Inverse Brascamp-Lieb inequalities along the Heat equation

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Abstract

Adapting Borell's proof of Ehrhard's inequality for general sets, we provide a semi-group approach to the reverse Brascamp-Lieb inequality, in its "convexity" version.

1 Introduction

We work in the Euclidean space $(\mathbb{R}^n, \langle\cdot,\cdot\rangle, |\cdot|)$. Given m vectors $(u_i)_{i=1}^m$ in $\mathbb{R}^n$ and m numbers $(c_i)_{i=1}^m$, we shall say that they decompose the identity of $\mathbb{R}^n$ if

$$c_i > 0, \quad |u_i| = 1, \quad \sum_{i=1}^m c_i\, u_i \otimes u_i = I_n. \tag{1}$$

This relation ensures that for every $x \in \mathbb{R}^n$,

$$x = \sum_{i=1}^m c_i \langle x, u_i\rangle u_i, \quad \text{and} \quad |x|^2 = \sum_{i=1}^m c_i \langle x, u_i\rangle^2.$$

Such a decomposition appears in John's description of the maximal volume ellipsoid. Keith Ball was able to adapt a family of inequalities by Brascamp and Lieb [9] to this setting. He used it in order to derive several optimal estimates of volume ratios and volumes of sections of convex bodies [2, 3]. His version of the Brascamp-Lieb inequality is as follows. Consider for $i = 1, \dots, m$, vectors $u_i \in \mathbb{R}^n$, and numbers c_i satisfying the decomposition relation (1), then for all non-negative integrable functions $f_i : \mathbb{R} \to \mathbb{R}^+$ one has

$$\int_{\mathbb{R}^n} \prod_{i\le m} f_i^{c_i}(\langle x, u_i\rangle)\, dx \le \prod_{i\le m} \left(\int_{\mathbb{R}} f_i\right)^{c_i}. \tag{2}$$

He also conjectured a reverse inequality, which was established by the first-named author [5, 4], and had also several applications in convex geometry. It asserts that under the same assumptions, if a measurable $h : \mathbb{R}^n \to \mathbb{R}^+$ verifies for all $(\theta_1, \dots, \theta_m) \in \mathbb{R}^m$

$$h\Big(\sum_{i\le m} c_i \theta_i u_i\Big) \ge \prod_{i\le m} f_i^{c_i}(\theta_i), \tag{3}$$

then

$$\int_{\mathbb{R}^n} h \ge \prod_{i\le m} \left(\int_{\mathbb{R}} f_i\right)^{c_i}. \tag{4}$$

Up to now the shortest proof of (2) and (4) stands on measure transportation [4]. This tool recently appeared as a powerful challenger to the semi-group interpolation method and provided new approaches and extensions of Sobolev type inequalities and other inequalities related to concentrations. For a presentation of the semi-group and of the mass transport methods, one can consult [11, 1] and [12], respectively. The transport method seemed more powerful for Brascamp-Lieb and Brunn-Minkowski type inequalities.

But we shall see that the semi-group approach is still alive! In the next section we adapt a recent argument of Christer Borell [6]. He gave an impressive proof of Ehrhard's inequality for general sets based on the precise study of inequalities along the Heat semi-group. We recover by similar arguments the reverse Brascamp-Lieb inequality (4). It is to be mentioned that Borell already used (somewhat different) diffusion semi-groups arguments in the setting of Prékopa–Leindler type inequalities [7, 8]. It is natural to ask whether there exists a semi-group proof of (2) since the two inequalities (2) and (4) were obtained simultaneously in the transport method. The answer is yes. We learned from Eric Carlen of a major work in preparation of Carlen, Lieb and Loss [10] containing a semi-group approach of quite general Brascamp-Lieb inequalities with striking applications to kinetic theory. The (surprisingly) short argument we present in the third section is nothing else but a particular case of their work where "geometry" provides welcome simplifications.

2 Proof of the inverse Brascamp-Lieb inequality

We work with the heat semigroup defined for $f : \mathbb{R}^n \to \mathbb{R}^+$ and $t \geq 0$ by

$$P_t f(x) = \int_{\mathbb{R}^n} f(x + \sqrt{t}\, z)\, d\gamma_n(z),$$

where γ_n is the standard Gaussian measure on $\mathbb{R}^n$, with density $(2\pi)^{-n/2}\exp(-|z|^2/2)$, $z \in \mathbb{R}^n$ with respect to Lebesgue's measure. Under appropriate assumptions, $g(t,x) = P_t f(x)$ solves the equation

$$\begin{cases} \dfrac{\partial g}{\partial t} = \dfrac{1}{2}\Delta g \\[2ex] g(0,\cdot) = f \end{cases}$$

In this equation and in the rest of the paper, the Laplacian and the gradient act on space variables only. Note that the function $(t,x) \mapsto F^{(t)}(x) = \log P_t f(x)$ satisfies

$$\begin{cases} 2\dfrac{\partial F^{(t)}}{\partial t}(x) = \Delta F^{(t)}(x) + |\nabla F^{(t)}(x)|^2 \\[2ex] F^{(0)} = \log f \end{cases} \tag{5}$$

Let $(u_i)_{i=1}^m$ be vectors in $\mathbb{R}^n$ and $(c_i)_{i=1}^m$ be non-negative numbers, inducing a decomposition of the identity (1). Let $(f_i)_{i=1}^m$ be non-negative integrable functions on $\mathbb{R}$ and let h on $\mathbb{R}^n$ satisfy (3). Introduce $H^{(t)} = \log P_t h : \mathbb{R}^n \to \mathbb{R}$ and $F_i^{(t)} = \log P_t f_i : \mathbb{R} \to \mathbb{R}$, where we use the same notation for the n-dimensional and the one dimensional semigroups. We define

$$\begin{aligned} C(t,\theta_1,\ldots,\ \theta_m) \ &:= \ H^{(t)}\left(\sum_{i=1}^m c_i\theta_i u_i\right) - \sum_{i=1}^m c_i F_i^{(t)}(\theta_i) \qquad (6) \\ &= \ H^{(t)}(\Theta) - \sum_{i=1}^m c_i F_i^{(t)}(\theta_i) \end{aligned}$$

with the notation $\Theta = \Theta(\theta_1,\ldots,\theta_m) = \sum_{i=1}^m c_i\theta_i\, u_i \in \mathbb{R}^n$. The evolution equations (5) satisfied by $F_i^{(t)}$ and $H^{(t)}$ give

$$\begin{aligned} 2\frac{\partial C}{\partial t}(t,\theta_1,\ldots,\ \theta_m) \ &= \ \Delta H^{(t)}(\Theta) - \sum_{i=1}^m c_i (F_i^{(t)})''(\theta_i) \\ &+ \ |\nabla H^{(t)}|^2(\Theta) - \sum_{i=1}^m c_i (F_i^{(t)})'^2(\theta_i). \end{aligned}$$

For simplicity, we shall write $H = H^{(t)}$ and $F_i = F_i^{(t)}$, keeping in mind that we are working at a fixed t. Following Borell we try to express the right hand side in terms of C. We start with the second order terms. From the definition (6), we get

$$\frac{\partial^2 C}{\partial\theta_i\partial\theta_j}(t,\theta_1,\dots,\theta_m) = c_i c_j \langle D^2 H(\Theta)u_i, u_j\rangle - \delta_{i,j}\, c_i F''(\theta_i).$$

By the decomposition of the identity (1), we have $D^2H = \sum c_i (D^2 H u_i)\otimes u_i$ and

$$\Delta H = \mathrm{tr} D^2 H = \sum_{i\le m} c_i \langle D^2 H u_i, u_i\rangle = \sum_{i,j\le m} c_i c_j \langle u_i, u_j\rangle \langle D^2 H u_i, u_j\rangle.$$

Therefore if we denote by $\mathcal{E}$ the degenerate elliptic operator

$$\mathcal{E} = \sum_{i,j\le m} \langle u_i, u_j\rangle \frac{\partial^2}{\partial\theta_i\partial\theta_j},$$

we have, since $|u_i| = 1$,

$$\mathcal{E}C(t,\theta_1,\dots,\theta_m) = \Delta H(\Theta) - \sum_{i=1}^m c_i F_i''(\theta_i). \tag{7}$$

Next we turn to first order terms. The gradient of C has coordinates

$$\frac{\partial C}{\partial\theta_i}(t,\theta_1,\dots,\theta_m) = c_i\left(\langle\nabla H(\Theta), u_i\rangle - F_i'(\theta_i)\right).$$

We introduce the vector valued function $b : \mathbb{R}\times\mathbb{R}^m \to \mathbb{R}^m$ with coordinates

$$b_i(t,\theta_1,\dots,\theta_m) = \langle\nabla H(\Theta), u_i\rangle + F_i'(\theta_i), \tag{8}$$

so that

$$\begin{aligned}\langle b, \nabla C\rangle(t,\theta_1,\dots,\theta_m) &= \sum_{i\le m} c_i\left(\langle\nabla H(\Theta), u_i\rangle^2 - F_i'^2(\theta_i)\right)\\ &= |\nabla H|^2(\Theta) - \sum_{i\le m} c_i F_i'^2(\theta_i),\end{aligned}$$

where we have used again the decomposition of identity (1). Finally, the evolution equation for C can be rewritten as

$$2\frac{\partial C}{\partial t} = \mathcal{E}C + \langle b, \nabla C\rangle. \tag{9}$$

Note that we are now working in the m-dimensional Euclidean space. It follows from the theory of parabolic-elliptic evolution equations that when $C(0,\cdot)$ is a non-negative function, then for all $t > 0$, $C(t,\cdot)$ remains non-negative. Alternately, an explicit expression of C can be given in terms of stochastic processes, on which this property can be read. See the remark after the proof.

By the hypothesis (3) we know that $C(0,\cdot) \ge 0$. At time $\sigma > 0$, $C(\sigma, 0, \dots, 0) \ge 0$ reads as

$$P_\sigma h(0) \ge \prod_{i\le m} (P_\sigma f_i)^{c_i}(0),$$

or equivalently,

$$\int_{\mathbb{R}^n} h(z)\, \frac{e^{-|z|^2/(2\sigma)}\, dz}{(\sqrt{2\pi\sigma})^n} \ge \prod_{i\le m}\left(\int_{\mathbb{R}} f_i(s)\, \frac{e^{-|s|^2/(2\sigma)}\, ds}{\sqrt{2\pi\sigma}}\right)^{c_i}.$$

Since (1) implies $\sum c_i = n$, we find, letting $\sigma \to +\infty$, $\displaystyle\int_{\mathbb{R}^n} h \ge \prod_{i\le m}\left(\int_{\mathbb{R}} f_i\right)^{c_i}$. □

Remark. In order to justify that a solution of (9) preserves the non-negativity (maximum principle), or in order to express the solution in terms of a stochastic process, one has to be a little bit careful. It is convenient to assume first that the functions h and $(f_i)_{i\le m}$ are "nice" enough (this will guarantee that b defined in (8) is also nice), and to conclude by approximation. This procedure is described in details in Borell's papers. Assuming for instance that $b(t,\cdot)$ is Lipschitz with Lipschitz constant uniformly bounded in $t\in[0,T]$, the solution of (9) at time T can be expressed as

$$C(T,\theta_1,\dots,\theta_m)=\mathbb{E}\left[C(0,W(T)\right]. \tag{10}$$

Here $W(t)$ is the m-dimensional (degenerate) process with initial value $W(0)=(\theta_1,\dots,\theta_m)$ satisfying

$$dW(t)=\sigma dB(t)+\frac{1}{2}b(T-t,W(t))\,dt, \qquad t\in[0,T]$$

where B is a standard n-dimensional Brownian motion and σ is the $m\times n$ matrix whose i-th row is given by the vector u_i. It follows from (10) that $C(T,\cdot)\ge 0$ when $C(0,\cdot)\ge 0$.

3 The Brascamp-Lieb inequality

As explained in the introduction, this section is a particular case of the recent work of Carlen, Lieb and Loss.

Instead of working with the heat semi-group, it will be more convenient to introduce the Ornstein-Uhlenbeck semi-group with generator $L:=\Delta-\langle x,\nabla\ \rangle$:

$$P_tf(x)=\int_{\mathbb{R}^n} f(e^{-t}x+\sqrt{1-e^{-2t}}z)\,d\gamma_n(z).$$

The function $g(t,x)=P_tf(x)$ solves the equation

$$\begin{cases}\dfrac{\partial g}{\partial t}(t,x)=\Delta g(t,x)-\langle x,\nabla g(t,x)\rangle=Lg(t,x)\\[2ex] g(0,\cdot)=f\end{cases}$$

Thus, $F^{(t)}(x):=\log P_tf(x)$ satisfies

$$\begin{cases}\dfrac{\partial F^{(t)}}{\partial t}(x)=LF^{(t)}(x)+|\nabla F^{(t)}(x)|^2\\[2ex] F(0,\cdot)=\log f\end{cases}$$

We recall that under appropriate assumptions, $P_tf\longrightarrow\int f\,d\gamma_n$ almost surely and that

$$\int (Lu)v\,d\gamma_n=-\int\langle\nabla u,\nabla v\rangle\,d\gamma_n \tag{11}$$

whenever it makes sense.

We introduce the *one-dimensional* semi-groups $P_t(f_i)$ and $F_i^{(t)}=\log P_tf_i$, together with

$$\alpha(t):=\int\prod_{i\le m}(P_tf_i)^{c_i}(\langle x,u_i\rangle)\,d\gamma_n(x)=\int e^{\sum_{i\le m}c_iF_i^{(t)}(\langle x,u_i\rangle)}\,d\gamma_n(x).$$

We have

$$\alpha'(t)=\int\sum_{i\le m}c_i\left(LF_i(\langle x,u_i\rangle)+(F_i')^2(\langle x,u_i\rangle)\right)e^{\sum c_iF_i(\langle x,u_i\rangle)}\,d\gamma_n(x),$$

where we have written F_i instead of $F_i^{(t)}$ (after differentiating in t, we are working at a fixed t). For a function F on $\mathbb{R}$, the function $\tilde{F}(x) := F(\langle x, u_i\rangle)$ on $\mathbb{R}^n$ verifies $L\tilde{F}(x) = LF(\langle x, u_i\rangle)$ since $|u_i| = 1$ (here we used the same notation for the 1 and n dimensional generators). Thus we can use the integration by parts formula (11) in $\mathbb{R}^n$ and get

$$\alpha'(t) = \int \Big(- \sum_{i,j\le m} c_i c_j \langle u_i, u_j\rangle F_i'(\langle x, u_i\rangle) F_j'(\langle x, u_j\rangle) + \sum_{i\le m} c_i (F_i')^2(\langle x, u_i\rangle)\Big) e^{\sum c_i F_i(\langle x,u_i\rangle)}\, d\gamma_n(x).$$

This can be rewritten as

$$\alpha'(t) = \int \Big(- \Big|\sum_{i\le m} c_i F_i'(\langle x, u_i\rangle) u_i\Big|^2 + \sum_{i\le m} c_i (F_i')^2(\langle x, u_i\rangle)\Big) e^{\sum c_i F_i(\langle x,u_i\rangle)}\, d\gamma_n(x).$$

We can deduce that $\alpha'(t) \ge 0$ from the inequality

$$\Big|\sum_{i\le m} c_i\, \theta_i\, u_i\Big|^2 \le \sum_{i\le m} c_i\, \theta_i^2, \qquad \forall (\theta_i)_{i\le m} \in \mathbb{R}^m,$$

which is an easy consequence of the decomposition of the identity (1). To see this, set $v = \sum_{i\le m} c_i\, \theta_i\, u_i$ and write

$$\begin{aligned} |v|^2 &= \langle v, \sum_{i\le m} c_i\, \theta_i\, u_i\rangle = \sum_{i\le m} c_i\, \theta_i \langle v, u_i\rangle \\ &\le \Big(\sum_{i\le m} c_i\, \theta_i^2\Big)^{\frac{1}{2}} \Big(\sum_{i\le m} c_i \langle v, u_i\rangle^2\Big)^{\frac{1}{2}} = \Big(\sum_{i\le m} c_i\, \theta_i^2\Big)^{\frac{1}{2}} |v|. \end{aligned}$$

The inequality $\alpha(0) \le \alpha(+\infty)$ reads as

$$\int_{\mathbb{R}^n} \prod_{i\le m} f_i^{c_i}(\langle x, u_i\rangle)\, d\gamma_n(x) \le \prod_{i\le m} \Big(\int_{\mathbb{R}} f_i\, d\gamma_1\Big)^{c_i}.$$

Setting $g_i(s) := f_i(s) e^{-|s|^2/2}$ and using (1) (and $\sum c_i = n$) we end up with the classical form

$$\int_{\mathbb{R}^n} \prod_{i\le m} g_i^{c_i}(\langle x, u_i\rangle)\, dx \le \prod_{i\le m} \Big(\int_{\mathbb{R}} g_i\Big)^{c_i}.$$

□

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