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On Berndtsson's generalization of Prékopa's theorem

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Abstract. We present a new approach to Berndtsson's complex extension of Prékopa's theorem. This approach is inspired by the recent local proof of Prékopa's theorem obtained by Ball, Barthe and Naor. In the complex setting, this approach leads to a formula expressing $\Delta\Phi$ in terms of (derivatives of) φ where $e^{-\Phi(z)} = \int e^{-\varphi(z,w)}\, d\lambda(w)$. As a consequence, we obtain new conditions ensuring that Φ is subharmonic when φ is plurisubharmonic.

1. Introduction

Prékopa's theorem [13] asserts that marginals of log-concave functions are log-concave. In other words, if $\varphi : \mathbb{R}^{n+1} \to \mathbb{R} \cup \{+\infty\}$ is convex, then the function ϕ given by

$$e^{-\phi(t)} = \int_{\mathbb{R}^n} e^{-\varphi(t,x)}\, dx$$

is convex on $\mathbb{R}$. There exists many proofs and applications of this result, in particular in connection with the Brunn-Minkowski theory (see e.g. [8, 11, 12]). Of course, by modifying φ if necessary, we can assume that the integration is performed on sections of a convex set $K \subset \mathbb{R}^{n+1}$,

$$e^{-\phi(t)} = \int_{K(t)} e^{-\varphi(t,x)}\, dx$$

where $K(t) := \{x \in \mathbb{R}^n \ ;\ (t,x) \in K\}$.

Recently, Berndtsson obtained [3] a complex version of Prékopa's theorem, where, as expected, convexity is replaced by plurisubharmonicity. It is explained by Berndtsson how to recover from this version Kiselman's minimum principle. Another application to geometric functional analysis can be found in [7]. We shall denote by $d\lambda$ the Lebesgue measure on $\mathbb{C}^n \cong \mathbb{R}^{2n}$.

Theorem 1 (Berndtsson). *Let U be an open set of $\mathbb{C}$, V be a pseudo-convex domain of $\mathbb{C}^n$ and $\varphi : U \times V \longrightarrow \mathbb{R}$ be a plurisubharmonic function. Assume that one of the following conditions holds*

i) $0 \in V$ *and for every* $z \in U$, $w \in V$, $\theta \in \mathbb{R}$ *we have*

$$e^{i\theta} w \in V \quad \text{and} \quad \varphi(z, e^{i\theta} w) = \varphi(z, w).$$

ii) V *is a connected Reinhardt domain and for every* $z \in U$ *and* $w = (w_1, \dots, w_n) \in V$,

$$\varphi(z, w) = \varphi(z, |w_1|, \dots, |w_n|).$$

Then the function Φ *defined on* U *by*

$$e^{-\Phi(z)} = \int_V e^{-\varphi(z,w)} \, d\lambda(w) \tag{1}$$

is subharmonic.

Compared to the real case, the invariance requirements are new. But the example of the plurisubharmonic function $\varphi(z, w) = |z - \overline{w}|^2 - |z|^2$ on $\mathbb{C} \times \mathbb{C}$ given by Berndtsson shows that the conclusion cannot hold without further assumptions. As mentioned by Berndtsson, the result also applies if the integration is performed on "sections" of a pseudo-convex set $\Omega \subset \mathbb{C}^{n+1}$,

$$e^{-\Phi(z)} = \int_{\Omega(z)} e^{-\varphi(z,w)} \, d\lambda(w),$$

provided $\Omega(z) := \{w \in \mathbb{C}^n \, ; \ (z, w) \in \Omega\}$ satisfies the invariance conditions imposed on V. The proof of Berndtsson uses Hörmander's L^2-estimates for the $\overline{\partial}$-operator (cf. [9]), a partial converse of these estimates, and an iteration procedure consisting in adding extra dimensions. Surprisingly enough, this proof does not follow any existing proof of Prékopa's theorem. And although it would be possible to adapt Berndtsson's proof to the real case, the real counterpart of Hörmander's L^2-estimates being nothing else but (one of) the so-called Brascamp-Lieb inequality [5], this would lead to an unnecessarily involved approach of Prékopa's theorem. The aim of this note is to give a new approach to Berndtsson's result based on a recent "local" proof of Prékopa's theorem discovered by Ball, Barthe and Naor [2] (see also [1]). It is to be noted that we will also use the L^2-theory for the $\overline{\partial}$-operator.

By adding $\varepsilon|w|^2$ ($\varepsilon \to 0$) to φ if necessary, we can assume without loss of generality that $\phi(w) := \varphi(z, w)$ is uniformly plurisubharmonic on V in the sense of (3). Also, we can assume that φ is smooth (by smooth we always mean C^∞-smooth) up to the boundary of V.

The local proof consists in finding a formula for computing $\Delta\Phi(z)$ from (1). Of course, it is enough to consider the case $z = 0$. This will be the content of Theorem 4. Let $\mathbb{C}_0$ denote an arbitrary open subset of $\mathbb{C}$ containing 0. As a consequence of Theorem 4, we will get the following variant of Berndtsson's result.

Theorem 2. *Let $V \subset \mathbb{C}^n$ be a strictly pseudo-convex domain and $\varphi : \mathbb{C}_0 \times \overline{V} \longrightarrow \mathbb{R}$ be a smooth plurisubharmonic function such that $\phi(w) := \varphi(0, w)$ is uniformly plurisubharmonic* (3) *on V. We assume that the function*

$$w \longrightarrow \frac{\partial\varphi}{\partial z}(0, w)$$

is orthogonal in $L^2(e^{-\phi})$ to

$$\left\{ h \in L^2(e^{-\phi})\ ;\ \overline{\partial} h = 0 \text{ and } \int h(w) e^{-\phi(w)}\, d\lambda(w) = 0 \right\}. \qquad (2)$$

Then the function Φ given by (1) *verifies* $\Delta\Phi(0) \geq 0$.

It will be clear after Lemma 5 that if φ and V satisfy one of the conditions *i)* or *ii)* of Theorem 1, then $\frac{\partial\varphi}{\partial z}(0, \cdot)$ will satisfy the hypothesis of Theorem 2. But among the advantages of this new formulation, one can note that the hypotheses in Theorem 2 make sense in more general settings (for instance on complex manifolds). As a matter of fact, we will see in the next section that one can prove a general formula for the computation of $\Delta\Phi$ without further assumptions on φ.

The next section contains some classical L^2-formalism and our main statement (Theorem 4). The last section contains the proof of this statement and of Theorem 2. It also describes how to recover Berndtsson's result from Theorem 2.

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2. Classical background from the L^2-theory of the $\overline{\partial}$-equation and main result

We fix some *strictly* pseudo-convex domain $V \subset \mathbb{C}^n$ and a smooth plurisubharmonic function $\phi : V \longrightarrow \mathbb{R}$. For a C^2-smooth function u on $\mathbb{C}^n$, we will denote by $\mathcal{L}u_w$ the Levi form (i.e. the quadratic form associated to the complex Hessian) of u at the point w: for $\xi \in \mathbb{C}^n$,

$$\mathcal{L}u_w(\xi) := \sum_{j,k=1}^{n} \frac{\partial^2 u}{\partial w_j \partial \overline{w_k}}(w)\, \xi_j \overline{\xi_k}.$$

We will throughout the paper assume that ϕ is *uniformly plurisubharmonic* on V in the sense that

$$\forall w \in V,\ \ \forall \xi \in \mathbb{C}^n, \qquad \mathcal{L}\phi_w(\xi) = \sum_{j,k=1}^{n} \frac{\partial^2 \phi}{\partial w_j \partial \overline{w_k}}(w)\, \xi_j \overline{\xi_k} \geq c|\xi|^2, \qquad (3)$$

for some $c > 0$. We will also assume that V has a C^2-smooth boundary. Hence, V is given by some defining $\rho : \mathbb{C}^n \longrightarrow \mathbb{R}$,

$$V = \{\rho < 0\}, \quad \rho\ C^2\text{-smooth and with } d\rho \neq 0 \text{ on } \partial V. \qquad (4)$$

In particular, we have $\partial\rho := \left(\frac{\partial\rho}{\partial w_1}, \dots, \frac{\partial\rho}{\partial w_n}\right) \neq 0$ on ∂V. Since V is pseudoconvex, we have

$$\mathcal{L}\rho_w(\xi) \geq 0 \ \text{ when } w \in \partial V \text{ and } \sum_{j=1}^{n} \frac{\partial\rho}{\partial w_j}(w)\xi_j = 0 \tag{5}$$

Let us denote by $L^2(e^{-\phi}) = L^2(V, e^{-\phi})$ the Hilbert space with scalar product

$$\int f(w)\overline{g(w)}e^{-\phi(w)}\,d\lambda(w).$$

The integrations will always be implicitly on V only. We also consider the corresponding Hilbert space of (0, 1)-forms,

$$L^2_{(0,1)}(e^{-\phi}) = \left\{\alpha = \sum_{j=1}^{n} \alpha_j d\overline{w_j}\ ;\ \alpha_j \in L^2(e^{-\phi})\right\},$$

with scalar product

$$\int \alpha \cdot \overline{\beta}\, e^{-\phi} = \int \left(\sum_{j=1}^{n} \alpha_j(w)\overline{\beta_j(w)}\right) e^{-\phi(w)}\,d\lambda(w).$$

For a smooth function f, we can define

$$\overline{\partial} f = \sum_{j=1}^{n} \frac{\partial f}{\partial \overline{w_j}}\, d\overline{w_j}.$$

More generally, for a function u on V, $\overline{\partial} u$ can be defined in the distributional sense on V (the test functions are the smooth functions compactly supported inside V). Then, $\overline{\partial}$ can be viewed as a closed (unbounded) operator from $L^2(e^{-\phi})$ to $L^2_{(0,1)}(e^{-\phi})$ with (dense) domain

$$\mathcal{D}(\overline{\partial}) = \{u \in L^2(e^{-\phi})\ ;\ \overline{\partial} u \in L^2_{(0,1)}(e^{-\phi})\}.$$

Its adjoint (in the usual operator sense) will be denoted by $\overline{\partial}^*_\phi$. On smooth (0, 1)-forms α compactly supported in V, we have

$$\overline{\partial}^*_\phi \alpha = -e^{\phi} \sum_{j=1}^{n} \frac{\partial(e^{-\phi}\alpha_j)}{\partial w_j}.$$

We have, whenever it makes sense, for a function f and a (0, 1)-form α, $\int \overline{\partial} f \cdot \overline{\alpha}\, e^{-\phi} = \int f \overline{\overline{\partial}^*_\phi \alpha}\, e^{-\phi}$. We now introduce the operator L on $L^2(e^{-\phi})$ defined by

$$L := -\overline{\partial}^*_\phi \circ \overline{\partial}. \tag{6}$$

The operator L is self-adjoint and has a dense domain. We have, for $f \in \mathcal{D}(\overline{\partial})$ and $g \in \mathcal{D}(L)$,

$$\int f\overline{Lg}\, e^{-\phi} = -\int \overline{\partial} f \cdot \overline{\overline{\partial} g}\, e^{-\phi}. \tag{7}$$

The main integration by parts formula of [10] (see also [4,6]) applied to $\alpha = \overline{\partial} u$ gives for every $g \in \mathcal{D}(L)$ smooth on $\overline{V}$,

$$\int |Lg|^2 e^{-\phi} = \int \left[\mathcal{L}\phi(\overline{\partial} g) + \sum_{j,k} \left| \frac{\partial^2 g}{\partial \overline{w}_j \partial \overline{w_k}} \right|^2 \right] e^{-\phi} + \int_{\partial V} \mathcal{L}\rho(\overline{\partial} g) \frac{e^{-\phi} d\sigma}{|\partial \rho|}. \tag{8}$$

where $d\sigma$ denotes the element of volume on ∂V (4). Here, for simplicity, we have made the identification

$$\overline{\partial} g = \left(\frac{\partial g}{\partial \overline{w}_1}, \dots, \frac{\partial g}{\partial \overline{w}_n} \right) \in \mathbb{C}^n.$$

Note that for a smooth $g \in \mathcal{D}(L)$, one has, since, $\overline{\partial} g \in \mathcal{D}(\overline{\partial}^*_\phi)$,

$$\sum_{j=1}^{n} \frac{\partial g}{\partial \overline{w}_j} \frac{\partial \rho}{\partial w_j} = 0 \quad \text{on } \partial V. \tag{9}$$

Combining (5) and (9), we have, for $g \in \mathcal{D}(L)$:

$$\mathcal{L}\rho(\overline{\partial} g) \geq 0 \quad \text{on } \partial V. \tag{10}$$

We denote by H_ϕ the closed subspace of all the holomorphic functions of $L^2(e^{-\phi})$:

$$H_\phi = \{h \in L^2(e^{-\phi}) \ ; \ \overline{\partial} h = 0\} = \ker L. \tag{11}$$

Let us recall the following classical existence result:

Lemma 3. *Let $f \in L^2(e^{-\phi})$ be a function belonging to the orthogonal of H_ϕ. Then there exists $g \in \mathcal{D}(L)$ such that $Lg = f$. Furthermore, if f is smooth on $\overline{V}$, then g is smooth on $\overline{V}$.*

The existence of g is mainly due to Hörmander [9,10]. The regularity of the solution (up to the boundary of V) follows from the regularity theory for the $\overline{\partial}$-Neumann problem (see for instance [6]).

Note that constant functions belong to H_ϕ. The subspace of H_ϕ defined by

$$\mathcal{H}_\phi := \left\{ h \in L^2(e^{-\phi}) \ ; \ \overline{\partial} h = 0 \text{ and } \int h(w) e^{-\phi(w)}\, d\lambda(w) = 0 \right\}. \tag{12}$$

is precisely the orthogonal *in* H_ϕ of the constant functions: $H_\phi = \mathbb{C}1 \oplus \mathcal{H}_\phi$. In fact, it should be noted that for every $u \in L^2(e^{-\phi})$,

$$u \in H_\phi \Longleftrightarrow u - \frac{\int u\, e^{-\phi}}{\int e^{-\phi}} \in \mathcal{H}_\phi.$$

Thus every function $f \in L^2(e^{-\phi})$ admits the following decomposition:

$$f(w) = \frac{\int f\, e^{-\phi}}{\int e^{-\phi}} + H(w) + F(w)$$

with H in $\mathcal{H}_\phi$ (12) and $F \in L^2(e^{-\phi})$ orthogonal to H_ϕ (11). In view of Lemma 3, such an F can be written as $F = Lg$ for some $g \in L^2(e^{-\phi})$, g smooth when f is smooth. Thus every $f \in L^2(e^{-\phi})$ has a decomposition as

$$f(w) = \frac{\int f\, e^{-\phi}}{\int e^{-\phi}} + H(w) + Lg(w) \tag{13}$$

with $H \in \mathcal{H}_\phi$ (12), $g \in L^2(e^{-\phi})$. The two first terms of this decomposition give the holomorphic component of f. We can now state our main result, which can be viewed as a complex extension of the "basic formula" of [1,2].

Theorem 4. *Let* $\varphi : \mathbb{C}_0 \times \overline{V} \longrightarrow \mathbb{R}$ *be a smooth function where* $V \subset \mathbb{C}^n$ *is a strictly pseudo-convex domain given by* (4). *Assume that the function* $\phi := \varphi(0, \cdot) : \overline{V} \longrightarrow \mathbb{R}$ *is uniformly plurisubharmonic* (3) *on* V. *Consider the decomposition in* $L^2(e^{-\phi})$ *of* $w \longrightarrow \frac{\partial\varphi}{\partial z}(0, w)$ *as*

$$\frac{\partial\varphi}{\partial z}(0, w) = \frac{\int \frac{\partial\varphi}{\partial z}(0, \cdot)\, e^{-\phi}}{\int e^{-\phi}} + H(w) + Lg(w), \tag{14}$$

with $H \in \mathcal{H}_\phi$ (12) *and* $g \in L^2(e^{-\phi})$ *smooth on* $\overline{V}$. *If* Φ *denotes the function given by*

$$e^{-\Phi(z)} = \int_V e^{-\varphi(z,w)}\, d\lambda(w),$$

then, $e^{-\Phi(0)}\Delta\Phi(0)/4$ *is equal to*

$$\int \left[\mathcal{L}\varphi_{(0,\cdot)}\big(1, \overline{\partial} g\big) + \sum_{j,k} \left| \frac{\partial^2 g}{\partial \overline{w}_j \partial \overline{w}_k} \right|^2 - |H|^2 \right] e^{-\phi} + \int_{\partial V} \mathcal{L}\rho(\overline{\partial} g) \frac{e^{-\phi} d\sigma}{|\partial\rho|}. \tag{15}$$

In (15), $(1, \overline{\partial} g)$ stands for the $(n+1)$-dimensional vector $\left(1, \frac{\partial g}{\partial \overline{w}_1}, \dots, \frac{\partial g}{\partial \overline{w}_n}\right)$. For the result to make sense, we have of course implicitly assumed that $\frac{\partial\varphi}{\partial z}(0, \cdot) \in L^2(e^{-\phi})$ and that all the second derivatives $w \longrightarrow \partial^2_{\alpha\overline{\beta}}\, \varphi(0, w)$ belong to $L^2(e^{-\phi})$

for $\alpha, \beta = 1, \dots, n+1$. We can note that the result can as well be stated using the (0, 1)-form $\alpha = \overline{\partial} g$ coming from the decomposition

$$\frac{\partial \varphi}{\partial z}(0, w) = \frac{\int \frac{\partial \varphi}{\partial z}(0, \cdot)\, e^{-\phi}}{\int e^{-\phi}} + H(w) + \overline{\partial}^*_\phi \alpha(w).$$

The boundary term in (15) is non-negative, in view of (10). In the case $V = \mathbb{C}^n$, if we add the hypothesis that $H = 0$ (or equivalently that $\frac{\partial \varphi}{\partial z}(0, \cdot)$ is orthogonal to $\mathcal{H}_\phi$ (12)), then we have

$$e^{-\Phi(0)} \Delta \Phi(0)/4 = \int \left[\mathcal{L}\varphi_{(0,\cdot)}\big(1, \overline{\partial} g\big) + \sum_{j,k} \left| \frac{\partial^2 g}{\partial \overline{w}_j \partial \overline{w_k}} \right|^2 \right] e^{-\phi}.$$

3. Proofs of the Theorems

Proof of Theorem 4. We start by trying to adapt the real proof of Artstein, Ball, Barthe and Naor [1,2].

For a complex function u, we have $\frac{\partial u}{\partial \overline{z}} = \overline{\frac{\partial \overline{u}}{\partial z}}$. Thus, if $u : \mathbb{C}_0 \to (0, +\infty)$, we have,

$$u(0) \frac{\partial^2(-\log u)}{\partial z \partial \overline{z}}(0) = -\frac{\partial^2 u}{\partial z \partial \overline{z}}(0) + \frac{1}{u(0)} \left| \frac{\partial u}{\partial z}(0) \right|^2. \tag{16}$$

For $\varphi : \mathbb{C}_0 \times V \to \mathbb{R}$ and $\phi(w) = \varphi(0, w)$ as in the theorem, we introduce

$$u(z) := e^{-\Phi(z)} = \int_V e^{-\varphi(z,w)}\, d\lambda(w).$$

We are looking for a formula for

$$X := u(0) \frac{\partial^2(-\log u)}{\partial z \partial \overline{z}}(0) = e^{-\Phi(0)} \Delta \Phi(0)/4$$

We have, in view of (16),

$$\begin{aligned} X &= -\int \frac{\partial^2(e^{-\varphi})}{\partial z \partial \overline{z}}(0, w)\, d\lambda(w) + \frac{1}{\int e^{-\phi}} \left| \int \frac{\partial \varphi}{\partial z}(0, w)\, e^{-\phi(w)}\, d\lambda(w) \right|^2 \\ &= \int \left(\frac{\partial^2 \varphi}{\partial z \partial \overline{z}}(0, w) - \left| \frac{\partial \varphi}{\partial z}(0, w) \right|^2 \right) e^{-\phi(w)}\, d\lambda(w) \\ &\quad + \frac{1}{\int e^{-\phi}} \left| \int \frac{\partial \varphi}{\partial z}(0, w)\, e^{-\phi(w)}\, d\lambda(w) \right|^2 \end{aligned} \tag{17}$$

We will now use the decomposition

$$\frac{\partial \varphi}{\partial z}(0, w) = \frac{\int \frac{\partial \varphi}{\partial z}(0, \cdot)\, e^{-\phi}}{\int e^{-\phi}} + H(w) + Lg(w),$$

with $g \in \mathcal{D}(L)$ smooth on $\overline{V}$ and $H \in \mathcal{H}_\phi$ (12). By integration of

$$\left| \frac{\int \frac{\partial\varphi}{\partial z}(0,\cdot)e^{-\phi}}{\int e^{-\phi})} + H(w) \right|^2 = \left| \frac{\partial \varphi}{\partial z}(0,w) - Lg(w) \right|^2$$

with respect to $e^{-\phi(w)}\, d\lambda(w)$, we obtain, using that H is orthogonal to constants,

$$\int \left| \frac{\int \frac{\partial\varphi}{\partial z}(0,\cdot)e^{-\phi}}{\int e^{-\phi}} \right|^2 e^{-\phi(w)}\, d\lambda(w) + \int |H(w)|^2 e^{-\phi(w)}\, d\lambda(w)$$
$$= \int \left| \frac{\partial \varphi}{\partial z}(0,w) - Lg(w) \right|^2 e^{-\phi(w)}\, d\lambda(w).$$

We therefore have

$$\frac{1}{\int e^{-\phi}} \left| \int \frac{\partial\varphi}{\partial z}(0,w) e^{-\phi(w)}\, d\lambda(w) \right|^2$$
$$= \int \left(\left| \frac{\partial \varphi}{\partial z}(0,w) - Lg(w) \right|^2 - |H(w)|^2 \right) e^{-\phi(w)}\, d\lambda(w). \tag{18}$$

Combining (17) and (18) we find

$$X = \int \left[\frac{\partial^2\varphi}{\partial z \partial \overline{z}}(0,w) - 2\Re \left(\frac{\partial\varphi}{\partial z}(0,w)\, \overline{Lg(w)} \right) + |Lg|^2 - |H|^2 \right] e^{-\phi(w)}\, d\lambda(w).$$

By (7), we have

$$X = \int \left[\frac{\partial^2\varphi}{\partial z \partial \overline{z}}(0,w) + 2\Re \left(\sum_{j=1}^n \frac{\partial^2\varphi}{\partial \overline{w}_j \partial z}(0,w)\, \overline{\frac{\partial g}{\partial \overline{w}_j}} \right) + |Lg|^2 - |H|^2 \right] e^{-\phi(w)}\, d\lambda(w).$$

Using (8) we then conclude that

$$X = \int \left[\mathcal{L}\varphi_{(0,\cdot)}\big(1, \overline{\partial} g\big) + \sum_{j,k} \left| \frac{\partial^2 g}{\partial \overline{w}_j \partial \overline{w_k}} \right|^2 - |H|^2 \right] e^{-\phi} + \int_{\partial V} \mathcal{L}\rho(\overline{\partial} g) \frac{e^{-\phi} d\sigma}{|\partial\rho|}.$$

□

Proof of Theorem 2. We apply Theorem 4. The assumption on $\frac{\partial\varphi}{\partial z}(0,\cdot)$ ensures that $H = 0$ in the decomposition (14). Therefore we have

$$e^{-\Phi(0)} \Delta\Phi(0)/4 = \int \left[\mathcal{L}\varphi_{(0,\cdot)}\big(1, \overline{\partial} g\big) + \sum_{j,k} \left| \frac{\partial^2 g}{\partial \overline{w}_j \partial \overline{w_k}} \right|^2 \right] e^{-\phi} + \int_{\partial V} \mathcal{L}\rho(\overline{\partial} g) \frac{e^{-\phi} d\sigma}{|\partial\rho|}.$$

The boundary term is non-negative in view of (10). Since φ is plurisubharmonic, we have $\mathcal{L}\varphi_{(0,w)}\Big(1, \frac{\partial g}{\partial \overline{w}_1}(w), \dots, \frac{\partial g}{\partial \overline{w}_n}(w)\Big) \geq 0$, and thus we can conclude that $\Delta\Phi(0) \geq 0$. □

Finally, it is time to explain how to recover Berndtsson's theorem from Theorem 2. First, we note that it costs no generality to assume that V is a strictly pseudo-convex domain given by (4) and that φ is smooth (and uniformly plurisubharmonic in w on V). For a function u and $\theta_1, \dots, \theta_n \in \mathbb{R}$, let us introduce (whenever it makes sense) the function $u^{\theta_1,\dots,\theta_n}$ given by

$$u^{\theta_1,\dots,\theta_n}(w) = u(e^{i\theta_1}w_1, \dots, e^{i\theta_n}w_n).$$

Let us also set, for $\theta \in \mathbb{R}$, $u^\theta = u^{\theta,\dots,\theta}$. We recall that a set V is said to be *circled* if

$$w \in V \Longrightarrow \forall\theta \in \mathbb{R}, \quad e^{i\theta}w \in V.$$

Theorem 1 follows from Theorem 2 and from the next lemma applied to $\phi(w) = \varphi(z, w)$ and $f(w) = \frac{\partial\varphi}{\partial z}(z, w)$.

Lemma 5. *Let $V \subset \mathbb{C}^n$ be a pseudo-convex domain, ϕ be a uniformly plurisubharmonic function* (3) *on V, and $f : V \longrightarrow \mathbb{C}$ be an element of $L^2(e^{-\phi})$. Assume that one of the following situations holds*

i) V is circled, $0 \in V$ and for every $\theta \in \mathbb{R}$, $\phi^\theta = \phi$, $f^\theta = f$ on V.

ii) V is a connected Reinhardt domain and for every $\theta_1, \dots, \theta_n \in \mathbb{R}$, $\phi^{\theta_1,\dots,\theta_n} = \phi$ and $f^{\theta_1,\dots,\theta_n} = f$ on V.

Then, f is orthogonal in $L^2(e^{-\phi})$ to the space $\mathcal{H}_\phi$ (12).

Proof. Let $F_0 \in H_\phi$ and F_1 orthogonal to H_ϕ be such that

$$f = F_0 + F_1.$$

Note that, in each situation, if a given h is holomorphic on V, then so are h^θ and $h^{\theta_1,\dots,\theta_n}$, respectively. Furthermore, in both situations f and the "metric" $e^{-\phi}$ satisfy the same invariances. Thus the projections F_0 and F_1 also satisfy these invariances. Therefore, we are in one of the following situations:

i) V is circled, $0 \in V$ and for every $\theta \in \mathbb{R}$, $F_0^\theta = F_0$ on V.

ii) V is a connected Reinhardt domain and for every $\theta_1, \dots, \theta_n \in \mathbb{R}$, $F_0^{\theta_1,\dots,\theta_n} = F_0$ on V.

It was noted and used by Berndtsson [3] that a function F_0 holomorphic on V and satisfying one of these conditions is constant. By construction, this implies that $f = (cst) + F_1$ belongs to the orthogonal of $\mathcal{H}_\phi$ (12). □

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