Divergent trajectories in a windtree model

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Paříž VII, Géométrie et Dynamique

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The windtree model

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In the euclidean plane, we place a square with side $1/2$ centered at each integer point. A particule (identified to a point) bounds on square scatterers with the standard law of reflexion.

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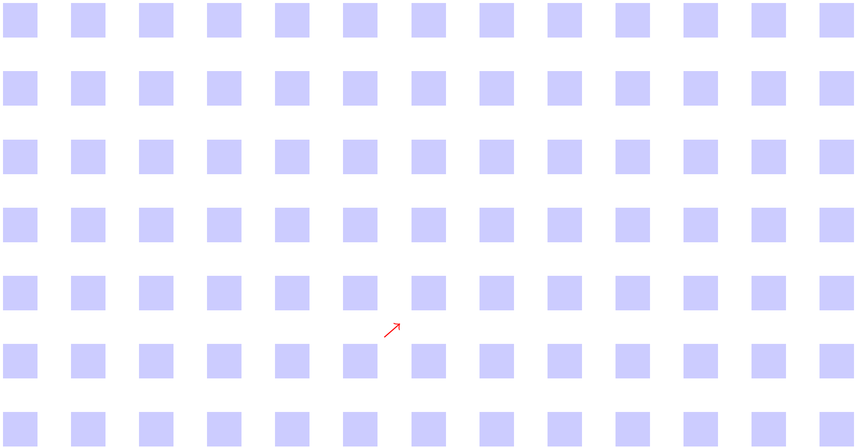

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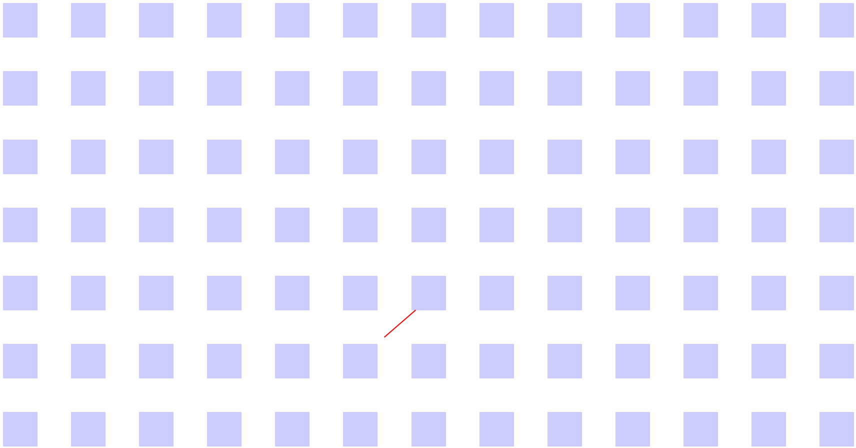

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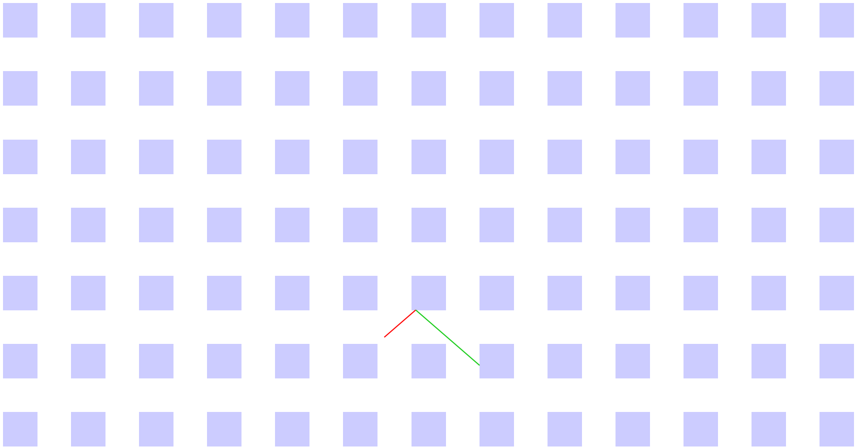

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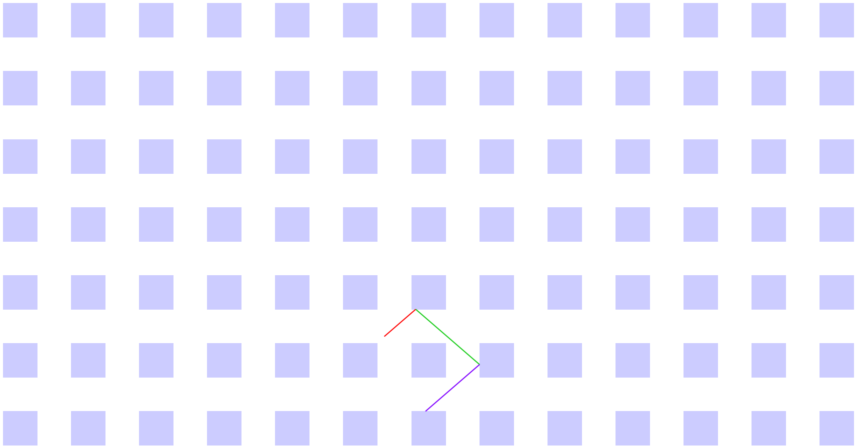

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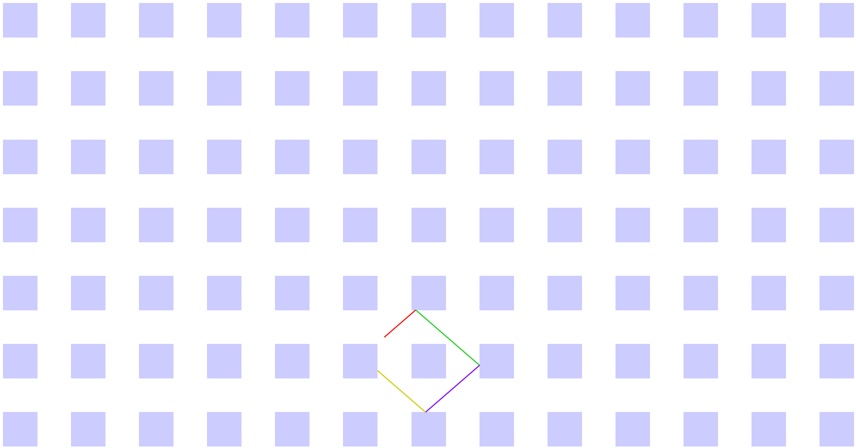

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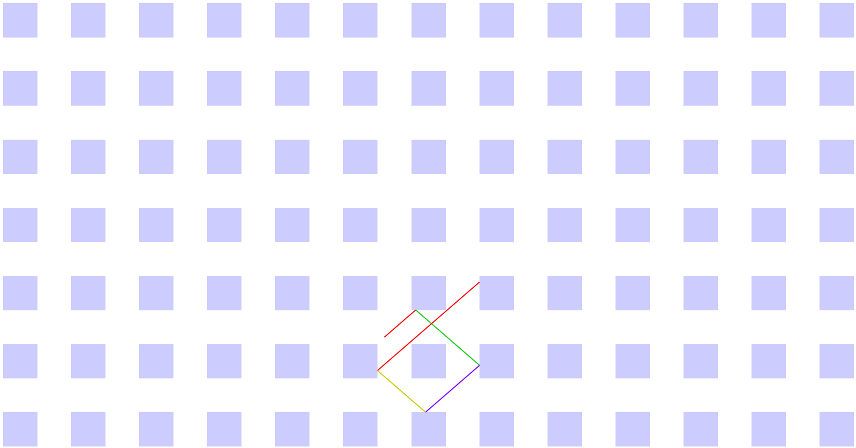

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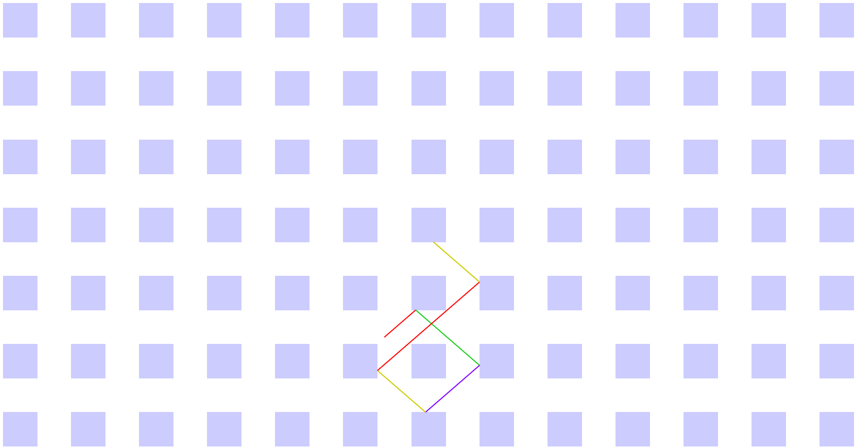

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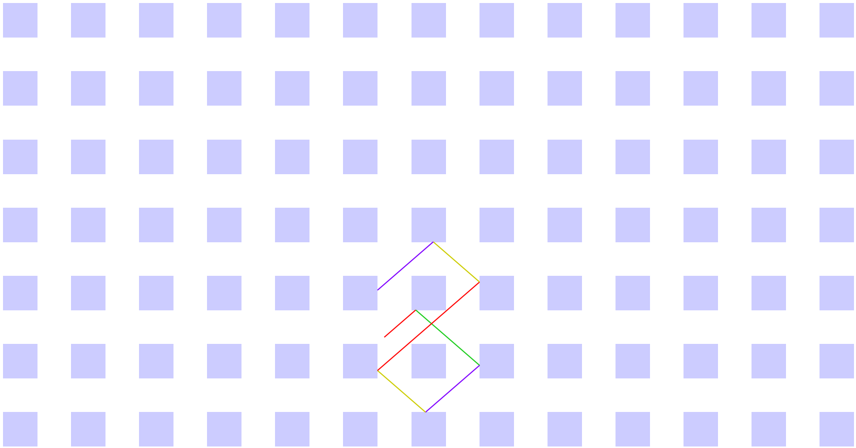

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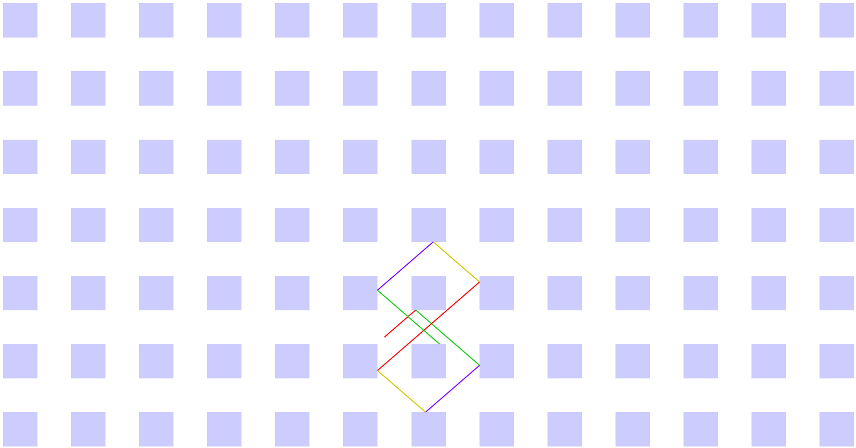

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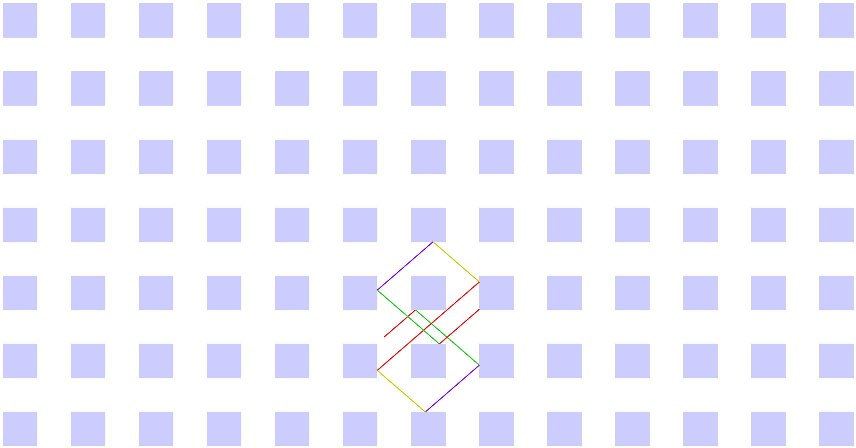

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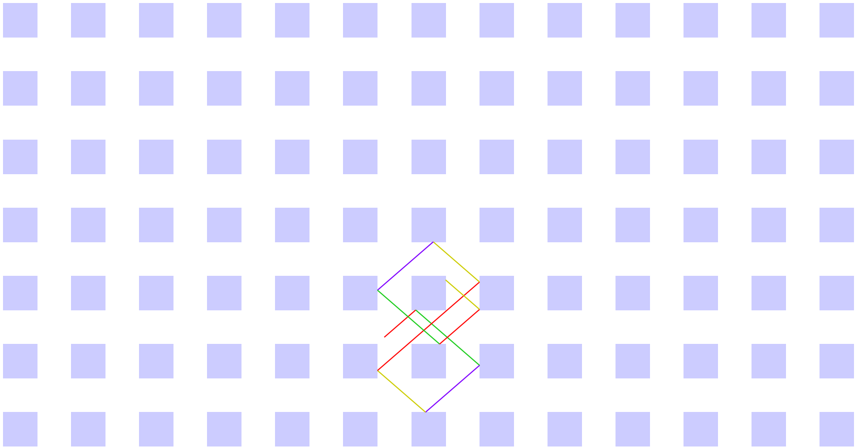

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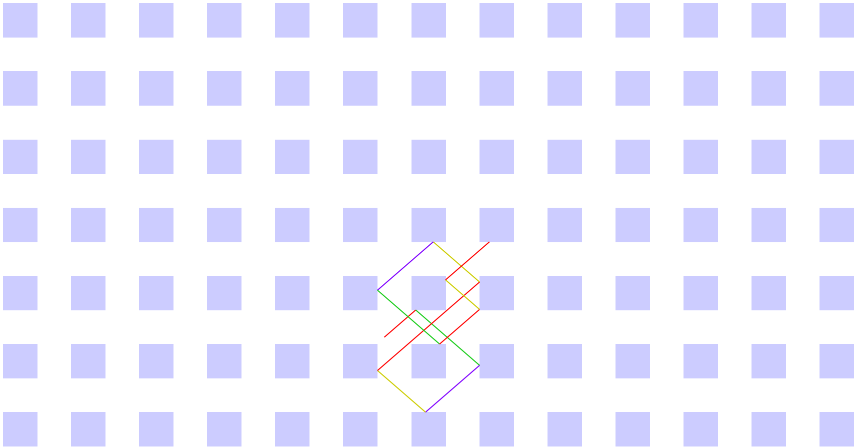

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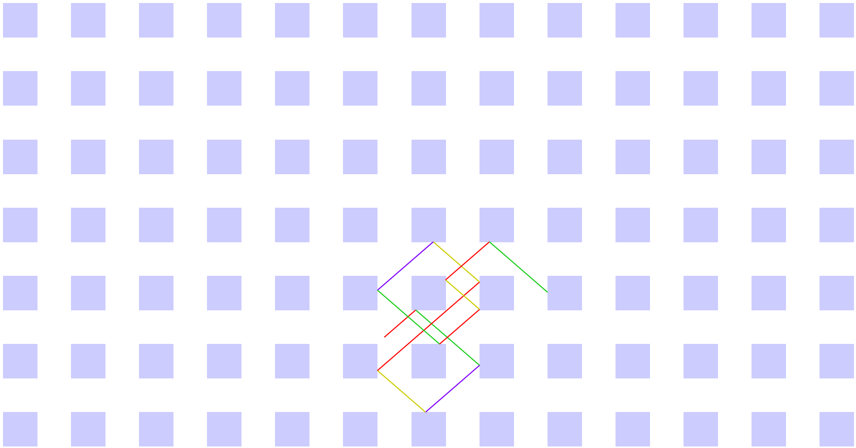

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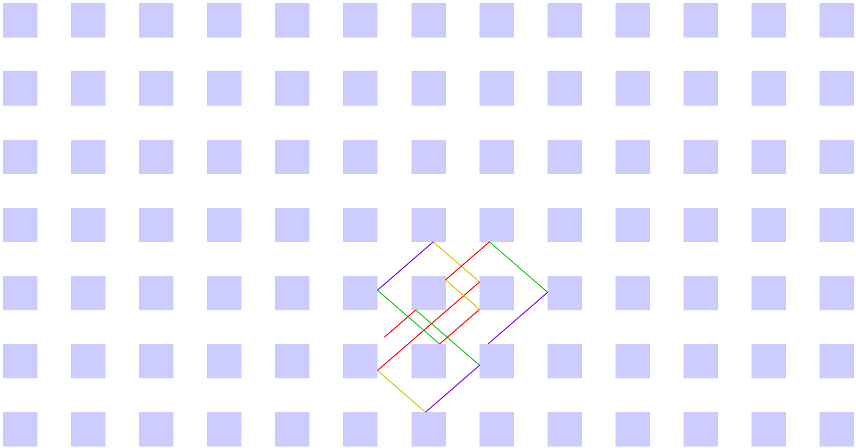

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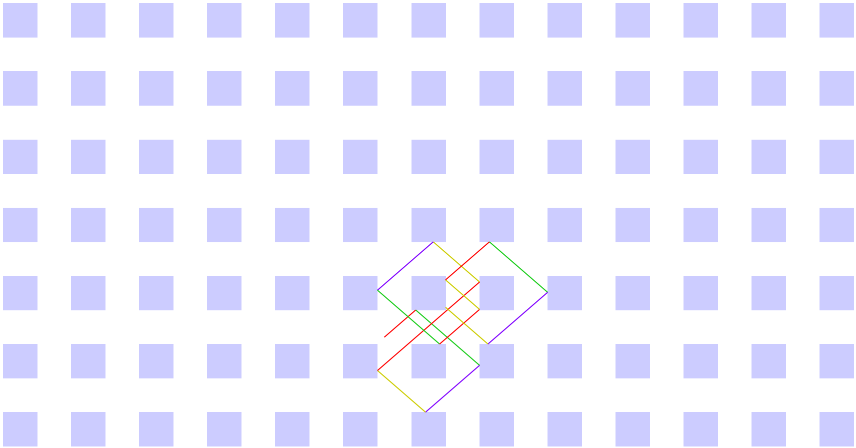

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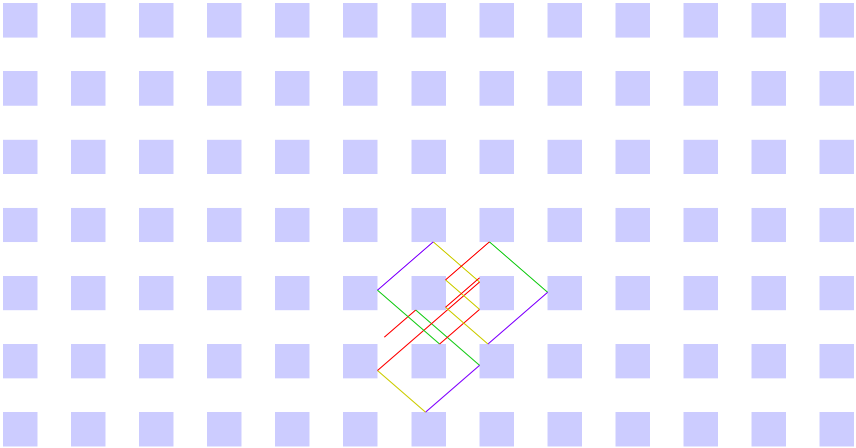

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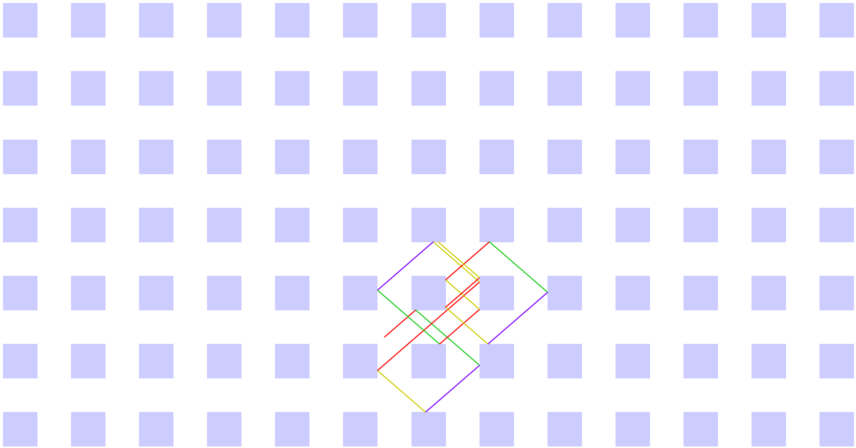

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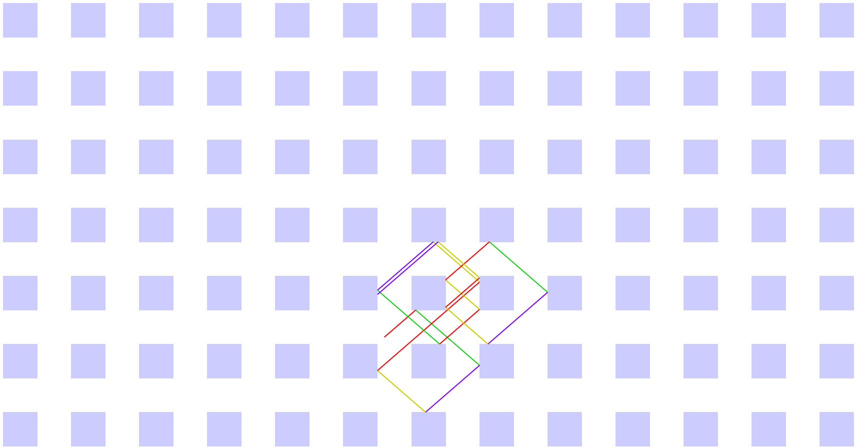

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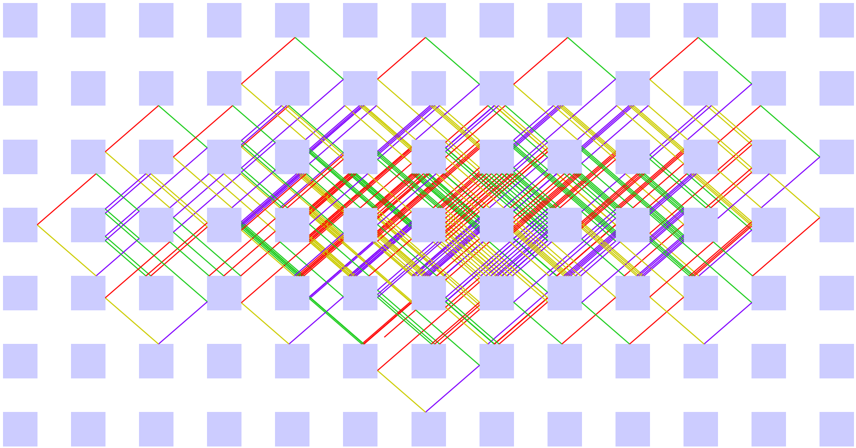

Phase space and flow of the billiard

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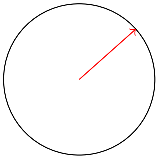

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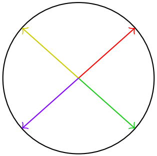

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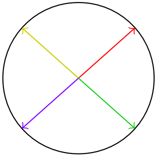

The *flow* (in direction θ) of the billiard is the map $\phi^\theta : T_{a,b} \times K \times \mathbb{R} \to T_{a,b} \times K$ which associates to a state $(p, \tau) \in T_{a,b} \times K$ and a time t, the position of a particule starting from (p, τ) after time t.

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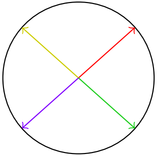

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Recurrence is generic

Theorem (Hubert-Lelièvre-Troubetzkoy)

Almost every angle θ is recurrent.

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Theorem (Delecroix)

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It follows that the set of divergent angles has a Hausdorff dimension bigger than $1/2$.

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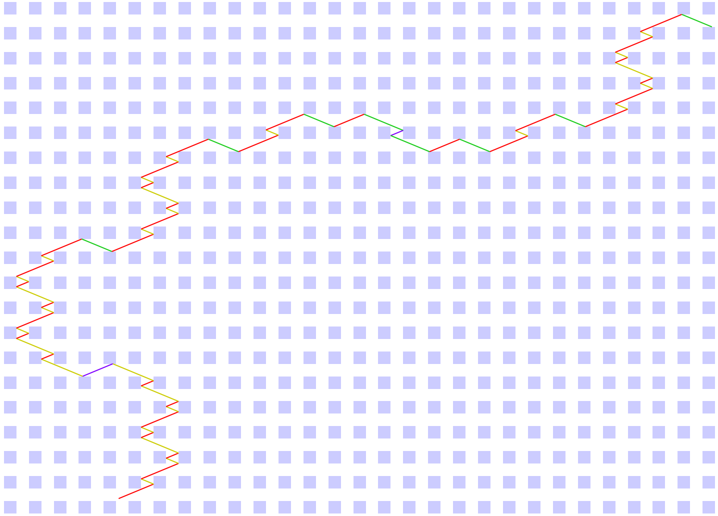
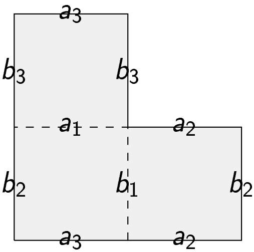
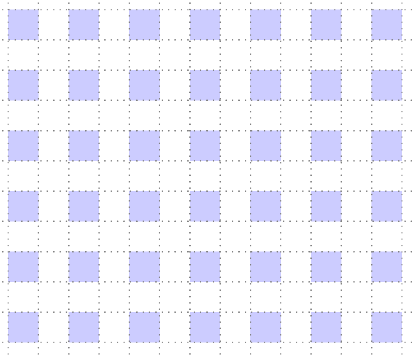


A simpler object : the L-shaped surface and a cocycle





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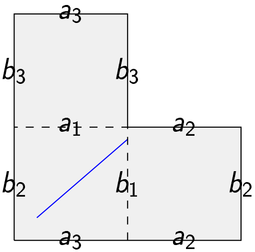

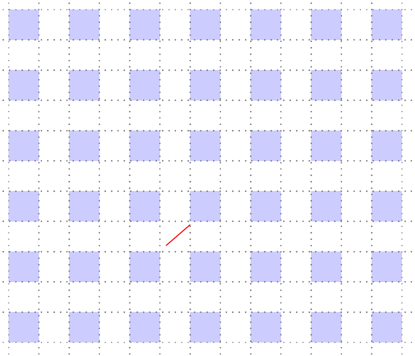

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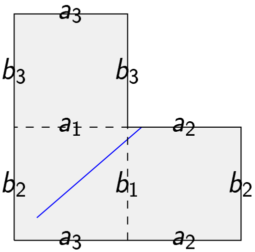

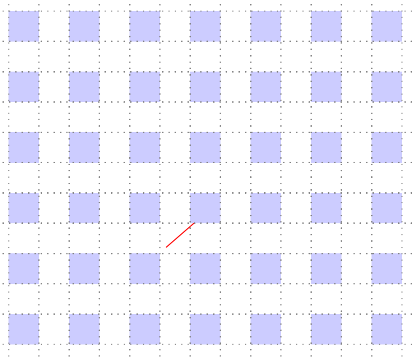

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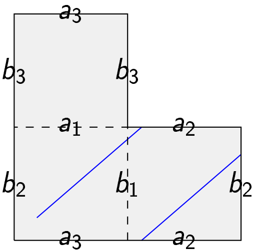

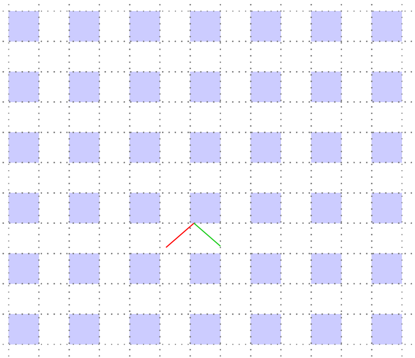

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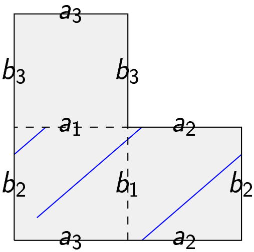

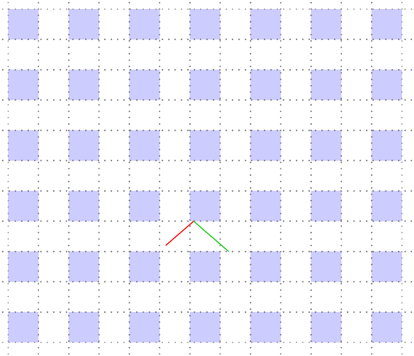

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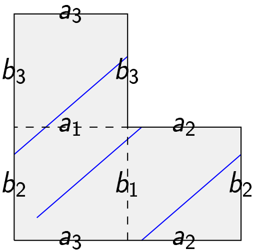

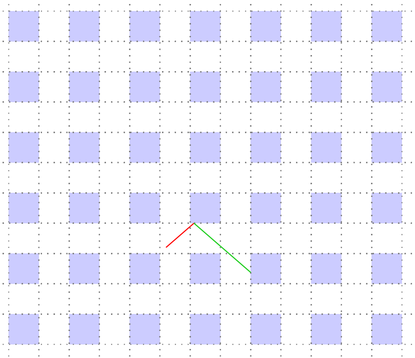

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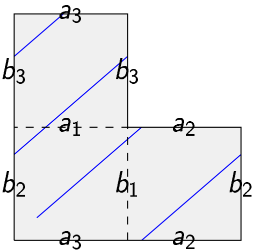

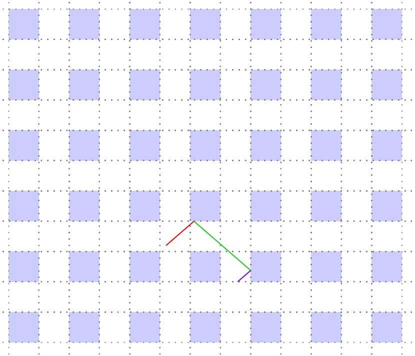

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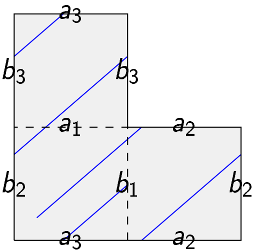

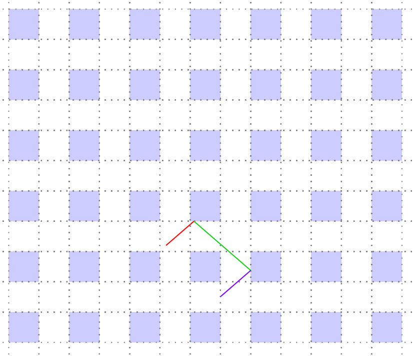

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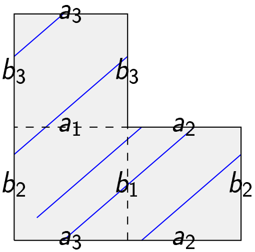

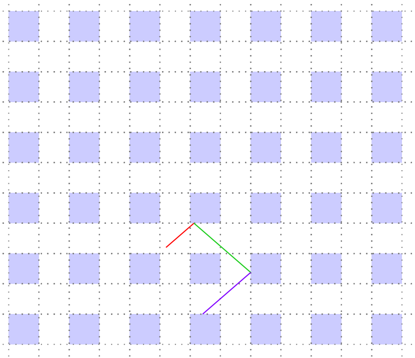

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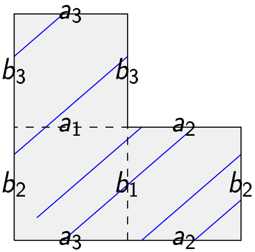

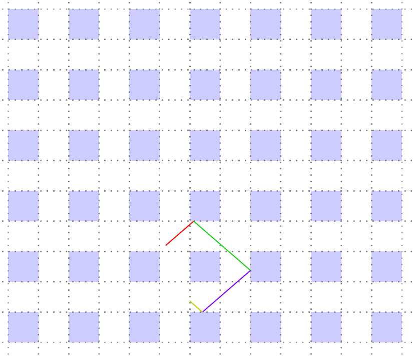

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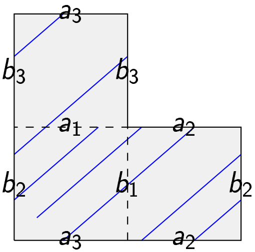

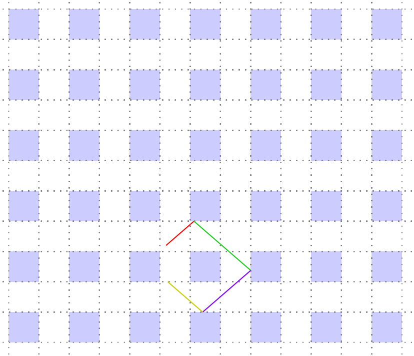

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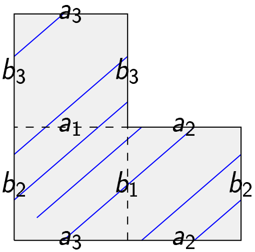

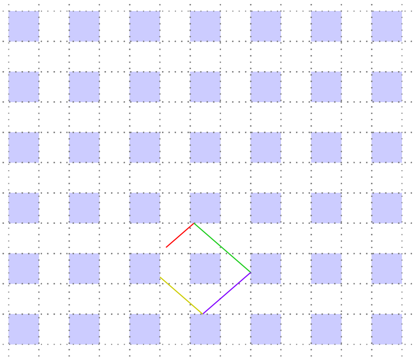

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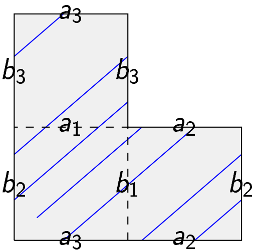

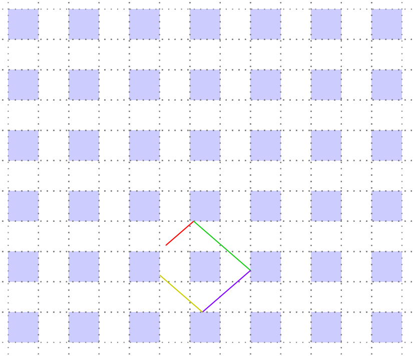

Remark that :

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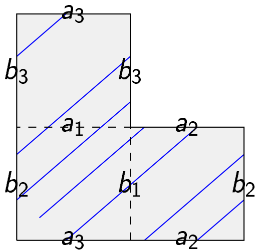

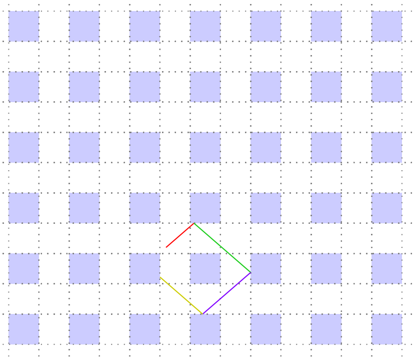

Remark that :

- Each time the side b_2 is crossed, the trajectory is reflected horizontally

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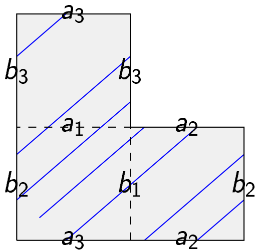

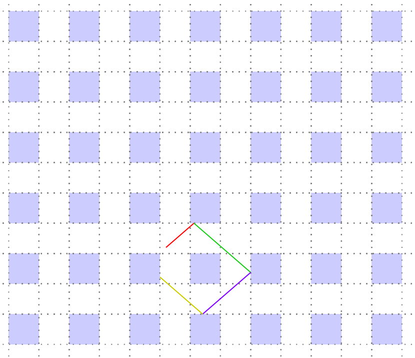

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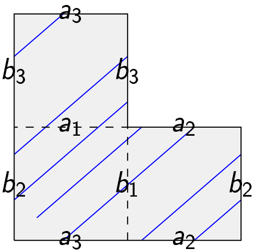

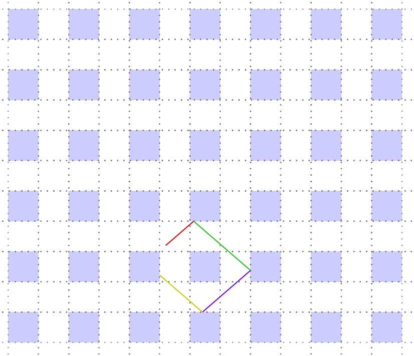

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- Each time the side b_2 is crossed, the trajectory is reflected horizontally
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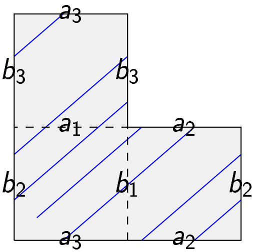

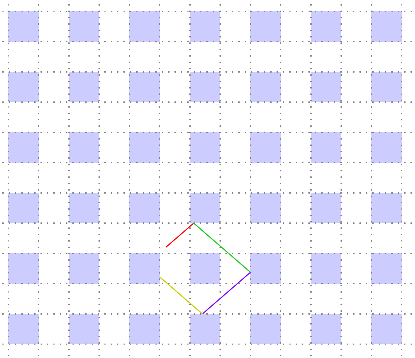

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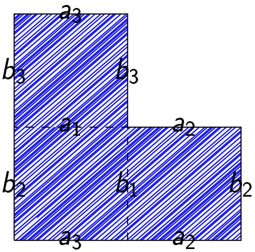
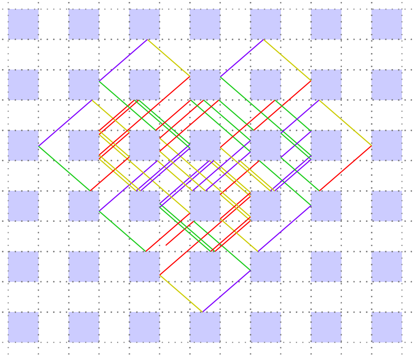
- Each time the side b_3 is crossed, the "level" changes vertically

A simpler object : the L-shaped surface and a cocycle





Remark that :

- Each time the side b_2 is crossed, the trajectory is reflected horizontally
- Each time the side a_3 is crossed, the trajectory is reflected vertically
- Each time the side a_2 is crossed, the "level" change horizontally
- Each time the side b_3 is crossed, the "level" changes vertically