

On a conjecture of Mirkovic and Vilonen  
Modular representations and minimal degenerations in affine  
Grassmannians

Daniel Juteau  
CNRS, LMNO, Université de Caen Basse-Normandie

April 2, 2009

# Outline

## Perverse sheaves

Derived categories of constructible complexes

Perverse sheaves

Intersection cohomology complexes

A first example: the nilpotent cone of  $\mathfrak{sl}_2$

# Outline

## Perverse sheaves

Derived categories of constructible complexes

Perverse sheaves

Intersection cohomology complexes

A first example: the nilpotent cone of  $\mathfrak{sl}_2$

## Geometric Satake correspondence

The big problem of modular representation theory

The geometric picture

The theorem of Mirkovic and Vilonen

A conjecture on the stalks

# Outline

## Perverse sheaves

Derived categories of constructible complexes

Perverse sheaves

Intersection cohomology complexes

A first example: the nilpotent cone of  $\mathfrak{sl}_2$

## Geometric Satake correspondence

The big problem of modular representation theory

The geometric picture

The theorem of Mirkovic and Vilonen

A conjecture on the stalks

## Minimal degenerations in the affine Grassmannians

Classification

Simple singularities

Minimal singularities

Quasi-minimal singularities

# Outline

## Perverse sheaves

Derived categories of constructible complexes

Perverse sheaves

Intersection cohomology complexes

A first example: the nilpotent cone of  $\mathfrak{sl}_2$

## Geometric Satake correspondence

The big problem of modular representation theory

The geometric picture

The theorem of Mirkovic and Vilonen

A conjecture on the stalks

## Minimal degenerations in the affine Grassmannians

Classification

Simple singularities

Minimal singularities

Quasi-minimal singularities

## Local systems

Let  $X$  be a complex algebraic variety.

## Local systems

Let  $X$  be a complex algebraic variety.

Let  $k$  be a ring of coefficients such as  $\mathbb{C}$ ,  $\mathbb{Z}$ ,  $\overline{\mathbb{F}}_p \dots$

## Local systems

Let  $X$  be a complex algebraic variety.

Let  $k$  be a ring of coefficients such as  $\mathbb{C}$ ,  $\mathbb{Z}$ ,  $\overline{\mathbb{F}}_p \dots$

A  $k$ -local system on a variety  $X$  is a locally constant  $k$ -sheaf on  $X$  with finitely generated stalks.

## Local systems

Let  $X$  be a complex algebraic variety.

Let  $k$  be a ring of coefficients such as  $\mathbb{C}$ ,  $\mathbb{Z}$ ,  $\overline{\mathbb{F}}_p \dots$

A  $k$ -local system on a variety  $X$  is a locally constant  $k$ -sheaf on  $X$  with finitely generated stalks.

If  $X$  is connected, for  $x \in X$  we have an equivalence of categories

$$\begin{array}{ccc} \text{Loc}(X, k) & \xrightarrow{\sim} & \text{Representations of } \pi_1(X, x) \text{ over } k \\ \mathcal{F} & \longmapsto & \mathcal{F}_x \end{array}$$

## Constructible sheaves

Let  $X$  be a complex algebraic variety, with a “good” (Whitney) stratification

$$X = \bigsqcup_{S \in \mathfrak{X}} S$$

into finitely many smooth connected locally closed pieces.

## Constructible sheaves

Let  $X$  be a complex algebraic variety, with a “good” (Whitney) stratification

$$X = \bigsqcup_{S \in \mathfrak{X}} S$$

into finitely many smooth connected locally closed pieces.

For  $S \in \mathfrak{X}$ , we denote by  $\iota_S : S \hookrightarrow X$  the inclusion.

## Constructible sheaves

Let  $X$  be a complex algebraic variety, with a “good” (Whitney) stratification

$$X = \bigsqcup_{S \in \mathfrak{X}} S$$

into finitely many smooth connected locally closed pieces.

For  $S \in \mathfrak{X}$ , we denote by  $\iota_S : S \hookrightarrow X$  the inclusion.

$\text{Sh}_{\mathfrak{X}}(X, k) \subset \text{Sh}(X, k)$  is the full subcategory of the  $\mathcal{F}$  such that

$$\forall S \in \mathfrak{X}, \iota_S^* \mathcal{F} \in \text{Loc}(S, k).$$

## Constructible sheaves

Let  $X$  be a complex algebraic variety, with a “good” (Whitney) stratification

$$X = \bigsqcup_{S \in \mathfrak{X}} S$$

into finitely many smooth connected locally closed pieces.

For  $S \in \mathfrak{X}$ , we denote by  $\iota_S : S \hookrightarrow X$  the inclusion.

$\mathrm{Sh}_{\mathfrak{X}}(X, k) \subset \mathrm{Sh}(X, k)$  is the full subcategory of the  $\mathcal{F}$  such that

$$\forall S \in \mathfrak{X}, \iota_S^* \mathcal{F} \in \mathrm{Loc}(S, k).$$

$\mathcal{D}_{\mathfrak{X}}^b(X, k) \subset \mathcal{D}^b(X, k)$  is the full subcategory of the  $\mathcal{F}^\bullet$  such that

$$\forall m \in \mathbb{Z}, \mathcal{H}^m(\mathcal{F}^\bullet) \in \mathrm{Sh}_{\mathfrak{X}}(X, k).$$

## The six operations of Grothendieck

We consider  $\mathcal{D}_c^b(X, k)$ , the limit over  $\mathfrak{X}$  of the  $\mathcal{D}_{\mathfrak{X}}^b(X, k)$ .

## The six operations of Grothendieck

We consider  $\mathcal{D}_c^b(X, k)$ , the limit over  $\mathfrak{X}$  of the  $\mathcal{D}_{\mathfrak{X}}^b(X, k)$ .

We have two internal operations on  $\mathcal{D}_c^b(X, k)$ :

$$\underline{\mathrm{RHom}}_{k_X}(-, -) : \mathcal{D}_c^b(X, k)^{\mathrm{op}} \times \mathcal{D}_c^b(X, k) \longrightarrow \mathcal{D}_c^b(X, k)$$

$$- \otimes_k^L - : \mathcal{D}_c^b(X, k) \times \mathcal{D}_c^b(X, k) \longrightarrow \mathcal{D}_c^b(X, k)$$

## The six operations of Grothendieck

We consider  $\mathcal{D}_c^b(X, k)$ , the limit over  $\mathfrak{X}$  of the  $\mathcal{D}_{\mathfrak{X}}^b(X, k)$ .

We have two internal operations on  $\mathcal{D}_c^b(X, k)$ :

$$\underline{\mathrm{RHom}}_{k_X}(-, -) : \mathcal{D}_c^b(X, k)^{\mathrm{op}} \times \mathcal{D}_c^b(X, k) \longrightarrow \mathcal{D}_c^b(X, k)$$

$$- \otimes_k^L - : \mathcal{D}_c^b(X, k) \times \mathcal{D}_c^b(X, k) \longrightarrow \mathcal{D}_c^b(X, k)$$

For a morphism  $f : X \rightarrow Y$ , we have functors (we don't write  $R$ 's)

$$\begin{array}{ccc} & f_*, f_! & \\ & \curvearrowright & \\ \mathcal{D}_c^b(X, k) & & \mathcal{D}_c^b(Y, k) \\ & \curvearrowleft & \\ & f^*, f^! & \end{array}$$

## The six operations of Grothendieck

We consider  $\mathcal{D}_c^b(X, k)$ , the limit over  $\mathfrak{X}$  of the  $\mathcal{D}_{\mathfrak{X}}^b(X, k)$ .

We have two internal operations on  $\mathcal{D}_c^b(X, k)$ :

$$\underline{\mathrm{RHom}}_{k_X}(-, -) : \mathcal{D}_c^b(X, k)^{\mathrm{op}} \times \mathcal{D}_c^b(X, k) \longrightarrow \mathcal{D}_c^b(X, k)$$

$$- \otimes_k^L - : \mathcal{D}_c^b(X, k) \times \mathcal{D}_c^b(X, k) \longrightarrow \mathcal{D}_c^b(X, k)$$

For a morphism  $f : X \rightarrow Y$ , we have functors (we don't write  $R$ 's)

$$\begin{array}{ccc} & \xrightarrow{f_*, f_!} & \\ \mathcal{D}_c^b(X, k) & & \mathcal{D}_c^b(Y, k) \\ & \xleftarrow{f^*, f^!} & \end{array}$$

We have adjunctions  $(f^*, f_*)$  and  $(f_!, f^!)$ .

## Grothendieck-Verdier duality

It is the functor

$$\mathbb{D} = \mathbb{D}_X := \underline{\mathrm{RHom}}(-, D_X) : \mathcal{D}_c^b(X, k)^{\mathrm{op}} \longrightarrow \mathcal{D}_c^b(X, k)$$

where  $D_X = f^! \underline{k}_{\mathrm{pt}}$  is the *dualizing complex*.

# Grothendieck-Verdier duality

It is the functor

$$\mathbb{D} = \mathbb{D}_X := \underline{\mathrm{RHom}}(-, D_X) : \mathcal{D}_c^b(X, k)^{\mathrm{op}} \longrightarrow \mathcal{D}_c^b(X, k)$$

where  $D_X = f^! \underline{k}_{\mathrm{pt}}$  is the *dualizing complex*.

We have isomorphisms

$$\mathbb{D}f_* \simeq f_! \mathbb{D}$$

$$\mathbb{D}f^* \simeq f^! \mathbb{D}$$

$$\mathbb{D}^2 \simeq \mathrm{Id}$$

# Grothendieck-Verdier duality

It is the functor

$$\mathbb{D} = \mathbb{D}_X := \underline{\mathrm{RHom}}(-, D_X) : \mathcal{D}_c^b(X, k)^{\mathrm{op}} \longrightarrow \mathcal{D}_c^b(X, k)$$

where  $D_X = f^! \underline{k}_{\mathrm{pt}}$  is the *dualizing complex*.

We have isomorphisms

$$\mathbb{D}f_* \simeq f_! \mathbb{D}$$

$$\mathbb{D}f^* \simeq f^! \mathbb{D}$$

$$\mathbb{D}^2 \simeq \mathrm{Id}$$

The first isomorphism generalizes Poincaré duality: for  $X$  smooth purely of dimension  $d$ , we have  $D_X \simeq \underline{k}_X[2d]$ .

# Outline

## Perverse sheaves

Derived categories of constructible complexes

### Perverse sheaves

Intersection cohomology complexes

A first example: the nilpotent cone of  $\mathfrak{sl}_2$

## Geometric Satake correspondence

The big problem of modular representation theory

The geometric picture

The theorem of Mirkovic and Vilonen

A conjecture on the stalks

## Minimal degenerations in the affine Grassmannians

Classification

Simple singularities

Minimal singularities

Quasi-minimal singularities

## Perverse sheaves

The perverse  $t$ -structure is obtained recursively on the strata by a glueing construction: on each  $S$ , we put the natural  $t$ -structure shifted by  $d_S := \dim S$ .

## Perverse sheaves

The perverse  $t$ -structure is obtained recursively on the strata by a glueing construction: on each  $S$ , we put the natural  $t$ -structure shifted by  $d_S := \dim S$ . We get

$${}^p\mathcal{D}_{\mathfrak{X}}^{\leq 0}(X, k) = \left\{ K \in \mathcal{D}_{\mathfrak{X}}^b(X, k) \mid \forall S \in \mathfrak{X}, \iota_S^* K \in \mathcal{D}^{\leq -d_S}(S, k) \right\}$$

$${}^p\mathcal{D}_{\mathfrak{X}}^{\geq 0}(X, k) = \left\{ K \in \mathcal{D}_{\mathfrak{X}}^b(X, k) \mid \forall S \in \mathfrak{X}, \iota_S^! K \in \mathcal{D}^{\geq -d_S}(S, k) \right\}$$

## Perverse sheaves

The perverse  $t$ -structure is obtained recursively on the strata by a glueing construction: on each  $S$ , we put the natural  $t$ -structure shifted by  $d_S := \dim S$ . We get

$${}^p\mathcal{D}_{\mathfrak{X}}^{\leq 0}(X, k) = \left\{ K \in \mathcal{D}_{\mathfrak{X}}^b(X, k) \mid \forall S \in \mathfrak{X}, \iota_S^* K \in \mathcal{D}^{\leq -d_S}(S, k) \right\}$$

$${}^p\mathcal{D}_{\mathfrak{X}}^{\geq 0}(X, k) = \left\{ K \in \mathcal{D}_{\mathfrak{X}}^b(X, k) \mid \forall S \in \mathfrak{X}, \iota_S^! K \in \mathcal{D}^{\geq -d_S}(S, k) \right\}$$

The *heart* of the perverse  $t$ -structure

$$\text{Perv}_{\mathfrak{X}}(X, k) = {}^p\mathcal{D}_{\mathfrak{X}}^{\leq 0}(X, k) \cap {}^p\mathcal{D}_{\mathfrak{X}}^{\geq 0}(X, k)$$

is the abelian category of perverse sheaves.

## Perverse sheaves

The perverse  $t$ -structure is obtained recursively on the strata by a glueing construction: on each  $S$ , we put the natural  $t$ -structure shifted by  $d_S := \dim S$ . We get

$$\begin{aligned} {}^p\mathcal{D}_{\mathfrak{X}}^{\leq 0}(X, k) &= \left\{ K \in \mathcal{D}_{\mathfrak{X}}^b(X, k) \mid \forall S \in \mathfrak{X}, \iota_S^* K \in \mathcal{D}^{\leq -d_S}(S, k) \right\} \\ {}^p\mathcal{D}_{\mathfrak{X}}^{\geq 0}(X, k) &= \left\{ K \in \mathcal{D}_{\mathfrak{X}}^b(X, k) \mid \forall S \in \mathfrak{X}, \iota_S^! K \in \mathcal{D}^{\geq -d_S}(S, k) \right\} \end{aligned}$$

The *heart* of the perverse  $t$ -structure

$$\text{Perv}_{\mathfrak{X}}(X, k) = {}^p\mathcal{D}_{\mathfrak{X}}^{\leq 0}(X, k) \cap {}^p\mathcal{D}_{\mathfrak{X}}^{\geq 0}(X, k)$$

is the abelian category of perverse sheaves.

If  $k$  is a field, then it is noetherian and artinian.

## Perverse sheaves

The perverse  $t$ -structure is obtained recursively on the strata by a glueing construction: on each  $S$ , we put the natural  $t$ -structure shifted by  $d_S := \dim S$ . We get

$$\begin{aligned} {}^p\mathcal{D}_{\mathfrak{X}}^{\leq 0}(X, k) &= \left\{ K \in \mathcal{D}_{\mathfrak{X}}^b(X, k) \mid \forall S \in \mathfrak{X}, \iota_S^* K \in \mathcal{D}^{\leq -d_S}(S, k) \right\} \\ {}^p\mathcal{D}_{\mathfrak{X}}^{\geq 0}(X, k) &= \left\{ K \in \mathcal{D}_{\mathfrak{X}}^b(X, k) \mid \forall S \in \mathfrak{X}, \iota_S^! K \in \mathcal{D}^{\geq -d_S}(S, k) \right\} \end{aligned}$$

The *heart* of the perverse  $t$ -structure

$$\text{Perv}_{\mathfrak{X}}(X, k) = {}^p\mathcal{D}_{\mathfrak{X}}^{\leq 0}(X, k) \cap {}^p\mathcal{D}_{\mathfrak{X}}^{\geq 0}(X, k)$$

is the abelian category of perverse sheaves.

If  $k$  is a field, then it is noetherian and artinian.

There are perverse truncation functors  ${}^p\tau_{\leq m}$ ,  ${}^p\tau_{\geq m}$  and cohomological functors  ${}^p\mathcal{H}^m : \mathcal{D}_{\mathfrak{X}}^b(X, k) \rightarrow \text{Perv}_{\mathfrak{X}}(X, k)$ .

## Diagram for a perverse sheaf

$$d = \dim X$$

$$S_m = \text{union of strata of dimension } m$$

$K \in \mathcal{D}_c^b(X, k)$  is perverse if and only if the cohomology sheaves of both  $K$  and  $\mathbb{D}K$  along strata are of the following form ( $k$  a field):

strata	...	$-d$	$-d + 1$	...	$-1$	$0$
$S_d$	0	*	0	0	0	0
$S_{d-1}$	0	*	*	0	0	0
$\vdots$	0	$\vdots$	$\vdots$	$\ddots$	0	0
$S_1$	0	*	*	...	*	0
$S_0$	0	*	*	...	*	*

where  $*$  denotes the possibility of a non-trivial cohomology sheaf.

# Outline

## Perverse sheaves

Derived categories of constructible complexes

Perverse sheaves

**Intersection cohomology complexes**

A first example: the nilpotent cone of  $\mathfrak{sl}_2$

## Geometric Satake correspondence

The big problem of modular representation theory

The geometric picture

The theorem of Mirkovic and Vilonen

A conjecture on the stalks

## Minimal degenerations in the affine Grassmannians

Classification

Simple singularities

Minimal singularities

Quasi-minimal singularities

## Intersection cohomology

If  $\mathcal{L}$  is an irreducible local system on  $S$ , there is a unique complex  $K$  supported by  $\overline{S}$  extending  $\mathcal{L}[d_S]$  (i.e. with  $\iota_S^* K \simeq \mathcal{L}[d_S]$ ) such that

$$\forall T < S, \begin{cases} \iota_T^* K \in \mathcal{D}^{< -\dim T}(T, k) \\ \iota_T^! K \in \mathcal{D}^{> -\dim T}(T, k) \end{cases}$$

## Intersection cohomology

If  $\mathcal{L}$  is an irreducible local system on  $S$ , there is a unique complex  $K$  supported by  $\bar{S}$  extending  $\mathcal{L}[d_S]$  (i.e. with  $\iota_S^* K \simeq \mathcal{L}[d_S]$ ) such that

$$\forall T < S, \begin{cases} \iota_T^* K \in \mathcal{D}^{< -\dim T}(T, k) \\ \iota_T^! K \in \mathcal{D}^{> -\dim T}(T, k) \end{cases}$$

The complex  $K =: \mathcal{IC}(\bar{S}, \mathcal{L})$  is called an intersection cohomology complex.

## Intersection cohomology

If  $\mathcal{L}$  is an irreducible local system on  $S$ , there is a unique complex  $K$  supported by  $\bar{S}$  extending  $\mathcal{L}[d_S]$  (i.e. with  $\iota_S^* K \simeq \mathcal{L}[d_S]$ ) such that

$$\forall T \subset S, \begin{cases} \iota_T^* K \in \mathcal{D}^{< -\dim T}(T, k) \\ \iota_T^! K \in \mathcal{D}^{> -\dim T}(T, k) \end{cases}$$

The complex  $K =: \mathcal{IC}(\bar{S}, \mathcal{L})$  is called an intersection cohomology complex.

We have  $\mathbb{D}\mathcal{IC}(\bar{S}, \mathcal{L}) \simeq \mathcal{IC}^+(\bar{S}, \mathcal{L}^\vee)$ . Hence, if we set

$$IH^m(\bar{S}, \mathcal{L}) = R\Gamma(\bar{S}, \mathcal{IC}(\bar{S}, \mathcal{L})[-d_S])$$

$$IH_c^m(\bar{S}, \mathcal{L}^\vee) = R\Gamma_c(\bar{S}, \mathcal{IC}(\bar{S}, \mathcal{L}^\vee)[-d_S])$$

we get invariants which satisfy Poincaré duality, even in the singular case.

## Intersection cohomology

If  $\mathcal{L}$  is an irreducible local system on  $S$ , there is a unique complex  $K$  supported by  $\bar{S}$  extending  $\mathcal{L}[d_S]$  (i.e. with  $\iota_S^* K \simeq \mathcal{L}[d_S]$ ) such that

$$\forall T \subset S, \begin{cases} \iota_T^* K \in \mathcal{D}^{< -\dim T}(T, k) \\ \iota_T^! K \in \mathcal{D}^{> -\dim T}(T, k) \end{cases}$$

The complex  $K =: \mathcal{IC}(\bar{S}, \mathcal{L})$  is called an intersection cohomology complex.

We have  $\mathbb{D}\mathcal{IC}(\bar{S}, \mathcal{L}) \simeq \mathcal{IC}^+(\bar{S}, \mathcal{L}^\vee)$ . Hence, if we set

$$IH^m(\bar{S}, \mathcal{L}) = R\Gamma(\bar{S}, \mathcal{IC}(\bar{S}, \mathcal{L})[-d_S])$$

$$IH_c^m(\bar{S}, \mathcal{L}^\vee) = R\Gamma_c(\bar{S}, \mathcal{IC}(\bar{S}, \mathcal{L}^\vee)[-d_S])$$

we get invariants which satisfy Poincaré duality, even in the singular case.

If  $k$  is a field, then the  $\mathcal{IC}(\bar{S}, \mathcal{L})$  are the simple objects of  $\text{Perv}_{\mathfrak{X}}(X, k)$ .

## Diagram for an IC complex

Let  $d_S$  denote the dimension of  $S$ , so that  $S \subset S_{d_S}$ .

The complex  $\mathcal{IC}(\bar{S}, \mathcal{L})$  has the following shape:

strata	$-d$	$\dots$	$-d_S - 1$	$-d_S$	$-d_S + 1$	$\dots$	$-1$	$0$
$S_d$	<b>0</b>	0	0	0	0	0	0	0
$\vdots$	0	<b>0</b>	0	0	0	0	0	0
$S_{d_S+1}$	0	0	<b>0</b>	0	0	0	0	0
$S_{d_S}$	0	0	0	$j_S! \mathcal{L}$	0	0	0	0
$S_{d_S-1}$	0	0	0	*	<b>0</b>	0	0	0
$\vdots$	0	0	0	*	*	<b>0</b>	0	0
$S_1$	0	0	0	*	*	*	<b>0</b>	0
$S_0$	0	0	0	*	*	*	*	<b>0</b>

where  $j_S : S \rightarrow S_{d_S}$  is the inclusion.

Note the zeroes on the diagonal.

## Deligne's intermediate extension functor

$S_m$  = union of strata of dimension  $m$ ,

$X_m$  = union of all strata of dimension  $\geq m$ .

We have a sequence of inclusions:

$$X_d \xrightarrow{j_{d-1}} X_{d-1} \xrightarrow{j_{d-2}} X_{d-2} \hookrightarrow \cdots \hookrightarrow X_1 \xrightarrow{j_0} X_0 = X.$$

Now let  $\mathcal{L}$  be an irreducible local system on  $S \in \mathfrak{X}$ . One has an isomorphism

$$\mathcal{IC}(\overline{S}, \mathcal{L}) \simeq (\tau_{\leq -1} \circ j_{0*}) \circ (\tau_{\leq -2} \circ j_{1*}) \circ \cdots \circ (\tau_{\leq -d_S} \circ j_{d_S-1*})(\mathcal{L}[d_S]).$$

## Deligne's intermediate extension functor

$S_m$  = union of strata of dimension  $m$ ,

$X_m$  = union of all strata of dimension  $\geq m$ .

We have a sequence of inclusions:

$$X_d \xrightarrow{j_{d-1}} X_{d-1} \xrightarrow{j_{d-2}} X_{d-2} \hookrightarrow \cdots \hookrightarrow X_1 \xrightarrow{j_0} X_0 = X.$$

Now let  $\mathcal{L}$  be an irreducible local system on  $S \in \mathfrak{X}$ . One has an isomorphism

$$\mathcal{IC}(\overline{S}, \mathcal{L}) \simeq (\tau_{\leq -1} \circ j_{0*}) \circ (\tau_{\leq -2} \circ j_{1*}) \circ \cdots \circ (\tau_{\leq -d_S} \circ j_{d_S-1*})(\mathcal{L}[d_S]).$$

So the IC complex is given by a functor, the intermediate extension functor  $j_{!*}$ , where  $j : S \rightarrow \overline{S}$ .

## Deligne's intermediate extension functor

$S_m$  = union of strata of dimension  $m$ ,

$X_m$  = union of all strata of dimension  $\geq m$ .

We have a sequence of inclusions:

$$X_d \xrightarrow{j_{d-1}} X_{d-1} \xrightarrow{j_{d-2}} X_{d-2} \hookrightarrow \cdots \hookrightarrow X_1 \xrightarrow{j_0} X_0 = X.$$

Now let  $\mathcal{L}$  be an irreducible local system on  $S \in \mathfrak{X}$ . One has an isomorphism

$$\mathcal{IC}(\overline{S}, \mathcal{L}) \simeq (\tau_{\leq -1} \circ j_{0*}) \circ (\tau_{\leq -2} \circ j_{1*}) \circ \cdots \circ (\tau_{\leq -d_S} \circ j_{d_S-1*})(\mathcal{L}[d_S]).$$

So the IC complex is given by a functor, the intermediate extension functor  $j_{!*}$ , where  $j : S \rightarrow \overline{S}$ . It can also be defined by

$$j_{!*} = \text{Im}(p_{j!} \rightarrow p_{j*})$$

where  $p_{j!} = {}^p\mathcal{H}^0 j_!$  and  $p_{j*} = {}^p\mathcal{H}^0 j_*$ .

# Outline

## Perverse sheaves

Derived categories of constructible complexes

Perverse sheaves

Intersection cohomology complexes

**A first example: the nilpotent cone of  $\mathfrak{sl}_2$**

## Geometric Satake correspondence

The big problem of modular representation theory

The geometric picture

The theorem of Mirkovic and Vilonen

A conjecture on the stalks

## Minimal degenerations in the affine Grassmannians

Classification

Simple singularities

Minimal singularities

Quasi-minimal singularities

## IC complex of a cone

Suppose  $X$  has only an isolated cone singularity.

Let  $i : \{0\} \hookrightarrow X$  be the vertex,  
 $j : U = X \setminus \{0\} \hookrightarrow X$ .



## IC complex of a cone

Suppose  $X$  has only an isolated cone singularity.

Let  $i : \{0\} \hookrightarrow X$  be the vertex,  
 $j : U = X \setminus \{0\} \hookrightarrow X$ .

Then  $j_*(\underline{k}_U)_0 \simeq R\Gamma(U, k)$ .



## IC complex of a cone

Suppose  $X$  has only an isolated cone singularity.

Let  $i : \{0\} \hookrightarrow X$  be the vertex,  
 $j : U = X \setminus \{0\} \hookrightarrow X$ .



Then  $j_*(\underline{k}_U)_0 \simeq R\Gamma(U, k)$ .

Actually

$$j_*(\underline{k}_U)_0 \simeq \lim_{V \ni 0} R\Gamma(V \setminus \{0\}, k)$$

and there is a basis of neighborhoods of 0 which are all homeomorphic to  $U$ .

## IC complex of a cone

Suppose  $X$  has only an isolated cone singularity.

Let  $i : \{0\} \hookrightarrow X$  be the vertex,  
 $j : U = X \setminus \{0\} \hookrightarrow X$ .



Then  $j_*(\underline{k}_U)_0 \simeq R\Gamma(U, k)$ .

Actually

$$j_*(\underline{k}_U)_0 \simeq \lim_{V \ni 0} R\Gamma(V \setminus \{0\}, k)$$

and there is a basis of neighborhoods of 0 which are all homeomorphic to  $U$ .

Then one can obtain  $\mathcal{IC}(X, k)$  by shifting by  $\dim X$  and truncating at  $-1$ .

## The nilpotent cone of $\mathfrak{sl}_2$

$$\mathfrak{g} = \mathfrak{sl}_2 \supset \mathcal{N} = \left\{ \begin{pmatrix} x & y \\ z & -x \end{pmatrix} \mid x^2 + yz = 0 \right\} = \mathcal{O}_{\text{reg}} \cup \{0\}$$

## The nilpotent cone of $\mathfrak{sl}_2$

$$\mathfrak{g} = \mathfrak{sl}_2 \supset \mathcal{N} = \left\{ \begin{pmatrix} x & y \\ z & -x \end{pmatrix} \mid x^2 + yz = 0 \right\} = \mathcal{O}_{\text{reg}} \cup \{0\}$$

We have  $\mathcal{N} \simeq \mathbb{C}^2 / \{\pm 1\}$  so

$$U = \mathcal{O}_{\text{reg}} \simeq (\mathbb{C}^2 \setminus \{0\}) / \{\pm 1\} \simeq \mathbb{R}_*^+ \times S^3 / \{\pm 1\} \sim \mathbb{RP}^3$$

and thus

$$R\Gamma(U, k) = (k \xrightarrow{0} k \xrightarrow{2} k \xrightarrow{0} k) = k \otimes_{\mathbb{Z}}^L R\Gamma(U, \mathbb{Z})$$

## The nilpotent cone of $\mathfrak{sl}_2$

$$\mathfrak{g} = \mathfrak{sl}_2 \supset \mathcal{N} = \left\{ \begin{pmatrix} x & y \\ z & -x \end{pmatrix} \mid x^2 + yz = 0 \right\} = \mathcal{O}_{\text{reg}} \cup \{0\}$$

We have  $\mathcal{N} \simeq \mathbb{C}^2 / \{\pm 1\}$  so

$$U = \mathcal{O}_{\text{reg}} \simeq (\mathbb{C}^2 \setminus \{0\}) / \{\pm 1\} \simeq \mathbb{R}_*^+ \times S^3 / \{\pm 1\} \sim \mathbb{RP}^3$$

and thus

$$R\Gamma(U, k) = (k \xrightarrow{0} k \xrightarrow{2} k \xrightarrow{0} k) = k \otimes_{\mathbb{Z}}^L R\Gamma(U, \mathbb{Z})$$

The cohomology of  $U$  with various coefficients is given below:

$k$	0	1	2	3
$\mathbb{Z}$	$\mathbb{Z}$	0	$\mathbb{Z}/2$	$\mathbb{Z}$
$\mathbb{Q}$	$\mathbb{Q}$	0	0	$\mathbb{Q}$
$\mathbb{F}_2$	$\mathbb{F}_2$	$\mathbb{F}_2$	$\mathbb{F}_2$	$\mathbb{F}_2$

## Tables for the nilpotent cone of $\mathfrak{sl}_2$

For  $k$  a field of characteristic  $p$ , we deduce that the stalks of  $\mathcal{IC}(\mathcal{N}, k)$  are given by

	-2	-1	0
$\mathcal{O}_{\text{reg}}$	$k$	0	0
$\{0\}$	$k$	$(k)_2$	0

where  $(k)_2$  means  $k$  if  $p = 2$ , and 0 otherwise.

## Tables for the nilpotent cone of $\mathfrak{sl}_2$

For  $k$  a field of characteristic  $p$ , we deduce that the stalks of  $\mathcal{IC}(\mathcal{N}, k)$  are given by

	-2	-1	0
$\mathcal{O}_{\text{reg}}$	$k$	0	0
$\{0\}$	$k$	$(k)_2$	0

where  $(k)_2$  means  $k$  if  $p = 2$ , and 0 otherwise.

Over  $\mathbb{Z}$ , the stalks of  $\mathcal{IC}(\mathcal{N}, \mathbb{Z})$ , resp.  $\mathcal{IC}^+(\mathcal{N}, \mathbb{Z})$  are given by

	-2	-1	0
$\mathcal{O}_{\text{reg}}$	$\mathbb{Z}$	0	0
$\{0\}$	$\mathbb{Z}$	0	$(\mathbb{Z}/2)_+$

where  $(\mathbb{Z}/2)_+$  means 0 for  $\mathcal{IC}$ , and  $\mathbb{Z}/2$  for  $\mathcal{IC}^+$ .

## Decomposition matrices

One can define decomposition matrices for perverse sheaves.  
The preceding calculation shows that here it is equal to

$$\begin{matrix} (1^2) \\ (2) \end{matrix} \begin{matrix} (1^2) & (2) \\ \begin{pmatrix} \mathbf{1} & 0 \\ \mathbf{1} & 1 \end{pmatrix} \end{matrix}$$

from which we can extract the decomposition matrix for  $\mathfrak{S}_2$  in characteristic 2:

$$\begin{matrix} S_{(2)} \\ S_{(1^2)} \end{matrix} \begin{matrix} D_{(2)} \\ \begin{pmatrix} \mathbf{1} \\ \mathbf{1} \end{pmatrix} \end{matrix}$$

**Explanation:** Springer correspondence, ordinary and modular !  
(Using a Fourier-Deligne transform.)

# Outline

## Perverse sheaves

Derived categories of constructible complexes

Perverse sheaves

Intersection cohomology complexes

A first example: the nilpotent cone of  $\mathfrak{sl}_2$

## Geometric Satake correspondence

The big problem of modular representation theory

The geometric picture

The theorem of Mirkovic and Vilonen

A conjecture on the stalks

## Minimal degenerations in the affine Grassmannians

Classification

Simple singularities

Minimal singularities

Quasi-minimal singularities

## Representations of reductive groups

$G$	a split (simple, simply connected and) connected reductive group scheme over $\mathbb{Z}$
$G \supset B \supset T$	split Borel subgroup, split maximal torus
$X(T) \supset X(T)^+$	weight lattice, dominant weights
$\Phi \supset \Phi^+ \supset \Delta$	root system, positive roots, simple roots
$\Delta(\lambda) \longrightarrow \nabla(\lambda)$	standard, costandard representations, $\lambda \in X(T)^+$

For  $k = \overline{\mathbb{F}}_p$ , the simple  $G_k$ -modules are the

$$L(\lambda) = \text{Im}(\Delta(\lambda)_k \longrightarrow \nabla(\lambda)_k)$$

## Representations of reductive groups

$G$	a split (simple, simply connected and) connected reductive group scheme over $\mathbb{Z}$
$G \supset B \supset T$	split Borel subgroup, split maximal torus
$X(T) \supset X(T)^+$	weight lattice, dominant weights
$\Phi \supset \Phi^+ \supset \Delta$	root system, positive roots, simple roots
$\Delta(\lambda) \longrightarrow \nabla(\lambda)$	standard, costandard representations, $\lambda \in X(T)^+$

For  $k = \overline{\mathbb{F}}_p$ , the simple  $G_k$ -modules are the

$$L(\lambda) = \text{Im}(\Delta(\lambda)_k \longrightarrow \nabla(\lambda)_k)$$

**Big problem:** compute  $d_{\lambda, \mu}^G = [\Delta(\lambda)_k : L(\mu)]$ .

(Note: Lusztig's conjecture, Andersen-Jantzen-Soergel, Fiebig)

## Representations of reductive groups

$G$	a split (simple, simply connected and) connected reductive group scheme over $\mathbb{Z}$
$G \supset B \supset T$	split Borel subgroup, split maximal torus
$X(T) \supset X(T)^+$	weight lattice, dominant weights
$\Phi \supset \Phi^+ \supset \Delta$	root system, positive roots, simple roots
$\Delta(\lambda) \longrightarrow \nabla(\lambda)$	standard, costandard representations, $\lambda \in X(T)^+$

For  $k = \overline{\mathbb{F}}_p$ , the simple  $G_k$ -modules are the

$$L(\lambda) = \text{Im}(\Delta(\lambda)_k \longrightarrow \nabla(\lambda)_k)$$

**Big problem:** compute  $d_{\lambda, \mu}^G = [\Delta(\lambda)_k : L(\mu)]$ .

(Note: Lusztig's conjecture, Andersen-Jantzen-Soergel, Fiebig)

This is 1 if  $\lambda = \mu$ , and 0 unless  $\lambda \geq \mu$ , that is,  $\lambda - \mu \in \mathbb{N}\Delta$ .

# Outline

## Perverse sheaves

Derived categories of constructible complexes

Perverse sheaves

Intersection cohomology complexes

A first example: the nilpotent cone of  $\mathfrak{sl}_2$

## Geometric Satake correspondence

The big problem of modular representation theory

**The geometric picture**

The theorem of Mirkovic and Vilonen

A conjecture on the stalks

## Minimal degenerations in the affine Grassmannians

Classification

Simple singularities

Minimal singularities

Quasi-minimal singularities

# The affine Grassmannian

Let  $G^\vee$  denote the Langlands dual group.

Let  $\mathcal{K} = \mathbb{C}((t))$  and  $\mathcal{O} = \mathbb{C}[[t]]$ .

We consider the complex affine Grassmannian of  $G^\vee$

$$\mathit{Gr} := G^\vee(\mathcal{K})/G^\vee(\mathcal{O})$$

# The affine Grassmannian

Let  $G^\vee$  denote the Langlands dual group.

Let  $\mathcal{K} = \mathbb{C}((t))$  and  $\mathcal{O} = \mathbb{C}[[t]]$ .

We consider the complex affine Grassmannian of  $G^\vee$

$$\mathcal{G}r := G^\vee(\mathcal{K})/G^\vee(\mathcal{O})$$

The  $G^\vee(\mathcal{O})$ -orbits are parametrized by  $X(T)^+$ .

We denote them by  $\mathcal{G}r_\lambda$ ,  $\lambda \in X(T)^+$ .

$\mathcal{G}r$  is an ind-scheme, direct limit of the finite dimensional projective varieties  $\overline{\mathcal{G}r}_\lambda$ .

## $G^\vee(\mathcal{O})$ -equivariant perverse sheaves on $\mathcal{G}r$

Let  $k$  be any noetherian commutative ring of finite global dimension.

We consider the category

$$\mathrm{Perv}_{G^\vee(\mathcal{O})}(\mathcal{G}r, k)$$

of  $G^\vee(\mathcal{O})$ -equivariant perverse sheaves with  $k$  coefficients on  $\mathcal{G}r$ .

## $G^\vee(\mathcal{O})$ -equivariant perverse sheaves on $\mathcal{G}r$

Let  $k$  be any noetherian commutative ring of finite global dimension.

We consider the category

$$\mathrm{Perv}_{G^\vee(\mathcal{O})}(\mathcal{G}r, k)$$

of  $G^\vee(\mathcal{O})$ -equivariant perverse sheaves with  $k$  coefficients on  $\mathcal{G}r$ .

Here, this is in fact equivalent to: perverse sheaves whose cohomology sheaves are constant along the  $G^\vee(\mathcal{O})$ -orbits.

## $G^\vee(\mathcal{O})$ -equivariant perverse sheaves on $\mathcal{G}r$

Let  $k$  be any noetherian commutative ring of finite global dimension.

We consider the category

$$\mathrm{Perv}_{G^\vee(\mathcal{O})}(\mathcal{G}r, k)$$

of  $G^\vee(\mathcal{O})$ -equivariant perverse sheaves with  $k$  coefficients on  $\mathcal{G}r$ .

Here, this is in fact equivalent to: perverse sheaves whose cohomology sheaves are constant along the  $G^\vee(\mathcal{O})$ -orbits.

There are canonical morphisms between standard and costandard objects

$$\mathcal{J}_!(\lambda, k) \longrightarrow \mathcal{J}_*(\lambda, k)$$

and if  $k$  is a field, the simple objects are the

$$\mathcal{J}_{!*}(\lambda, k) = \mathrm{Im}(\mathcal{J}_!(\lambda, k) \longrightarrow \mathcal{J}_*(\lambda, k))$$

# Outline

## Perverse sheaves

Derived categories of constructible complexes

Perverse sheaves

Intersection cohomology complexes

A first example: the nilpotent cone of  $\mathfrak{sl}_2$

## Geometric Satake correspondence

The big problem of modular representation theory

The geometric picture

**The theorem of Mirkovic and Vilonen**

A conjecture on the stalks

## Minimal degenerations in the affine Grassmannians

Classification

Simple singularities

Minimal singularities

Quasi-minimal singularities

## Theorem (Mirkovic-Vilonen)

*We have a equivalence of tensor categories*

$$(G_k\text{-mod}, \otimes_k) \simeq (\text{Perv}_{G^\vee(\mathcal{O})}(\mathcal{G}r, k), *)$$

*Under this equivalence, the morphisms*

$$\Delta(\lambda)_k \longrightarrow \nabla(\lambda)_k \quad \text{and} \quad \mathcal{J}_!(\lambda, k) \longrightarrow \mathcal{J}_*(\lambda, k)$$

*correspond.*

## Theorem (Mirkovic-Vilonen)

*We have an equivalence of tensor categories*

$$(G_k\text{-mod}, \otimes_k) \simeq (\text{Perv}_{G^\vee(\mathcal{O})}(\mathcal{G}r, k), *)$$

*Under this equivalence, the morphisms*

$$\Delta(\lambda)_k \longrightarrow \nabla(\lambda)_k \quad \text{and} \quad \mathcal{J}_!(\lambda, k) \longrightarrow \mathcal{J}_*(\lambda, k)$$

*correspond.*

*Besides, we have*

$$\begin{aligned}\mathcal{J}_!(\lambda, k) &= \mathcal{J}_!(\lambda, \mathbb{Z}) \otimes_{\mathbb{Z}}^L k \\ \mathcal{J}_*(\lambda, k) &= \mathcal{J}_*(\lambda, \mathbb{Z}) \otimes_{\mathbb{Z}}^L k\end{aligned}$$

## Theorem (Mirkovic-Vilonen)

*We have a equivalence of tensor categories*

$$(G_k\text{-mod}, \otimes_k) \simeq (\text{Perv}_{G^\vee(\mathcal{O})}(\mathcal{G}r, k), *)$$

*Under this equivalence, the morphisms*

$$\Delta(\lambda)_k \longrightarrow \nabla(\lambda)_k \quad \text{and} \quad \mathcal{J}_!(\lambda, k) \longrightarrow \mathcal{J}_*(\lambda, k)$$

*correspond.*

*Besides, we have*

$$\begin{aligned}\mathcal{J}_!(\lambda, k) &= \mathcal{J}_!(\lambda, \mathbb{Z}) \otimes_{\mathbb{Z}}^L k \\ \mathcal{J}_*(\lambda, k) &= \mathcal{J}_*(\lambda, \mathbb{Z}) \otimes_{\mathbb{Z}}^L k\end{aligned}$$

In particular, for  $k = \overline{\mathbb{F}}_p$ , we have

$$d_{\lambda\mu}^G = d_{\lambda\mu}^{\mathcal{G}r} =: [\mathcal{J}_!(\lambda, \mathbb{Z}) \otimes_{\mathbb{Z}}^L k : \mathcal{J}_!(\mu, k)]$$

## A Levi lemma

### Lemma (Malkin-Ostrik-Vybornov)

*Let  $I \subset \Delta$ . If  $\lambda - \mu \in \mathbb{N}I$ , then we have an equivalence of singularities (with “obvious” notations)*

$$\text{Sing}(\overline{\mathcal{G}r}_\lambda, \mathcal{G}r_\mu) = \text{Sing}(\overline{\mathcal{G}r}_{\lambda_I}^{L_I^\vee}, \mathcal{G}r_{\mu_I}^{L_I^\vee})$$

## A Levi lemma

### Lemma (Malkin-Ostrik-Vybornov)

*Let  $I \subset \Delta$ . If  $\lambda - \mu \in \mathbb{N}I$ , then we have an equivalence of singularities (with “obvious” notations)*

$$\text{Sing}(\overline{\mathcal{G}r}_\lambda, \mathcal{G}r_\mu) = \text{Sing}(\overline{\mathcal{G}r}_{\lambda_I}^{L_I^\vee}, \mathcal{G}r_{\mu_I}^{L_I^\vee})$$

### Corollary (see Jantzen’s book)

*If  $I \subset \Delta$  and  $\lambda - \mu \in \mathbb{Z}I$ , then we have (with “obvious” notations)*

$$[V(\lambda) : L(\mu)] = [V_I(\lambda) : L_I(\mu)]$$

# Outline

## Perverse sheaves

Derived categories of constructible complexes

Perverse sheaves

Intersection cohomology complexes

A first example: the nilpotent cone of  $\mathfrak{sl}_2$

## Geometric Satake correspondence

The big problem of modular representation theory

The geometric picture

The theorem of Mirkovic and Vilonen

**A conjecture on the stalks**

## Minimal degenerations in the affine Grassmannians

Classification

Simple singularities

Minimal singularities

Quasi-minimal singularities

## About the IC stalks : a conjecture of Mirkovic and Vilonen

- The IC stalks are a refinement of weight multiplicities.
- They are known for  $k = \mathbb{C}$  (Kazhdan-Lusztig polynomials).
- If we knew them for  $k = \overline{\mathbb{F}}_p$ , then we could solve the big problem.
- All the information is contained in the stalks and costalks of  $\mathcal{J}_!(\lambda, \mathbb{Z}) = \mathcal{J}_{!*}(\lambda, \mathbb{Z})$ .

## About the IC stalks : a conjecture of Mirkovic and Vilonen

- The IC stalks are a refinement of weight multiplicities.
- They are known for  $k = \mathbb{C}$  (Kazhdan-Lusztig polynomials).
- If we knew them for  $k = \overline{\mathbb{F}}_p$ , then we could solve the big problem.
- All the information is contained in the stalks and costalks of  $\mathcal{J}_!(\lambda, \mathbb{Z}) = \mathcal{J}_{!*}(\lambda, \mathbb{Z})$ .

### Conjecture (Mirkovic-Vilonen)

*The stalks of  $\mathcal{J}_{!*}(\lambda, \mathbb{Z})$  are torsion-free.*

## About the IC stalks : a conjecture of Mirkovic and Vilonen

- The IC stalks are a refinement of weight multiplicities.
- They are known for  $k = \mathbb{C}$  (Kazhdan-Lusztig polynomials).
- If we knew them for  $k = \overline{\mathbb{F}}_p$ , then we could solve the big problem.
- All the information is contained in the stalks and costalks of  $\mathcal{J}_!(\lambda, \mathbb{Z}) = \mathcal{J}_{!*}(\lambda, \mathbb{Z})$ .

### Conjecture (Mirkovic-Vilonen)

*The stalks of  $\mathcal{J}_{!*}(\lambda, \mathbb{Z})$  are torsion-free.*

- In type  $A$ , the singularities of  $\mathcal{G}r$  are nilpotent singularities. They checked that the stalks are torsion-free in nilpotent cones in type  $A_n$ ,  $n \leq 5$ .

## About the IC stalks : a conjecture of Mirkovic and Vilonen

- The IC stalks are a refinement of weight multiplicities.
- They are known for  $k = \mathbb{C}$  (Kazhdan-Lusztig polynomials).
- If we knew them for  $k = \overline{\mathbb{F}}_p$ , then we could solve the big problem.
- All the information is contained in the stalks and costalks of  $\mathcal{J}_!(\lambda, \mathbb{Z}) = \mathcal{J}_{!*}(\lambda, \mathbb{Z})$ .

### Conjecture (Mirkovic-Vilonen)

*The stalks of  $\mathcal{J}_{!*}(\lambda, \mathbb{Z})$  are torsion-free.*

- In type  $A$ , the singularities of  $\mathcal{G}r$  are nilpotent singularities. They checked that the stalks are torsion-free in nilpotent cones in type  $A_n$ ,  $n \leq 5$ .
- Bezrukavnikov noticed that this conjecture would imply a straightforward extension of his work on the unramified local geometric Langlands conjecture, from  $\mathbb{C}$  to  $\mathbb{Z}$  coefficients.

# Outline

## Perverse sheaves

Derived categories of constructible complexes

Perverse sheaves

Intersection cohomology complexes

A first example: the nilpotent cone of  $\mathfrak{sl}_2$

## Geometric Satake correspondence

The big problem of modular representation theory

The geometric picture

The theorem of Mirkovic and Vilonen

A conjecture on the stalks

## Minimal degenerations in the affine Grassmannians

### Classification

Simple singularities

Minimal singularities

Quasi-minimal singularities

## Covering relation of dominant weights

We say  $\lambda > \mu$  is a minimal degeneration if there is no  $\nu \in X(T)^+$  in between, and we write  $\lambda \rightsquigarrow \mu$ .

## Covering relation of dominant weights

We say  $\lambda > \mu$  is a minimal degeneration if there is no  $\nu \in X(T)^+$  in between, and we write  $\lambda \rightsquigarrow \mu$ .

We denote by  $\bar{\alpha}$  the highest short root.

## Covering relation of dominant weights

We say  $\lambda > \mu$  is a minimal degeneration if there is no  $\nu \in X(T)^+$  in between, and we write  $\lambda \rightsquigarrow \mu$ .

We denote by  $\bar{\alpha}$  the highest short root.

### Theorem (Stembridge)

*If  $\lambda \rightsquigarrow \mu$  then  $\beta := \lambda - \mu \in \Phi^+$ , and up to some Levi reduction, we have (with Bourbaki's numbering) either*

1.  $\beta \in \Delta$  (can reduce to type  $A_1$ )
2.  $\beta = \bar{\alpha}$  and  $\mu = 0$ , in any type
3.  $\beta = \bar{\alpha}$  and  $\mu = \varpi_n$ , in type  $B_n$
4.  $\beta = \alpha_1 + \alpha_2$  and  $\mu = 2\varpi_1$ , in type  $G_2$
5.  $\beta = \alpha_1 + \alpha_2$  and  $\mu = \varpi_1$ , in type  $G_2$

# Classification of the minimal degeneration singularities

## Theorem (Malkin-Ostrik-Vybornov)

In the first two cases, the singularity  $\text{Sing}(\overline{\mathcal{G}r}_\lambda, \mathcal{G}r_\mu)$  is

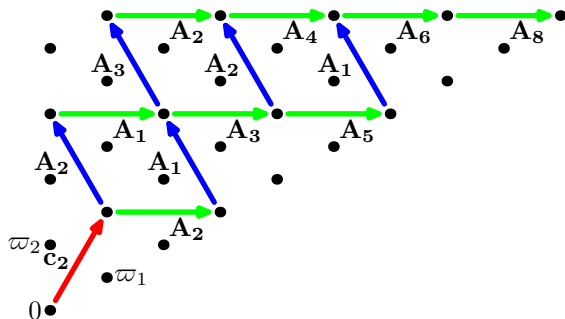
1. a **simple singularity** of type  $A_{m-1}$ , where  $m = \langle \lambda, \beta^\vee \rangle$ , equivalent to  $0 \in \mathbb{C}^2 / C_m$
2. a **minimal singularity** of the type of  $G^\vee$ , equivalent to  $0 \in \overline{\mathcal{O}}_{\min}$ , where  $\mathcal{O}_{\min} \subset \mathfrak{g}^\vee$ , one of  $a_n c_n b_n d_n e_6 e_7 e_8 f_4 g_2$

## Definition (Malkin-Ostrik-Vybornov)

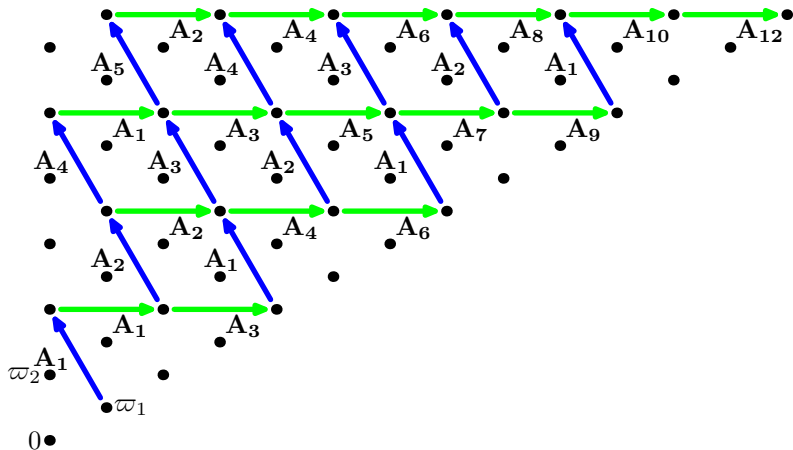
In the other cases, we call  $\text{Sing}(\overline{\mathcal{G}r}_\lambda, \mathcal{G}r_\mu)$  a **quasi-minimal singularity** of type

3.  $ac_n$ , which has the same IC stalks as  $a_n$  over  $\mathbb{Q}$
4.  $ag_2$ , which has the same IC stalks as  $a_2$  over  $\mathbb{Q}$
5.  $cg_2$ , which has the same IC stalks as  $c_2$  over  $\mathbb{Q}$

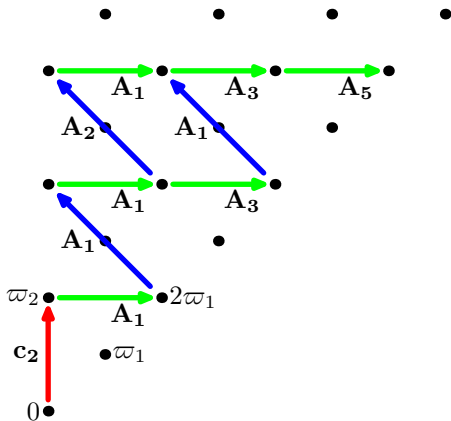
## Type $A_2$ , first component



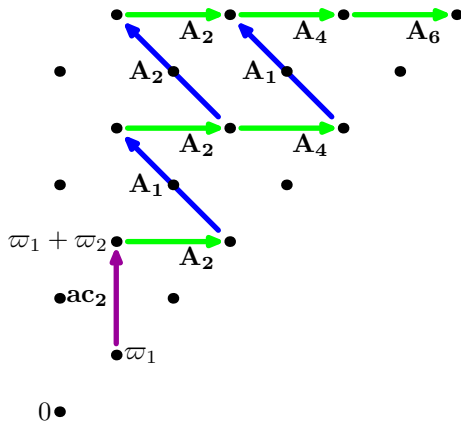
Type  $A_2$ , second component (the third one is similar)



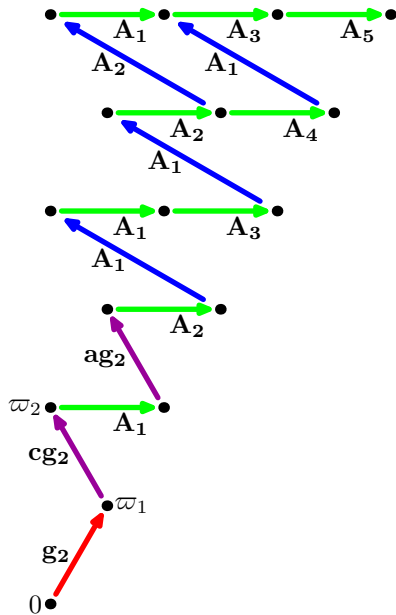
# Type $B_2$ , first component



# Type $B_2$ , second component



# Type $G_2$



# Outline

## Perverse sheaves

Derived categories of constructible complexes

Perverse sheaves

Intersection cohomology complexes

A first example: the nilpotent cone of  $\mathfrak{sl}_2$

## Geometric Satake correspondence

The big problem of modular representation theory

The geometric picture

The theorem of Mirkovic and Vilonen

A conjecture on the stalks

## Minimal degenerations in the affine Grassmannians

Classification

**Simple singularities**

Minimal singularities

Quasi-minimal singularities

## Simple singularities

The simple singularity of type  $A_{n-1}$  is  $\mathbb{C}^2/C_n$ . Say  $C_n = \langle \sigma \rangle$ .

As for  $A_1$ , but now with  $U = (\mathbb{C}^2 \setminus \{0\})/C_n \sim S^3/C_n$ , we have

$$R\Gamma(\mathbb{C}^2 \setminus \{0\}, k) = (kC_n \xrightarrow{1-\sigma} kC_n \xrightarrow{s} kC_n \xrightarrow{1-\sigma} kC_n)$$

where  $s = 1 + \sigma + \cdots + \sigma^{n-1}$ , and, taking derived invariants,

$$R\Gamma(U, k) = (k \xrightarrow{0} k \xrightarrow{n} k \xrightarrow{0} k)$$

## Simple singularities

The simple singularity of type  $A_{n-1}$  is  $\mathbb{C}^2/C_n$ . Say  $C_n = \langle \sigma \rangle$ .

As for  $A_1$ , but now with  $U = (\mathbb{C}^2 \setminus \{0\})/C_n \sim S^3/C_n$ , we have

$$R\Gamma(\mathbb{C}^2 \setminus \{0\}, k) = (kC_n \xrightarrow{1-\sigma} kC_n \xrightarrow{s} kC_n \xrightarrow{1-\sigma} kC_n)$$

where  $s = 1 + \sigma + \cdots + \sigma^{n-1}$ , and, taking derived invariants,

$$R\Gamma(U, k) = (k \xrightarrow{0} k \xrightarrow{n} k \xrightarrow{0} k)$$

The “critical” cohomology group over  $\mathbb{Z}$  is  $\mathbb{Z}/n$ , and we have

$$d_{\lambda\mu}^G = d_{\lambda\mu}^{Gr} = \begin{cases} 1 & \text{if } p \mid n \\ 0 & \text{otherwise} \end{cases}$$

## Simple singularities

The simple singularity of type  $A_{n-1}$  is  $\mathbb{C}^2/C_n$ . Say  $C_n = \langle \sigma \rangle$ .

As for  $A_1$ , but now with  $U = (\mathbb{C}^2 \setminus \{0\})/C_n \sim S^3/C_n$ , we have

$$R\Gamma(\mathbb{C}^2 \setminus \{0\}, k) = (kC_n \xrightarrow{1-\sigma} kC_n \xrightarrow{s} kC_n \xrightarrow{1-\sigma} kC_n)$$

where  $s = 1 + \sigma + \cdots + \sigma^{n-1}$ , and, taking derived invariants,

$$R\Gamma(U, k) = (k \xrightarrow{0} k \xrightarrow{n} k \xrightarrow{0} k)$$

The “critical” cohomology group over  $\mathbb{Z}$  is  $\mathbb{Z}/n$ , and we have

$$d_{\lambda\mu}^G = d_{\lambda\mu}^{Gr} = \begin{cases} 1 & \text{if } p \mid n \\ 0 & \text{otherwise} \end{cases}$$

More generally, for simple singularities of ADE types, we get  $P/Q$  for the “critical” cohomology group.

# Outline

## Perverse sheaves

Derived categories of constructible complexes

Perverse sheaves

Intersection cohomology complexes

A first example: the nilpotent cone of  $\mathfrak{sl}_2$

## Geometric Satake correspondence

The big problem of modular representation theory

The geometric picture

The theorem of Mirkovic and Vilonen

A conjecture on the stalks

## Minimal degenerations in the affine Grassmannians

Classification

Simple singularities

**Minimal singularities**

Quasi-minimal singularities

## Minimal singularities

In a simple complex Lie algebra  $\mathfrak{g} = \text{Lie } G$ , there is a unique minimal non-trivial nilpotent orbit  $\mathcal{O}_{\min}$ .

It is the orbit of a highest weight vector  $e_{\tilde{\alpha}}$  for the adjoint representation (the weight is the highest root  $\tilde{\alpha}$ ).

Its closure  $\overline{\mathcal{O}}_{\min} = \mathcal{O}_{\min} \cup \{0\}$  is a cone.

We have to compute  $H^*(\mathcal{O}_{\min}, \mathbb{Z})$ .

## The minimal singularity of type $c_n$

$G = Sp_{2n}$  is the subgroup of  $GL_{2n}$  stabilizing some symplectic form on  $\mathbb{C}^{2n}$  given by a matrix  $Q$  with respect to the canonical basis.

Let us take  $Q = \begin{pmatrix} 0 & 1 \\ J & 0 \end{pmatrix}$ , where  $J = \begin{pmatrix} & & & 1 \\ & & \ddots & \\ & & & \\ 1 & & & \end{pmatrix}$ .

We have a morphism  $q : \mathbb{C}^{2n} \rightarrow \overline{\mathcal{O}}_{\min}$  which sends the vector  $(x_1, \dots, x_n, y_1, \dots, y_n)$  to the matrix

$$\begin{pmatrix} -x_1 y_n & \cdots & -x_1 y_1 & x_1 x_n & \cdots & x_1^2 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ -x_n y_n & \cdots & -x_n y_1 & x_n^2 & \cdots & x_n x_1 \\ -y_1 y_n & \cdots & -y_1^2 & y_1 x_n & \cdots & y_1 x_1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ -y_n^2 & \cdots & -y_n y_1 & y_n x_n & \cdots & y_n x_1 \end{pmatrix}$$

This can be identified with the quotient by  $\{\pm 1\}$ .

This map induces an isomorphism  $(\mathbb{C}^{2n} \setminus \{0\})/\{\pm 1\} \xrightarrow{\sim} \mathcal{O}_{\min}$ .

Thus  $\mathcal{O}_{\min}$  is homotopic to  $\mathbb{R}P^{4n-1}$ , and we have

$$R\Gamma(\mathcal{O}_{\min}, k) = R\Gamma(\mathbb{R}P^{4n-1}, k) = (k \xrightarrow{0} k \xrightarrow{2} k \xrightarrow{0} \dots \xrightarrow{2} k \xrightarrow{0} k)$$

where the last complex has a copy of  $k$  in each degree between 0 and  $4n - 1$ , and the differential is alternatively 0 and 2.

Thus

$$H^m(\mathcal{O}_{\min}, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } m = 0, 4n - 1 \\ \mathbb{Z}/2 & \text{in all even degrees in between} \\ 0 & \text{otherwise} \end{cases}$$

## General strategy to compute the cohomology of $\mathcal{O}_{\min}$

Notation:  $\Phi \supset \Phi^+ \supset \Delta =$  root system, positive roots, simple roots

Let  $\tilde{I} \subset \Delta$  denote the set of simple roots orthogonal to  $\tilde{\alpha}$ .

Then  $W_{\tilde{I}} = \text{Stab}_W(\tilde{\alpha})$  and  $P_{\tilde{I}} = N_G(\mathbb{C}e_{\tilde{\alpha}})$ .

## General strategy to compute the cohomology of $\mathcal{O}_{\min}$

Notation:  $\Phi \supset \Phi^+ \supset \Delta =$  root system, positive roots, simple roots

Let  $\tilde{I} \subset \Delta$  denote the set of simple roots orthogonal to  $\tilde{\alpha}$ .

Then  $W_{\tilde{I}} = \text{Stab}_W(\tilde{\alpha})$  and  $P_{\tilde{I}} = N_G(\mathbb{C}e_{\tilde{\alpha}})$ .

The line bundle  $G \times_{P_{\tilde{I}}} \mathbb{C}_{\tilde{\alpha}}$  gives a resolution of  $\overline{\mathcal{O}}_{\min}$ , and  $\mathcal{O}_{\min}$  is the corresponding  $\mathbb{C}^*$ -bundle. Its cohomology fits in a Gysin sequence.

## General strategy to compute the cohomology of $\mathcal{O}_{\min}$

Notation:  $\Phi \supset \Phi^+ \supset \Delta =$  root system, positive roots, simple roots

Let  $\tilde{I} \subset \Delta$  denote the set of simple roots orthogonal to  $\tilde{\alpha}$ .

Then  $W_{\tilde{I}} = \text{Stab}_W(\tilde{\alpha})$  and  $P_{\tilde{I}} = N_G(\mathbb{C}e_{\tilde{\alpha}})$ .

The line bundle  $G \times_{P_{\tilde{I}}} \mathbb{C}\tilde{\alpha}$  gives a resolution of  $\overline{\mathcal{O}}_{\min}$ , and  $\mathcal{O}_{\min}$  is the corresponding  $\mathbb{C}^*$ -bundle. Its cohomology fits in a Gysin sequence.

By the Schubert calculus, we know explicitly the integral cohomology of  $G/P_{\tilde{I}}$  and how to multiply by the first Chern class of our line bundle, so everything can be done explicitly.

## General strategy to compute the cohomology of $\mathcal{O}_{\min}$

Notation:  $\Phi \supset \Phi^+ \supset \Delta =$  root system, positive roots, simple roots

Let  $\tilde{I} \subset \Delta$  denote the set of simple roots orthogonal to  $\tilde{\alpha}$ .

Then  $W_{\tilde{I}} = \text{Stab}_W(\tilde{\alpha})$  and  $P_{\tilde{I}} = N_G(\mathbb{C}e_{\tilde{\alpha}})$ .

The line bundle  $G \times_{P_{\tilde{I}}} \mathbb{C}_{\tilde{\alpha}}$  gives a resolution of  $\overline{\mathcal{O}}_{\min}$ , and  $\mathcal{O}_{\min}$  is the corresponding  $\mathbb{C}^*$ -bundle. Its cohomology fits in a Gysin sequence.

By the Schubert calculus, we know explicitly the integral cohomology of  $G/P_{\tilde{I}}$  and how to multiply by the first Chern class of our line bundle, so everything can be done explicitly.

Moreover, here there is a nice description in terms of root combinatorics.

$IC(\overline{O}_{\min}, \mathbb{Z})_0$

exceptional types

$E_6$

-22	$\mathbb{Z}$
-21	.
-20	.
-19	.
-18	.
-17	.
-16	$\mathbb{Z}$
-15	.
-14	$\mathbb{Z}$
-13	.
-12	.
-11	.
-10	$\mathbb{Z}$
-9	.
-8	$\mathbb{Z}$
-7	.
-6	$\mathbb{Z}/3$
-5	.
-4	$\mathbb{Z}/2$
-3	.
-2	$\mathbb{Z}$
-1	.

$E_7$

-34	$\mathbb{Z}$
-33	.
-32	.
-31	.
-30	.
-29	.
-28	.
-27	.
-26	$\mathbb{Z}$
-25	.
-24	.
-23	.
-22	$\mathbb{Z}$
-21	.
-20	.
-19	.
-18	$\mathbb{Z}$
-17	.
-16	$\mathbb{Z}/2$
-15	.
-14	$\mathbb{Z}$
-13	.
-12	.
-11	.
-10	$\mathbb{Z}$
-9	.
-8	$\mathbb{Z}/2$
-7	.
-6	$\mathbb{Z}/3$
-5	.
-4	$\mathbb{Z}/2$
-3	.
-2	$\mathbb{Z}$
-1	.

$E_8$

-58	$\mathbb{Z}$	-29	.
-57	.	-28	$\mathbb{Z}/2$
-56	.	-27	.
-55	.	-26	$\mathbb{Z}$
-54	.	-25	.
-53	.	-24	.
-52	.	-23	.
-51	.	-22	$\mathbb{Z}$
-50	.	-21	.
-49	.	-20	.
-48	.	-19	.
-47	.	-18	$\mathbb{Z}/3$
-46	$\mathbb{Z}$	-17	.
-45	.	-16	$\mathbb{Z}/2$
-44	.	-15	.
-43	.	-14	$\mathbb{Z}$
-42	.	-13	.
-41	.	-12	.
-40	.	-11	.
-39	.	-10	$\mathbb{Z}/5$
-38	$\mathbb{Z}$	-9	.
-37	.	-8	$\mathbb{Z}/2$
-36	.	-7	.
-35	.	-6	$\mathbb{Z}/3$
-34	$\mathbb{Z}$	-5	.
-33	.	-4	$\mathbb{Z}/2$
-32	.	-3	.
-31	.	-2	$\mathbb{Z}$
-30	.	-1	.

$F_4$

-16	$\mathbb{Z}$
-15	.
-14	.
-13	.
-12	.
-11	.
-10	$\mathbb{Z}/2$
-9	.
-8	$\mathbb{Z}$
-7	.
-6	.
-5	.
-4	$\mathbb{Z}/4$
-3	.
-2	$\mathbb{Z}/2$
-1	.

$G_2$

-6	$\mathbb{Z}$
-5	.
-4	.
-3	.
-2	$\mathbb{Z}/3$
-1	.

## General results about the cohomology of $\mathcal{O}_{\min}$

Let  $\Phi'$  denote the root subsystem generated by the long simple roots.

### Theorem (J.)

1. *The “critical” cohomology group is  $P(\Phi')/Q(\Phi')$ .*
2. *The other cohomology groups can have torsion (indeed in all types but in type A), but only for bad primes.*

## General results about the cohomology of $\mathcal{O}_{\min}$

Let  $\Phi'$  denote the root subsystem generated by the long simple roots.

### Theorem (J.)

1. *The “critical” cohomology group is  $P(\Phi')/Q(\Phi')$ .*
2. *The other cohomology groups can have torsion (indeed in all types but in type A), but only for bad primes.*

### Remark

*The second part is checked case by case !*

## General results about the cohomology of $\mathcal{O}_{\min}$

Let  $\Phi'$  denote the root subsystem generated by the long simple roots.

### Theorem (J.)

1. *The “critical” cohomology group is  $P(\Phi')/Q(\Phi')$ .*
2. *The other cohomology groups can have torsion (indeed in all types but in type A), but only for bad primes.*

### Remark

*The second part is checked case by case !*

### Corollary

1. *The decomposition number is  $\dim_{\mathbb{F}_p} \mathbb{F}_p \otimes_{\mathbb{Z}} P(\Phi')/Q(\Phi')$ .*
2. *The conjecture of Mirkovic and Vilonen is **wrong** as stated, except maybe in type A.*

## A modified conjecture

In view of our observations, we can propose the following modified conjecture:

### Conjecture

*If  $p$  is a good prime, then the stalks of  $\mathcal{J}_{!*}(\lambda, \mathbb{Z}_p)$  are torsion-free.*

# Outline

## Perverse sheaves

Derived categories of constructible complexes

Perverse sheaves

Intersection cohomology complexes

A first example: the nilpotent cone of  $\mathfrak{sl}_2$

## Geometric Satake correspondence

The big problem of modular representation theory

The geometric picture

The theorem of Mirkovic and Vilonen

A conjecture on the stalks

## Minimal degenerations in the affine Grassmannians

Classification

Simple singularities

Minimal singularities

Quasi-minimal singularities

# Proof of a conjecture of Malkin, Ostrik and Vybornov

## Conjecture (Malkin-Ostrik-Vybornov)

*The following singularities are pairwise non-equivalent:*

1.  $a_n$  and  $ac_n$
2.  $a_2$ ,  $ac_2$  and  $ag_2$
3.  $c_2$  and  $cg_2$

# Proof of a conjecture of Malkin, Ostrik and Vybornov

## Conjecture (Malkin-Ostrik-Vybornov)

*The following singularities are pairwise non-equivalent:*

1.  $a_n$  and  $ac_n$
2.  $a_2$ ,  $ac_2$  and  $ag_2$
3.  $c_2$  and  $cg_2$

PROOF - [J.] The decomposition numbers for  $G$  are:

1. 1 if  $p \mid n + 1$  (resp.  $2n + 1$ ) and 0 otherwise
2. 1 if  $p = 3$  (resp. 5, 7) and 0 otherwise
3. 1 if  $p = 2$  (resp. 3) and 0 otherwise

So in each case, the IC stalks have to be different for some prime  $p$ .

