\infty(a,b)$ under this $\mathbb{Z}^2$ action. As, the unfolding procedure of the billiard flow is equivariant with respect to the $\mathbb{Z}^2$ action $\mathrm{X}(a,b)$ can be also be seen as the unfolding of the billiard in a fundamental domain of $T(a,b)/\mathbb{Z}^2$ (see Figure 1).

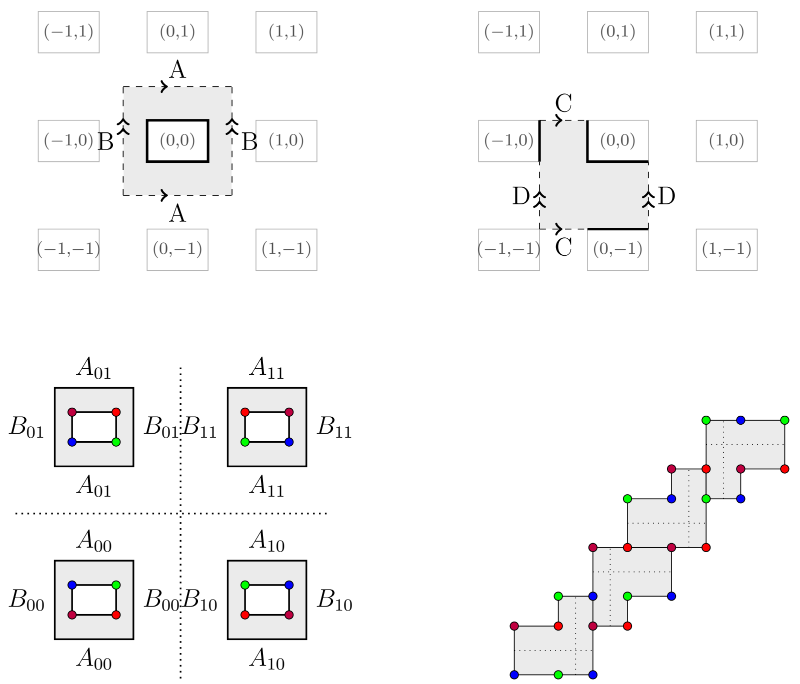

FIGURE 1. Two fundamental domains for the $\mathbb{Z}^2$ action and two versions of the associated unfolded surface $\mathrm{X}(a,b)$.

The position of the particle in $T(a,b)$ can be tracked from $\mathrm{X}(a,b)$. The position of the particle starting from x in direction θ can be approximated by the intersection of a geodesic in $X(a,b)$ with a cocycle $f \in H^1(X(a,b);\mathbb{Z}^2)$ describing the infinite cover $X_\infty(a,b)/X(a,b)$. Theorem 1 has an immediate translation in this language. The growth of such quantities has been studied since a long time by A. Zorich [13, 14] and G. Forni [5] (see also [8]) and are related to Lyapunov exponents of the Teichmüller flow. In our case which does not fit into the preceding general theory, we prove that the exponents do control the growth of the intersection. That's the main part of the paper. From results by M. Bainbridge [1] and A. Eskin, M. Kontsevich and A. Zorich [4], we deduce that the value of the Lyapunov exponent under consideration is $2/3$ which explains the right term in Theorem 1.

The surface $X(a,b)$ is a covering of the genus 2 surface $L(a,b)$ which is a so called L-shaped surface. By C. McMullen's fundamental work [9, 10, 11], the only $SL_2(\mathbb{R})$ invariant submanifolds of the stratum $\mathcal{H}(2)$ are the Teichmüller curves (cases 1 and 2 in Theorem 1) and the stratum itself (case 3). The only $SL_2(\mathbb{R})$ invariant probability measures are the Lebesgue measures on these loci. To prove Theorem 1 we use asymptotic theorems (namely Birkhoff and Oseledets ergodic theorem) with respect to those measure.

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