Exposé a Oxford en Avril 2000
Sur les orientaux de Street

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Dernière visite, le 13 novembre 2005

Le contenu de ce manuscript est celui d'un exposé fait à Oxford (University Computing Labotatory) en Avril 2000 dans le cadre de "Linear Logic in Computer Science (3^{rd} Annual Meeting, Merton College, Oxford, 15-18 April 2000)". L'exposé oral était en Français, tandis que ce texte était formé de "slides" projeté à l'intention de la partie anglophone du public. Il est donc écrit dans style abrégé aux intentions mi-techniques, mi-pédagogiques. Il contient de nombreuses fautes (pas seulement linguistiques). Je les ai laissées sans aucune correction. Il me semble que, tel qu'il est, il peut servir d'introduction rapide à l'article plus élaboré sur le même sujet et que je compte mettre sur ma page web incessamment.

OXFORD
Avril 2000

A new calculation of the orientals of Street

$\boxed{\text{objects}}$ orientals

(elementary ∞-catégories)

$\boxed{\text{aim}}$ I give a simple natural and algorithmic description of these objects, as normal forms w.r.t. an easy rewriting system

objects of study

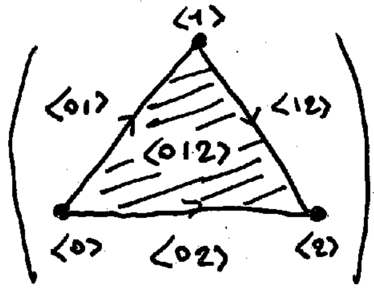
an <u>oriental</u> $\mathcal{O}^{(n)}$, of dimension n, is an object of ∞-$\underline{Cat}$ (the category of ∞-categories) similar to a simplicial model Δ_n in the category

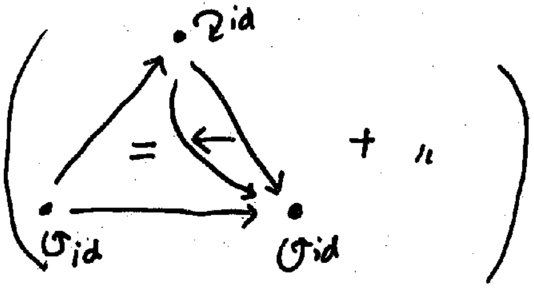
$$\underline{Simpl} = \underline{Set}^{\Delta^{op}}$$

of simplicial sets.

$$\Delta_0 = \Big(\overset{\langle 0\rangle}{\bullet}\Big) \mapsto \mathcal{O}^{(0)} = \Big(\bullet \circlearrowleft_{id} + \text{all identities of dim} \geq 2\Big)$$

$$\Delta_1 = \Big(\underset{\langle 0\rangle}{\bullet} \xrightarrow{\langle 01\rangle} \underset{\langle 1\rangle}{\bullet}\Big) \mapsto \mathcal{O}^{(1)} = \Big(\underset{\circlearrowleft id}{\bullet} \longrightarrow \underset{\circlearrowleft id}{\bullet} + \text{"}\Big)$$

$\Delta_2 =$ 

$\mapsto \mathcal{O}^{(2)} =$ 

$\boxed{\text{aim}}$

Recursive description (and definition) of orientals are considered as very difficult. Street gave a construction in 1987, but his paper is very difficult to read.

I give a natural (recursive) description. First, I recall the definition of the ∞-categories and construct a <u>monad</u> (Λ, η, μ) on ∞-<u>Cat</u> (cat. of the ∞-cat.)

Then, I obtain the orientals by

$$\mathcal{O}^{(n)} = \Lambda^{n+1}\phi$$

In fact I construct a <u>cosimplicial set</u> (nerve) in ∞-<u>Cat</u>:

$$\begin{tikzcd}
(\phi \arrow[r, dashed, "\eta\phi"] &)\ \Lambda\phi \arrow[r, shift left=3, "\eta\Lambda\phi"] \arrow[r, shift left=0, "\Lambda\eta\phi"] & \Lambda^2\phi \arrow[l, shift left=3, "\mu\phi"] \arrow[r, shift left=6, "\eta\Lambda^2\phi"] \arrow[r, shift left=3, "\Lambda\eta\Lambda\phi"] \arrow[r, "\Lambda^2\eta\phi"] & \Lambda^3\phi \arrow[l, shift left=3, "\mu\Lambda\phi"] \arrow[l, shift left=6, "\Lambda\mu\phi"] & \cdots
\end{tikzcd}$$

∞ - Graphs

data : $C = ((C_i)_{0 \le i}, (s_j^i, t_j^i)_{0 \le j \le i})$

axioms :

$$\boxed{\begin{aligned} s_j^k s_k^i = s_j^i = s_j^k t_k^i \\ t_j^k s_k^i = t_j^i = t_j^k t_k^i \end{aligned}} \qquad \boxed{s_i^i = t_i^i = id_{C_i}}$$

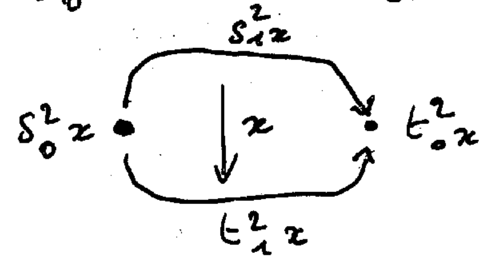
$$\begin{tikzcd} C_j \arrow[r, shift left, "s_j^k"] \arrow[r, shift right, "t_j^k"'] \arrow[rr, bend right, "s_j^i"'] \arrow[rr, bend right=50, "t_j^i"'] & C_k \arrow[r, shift left, "s_k^i"] \arrow[r, shift right, "t_k^i"'] & C_i \end{tikzcd}$$

geometrical representation :

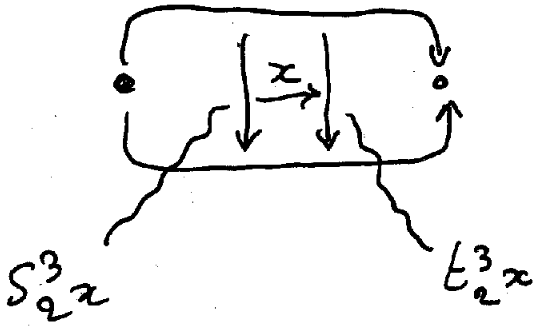
$x \in C_0$ 0-cell $\bullet\, x$

$x \in C_1$ 1-cell $s_0^1 x \bullet \xrightarrow{x} \bullet\, t_0^1 x$

$x \in C_2$ 2-cell



$x \in C_3$ 3-cell



<u>homomorphisms</u> : $F = (F_i)_{0 \le i} : C \to C'$

$$\begin{tikzcd} C_j \arrow[d, "F_j"'] & C_i \arrow[l, shift right, "s_j^i"'] \arrow[l, shift left, "t_j^i"] \arrow[d, "F_i"] \\ C'_j & C'_i \arrow[l, shift right, "s_j^i"'] \arrow[l, shift left, "t_j^i"] \end{tikzcd}$$

$$F_j s_j^i = s_j^i F_i$$
$$F_j t_j^i = t_j^i F_i$$

∞ - graphs and their homomorphisms form a category

∞ - <u>Graph</u> .

examples of ∞ - graphs

$e_0 = (\cdot)$ is $\{0\} \leftleftarrows \emptyset \leftleftarrows \emptyset \leftleftarrows \emptyset$

$e_1 = (\cdot \to \cdot)$ $\{0,1\} \leftleftarrows \{0\} \leftleftarrows \emptyset \leftleftarrows \emptyset$

$e_2 = (\cdot \underset{\to}{\overset{\to}{\downarrow}} \cdot)$ $\{0,1\} \leftleftarrows \{0,1\} \leftleftarrows \{0\} \leftleftarrows \emptyset$

$$\boxed{C_i = \mathrm{Hom}_{\infty\text{-}\underline{\mathrm{Graph}}}(e_i, C)}$$

∞-categories

data:

$$C = \Big(\underbrace{(C_i)_{0\le i},\ (s^i_j, t^i_j)_{0\le j\le i}}_{\substack{\text{geometrical side} \\ (\infty\text{-graph})}},\ \underbrace{(\mathrm{comp}^i_j)_{0\le j<i},\ (id^i_j)_{0\le j\le i}}_{\text{algebraic side}} \Big)$$

$$\mathrm{comp}^i_j : C_i \times_{C_j} C_i \longrightarrow C_i$$

$$C_i \times_{C_j} C_i = \{ (x,y) \in C_i \times C_i \mid s^i_j(x) = t^i_j(y) \}$$

usually we write: $y *^i_j x = \mathrm{comp}^i_j(x,y)$

$$id^i_j : C_j \longrightarrow C_i$$

axioms

(i) (positions)

$(j<k<i)$
$$\boxed{\begin{array}{c} s^i_k(y *^i_j z) = s^i_k(y) *^i_k s^i_k(z) \\ t^i_k(y *^i_j x) = t^i_k(y) *^i_k t^i_k(x) \\ \hline s^i_j(y *^i_j z) = s^i_j(x) \\ t^i_j(y *^i_j x) = t^i_j(x) \end{array}}$$
$(j<i)$

$(j<k<i)$
$(j<i)$
$$\boxed{\begin{array}{c} s^i_k(id^i_j u) = id^i_k(u) = t^i_k(id^i_j(u)) \\ \hline s^i_j(id^i_j u) = u = t^i_j(id^i_j(u)) \end{array}}$$

(ii) (associativity)

$$\boxed{(z *^i_j y) *^i_j x = z *^i_j (y *^i_j x)}$$

(iii) (neutrality)

$$\boxed{id^i_j(t^i_j x) *^i_j x = x = x *^i_j id^i_j(s^i_j x)}$$

(iv) (pure exchange)

$(j<k<i)$

$$\boxed{\begin{aligned}&(y' *^i_k y) *^i_j (x' *^i_k x)\\ = {}&(y' *^i_j x') *^i_k (y *^i_j x)\end{aligned}}$$

$(j<k<i)$

$$\boxed{id^k_j\,(id^i_k\, u) = id^i_j\, u}$$

(v) (mixt exchange)

$(k<j<i)$

$$\boxed{\begin{aligned}&id^i_j\,(v *^j_k u) = (id^i_j\, v) *^i_k (id^i_j)\\ &(id^i_j\, u) *^i_k (id^i_j\, u) = id^i_k\, u\end{aligned}}$$

∞-functors : $F : C \to C'$

homom. of ∞-graph
+ commutation with operations

$$\begin{tikzcd}
C_j \arrow[r, "id_j^i"] \arrow[d, "F_j"'] & C_i \arrow[d, "F_i"] & C_i \times_{C_j} C_i \arrow[l, "comp_j^i"'] \arrow[d, "F_i \times_{F_j} F_i"] \\
C'_j \arrow[r, "id_j^i"'] & C'_i & C'_i \times_{C'_j} C'_i \arrow[l, "comp_j^i"]
\end{tikzcd}$$

∞-categories and their functors form a category

∞-Cat

property needed in the sequel for ∞-Cat :

he admits arbitrary coproducts and pushouts ;

the underlying functor $\infty\text{-Cat} \xrightarrow{U} \infty\text{-Grph}$

admits a left adjoint $\infty\text{-Grph} \xrightarrow{L} \infty\text{-Cat}$.

In fact, we need $L(e_i)$ (for all integer i)

which has infinite number cells, of all dimensions, but the same cells as e_i which are not identities.

Two elementary constructions

We have two families of canonical morphisms:

$$\dot{e}_n \overset{j_n}{\rightarrowtail} e_n \overset{q_n}{\twoheadleftarrow} \dot{e}_{n+1}$$

where

$$\dot{e}_0 = (\ ) \qquad e_0 = (\bullet)$$

$$\dot{e}_1 = (\bullet \quad \bullet) \qquad e_1 = (\bullet \longrightarrow \bullet)$$

$$\dot{e}_2 = (\bullet \rightrightarrows \bullet) \qquad e_2 = (\bullet \overset{\longrightarrow}{\underset{\longrightarrow}{\Downarrow}} \bullet)$$

$$\vdots$$

which gives in ∞-Cat, by adjuntion:

$$L(\dot{e}_n) \overset{j_n}{\rightarrowtail} L(e_n) \overset{q_n}{\twoheadleftarrow} L(\dot{e}_{n+1})$$

Let I be a set, D an ∞-catgorie,

let
$$I \cdot D =. \coprod_{i \in I} D,$$

the coproduct of I copies of D.

For all ∞-functor of the form

$$\Sigma \cdot L(\dot{e}_n) \xrightarrow{\varphi} C$$

where Σ is a set,

we construct two pushouts :

$$\begin{tikzcd} \Sigma \cdot L(\dot{e}_n) \arrow[r, "\varphi"] \arrow[d, "\Sigma \cdot j_n"'] \arrow[dr, phantom, "po"] & C \arrow[d] \\ \Sigma \cdot L(e_n) \arrow[r] & C[\Sigma] \end{tikzcd}$$

called

n-<u>extention</u>

of C by (Σ, φ)

$$\begin{tikzcd} \Sigma \cdot L(\dot{e}_{n+1}) \arrow[r, "\varphi"] \arrow[d, "\Sigma \cdot q_n"'] \arrow[dr, phantom, "po"] & C \arrow[d] \\ \Sigma \cdot L(e_n) \arrow[r] & C/\Sigma \end{tikzcd}$$

called

n-<u>collapse</u> (quotient)

of C by (Σ, φ)

(n+1)-Layer of an ∞-category

Another form for the data

$$\Sigma \cdot L(e_{n+1}) \xrightarrow{\varphi} C$$

is a diagram in <u>Set</u> :

$$\Sigma \underset{\tau}{\overset{\sigma}{\rightrightarrows}} C_n \underset{t^n_{n-1}}{\overset{s^n_{n-1}}{\rightrightarrows}} C_{n-1}$$

with the commutation

$$s^n_{n-1}\,\sigma = s^n_{n-1}\,\tau$$
$$t^n_{n-1}\,\sigma = t^n_{n-1}\,\tau .$$

We call (Σ, φ) or (Σ, σ, τ) a

$(n+1)$ - <u>layer</u> for C

<u>In words</u> : we add $(n+1)$-cells

without compositions nor identities.

∞-presentation of an ∞-category.

First, let be a "pair" of ∞-categories

$$C \hookrightarrow D$$

We define relative ∞-presentation of (C,D)

as a construction of the form

$$\begin{tikzcd}
C^{(-1)} = C \arrow[d] \arrow[dr, tail, "\text{ext.}"] & \\
 & C^{(-1)}[G_0] \arrow[dl, two heads, "\text{col.}"] \\
C^{(0)} = C^{(-1)}[G_0]/R_0 \arrow[d] \arrow[dr, tail, "\text{ext.}"] & \\
 & C^{(0)}[G_1] \arrow[dl, two heads, "\text{col.}"] \\
C^{(1)} = C^{(0)}[G_1]/R_1 \arrow[d] \arrow[dr, tail] & \\
\vdots & \vdots
\end{tikzcd}$$

$\left.\begin{matrix} C^{(-1)}[G_0] \end{matrix}\right\}$ 0-generators and 0-relations

$\left.\begin{matrix} C^{(0)}[G_1] \end{matrix}\right\}$ 1-generators and 1-relations

$$D = \varinjlim_n C^{(n)}$$

An ∞-presentation of an ∞-category C

is a rel. pres. of the pair $\emptyset \hookrightarrow C$:

$$\emptyset \to C^{(0)} \to C^{(1)} \to C^{(2)} \to \dots \qquad C = \varinjlim_n C^{(n)}$$

Projective free ∞-Cone

Let C be an ∞-category, we construct the ∞-cone ΛC by induction.

More precisely we build, by ∞-presentation, a pair $C \overset{\eta^C}{\hookrightarrow} \Lambda C$

We set for the first steps

$$C^{(-1)} = C$$

$$C^{(0)} = C^{(-1)} \amalg \{\omega_C\} \quad \text{(we add one 0-cell to } C\text{)}$$

Suppose we have $C^{(n)}$ $(n \geq 0)$

We set

$$C^{(n+1)} = C^{(n)}[G_{n+1}] / R_{n+1}$$

(G_{n+1}, R_{n+1} defined below)

At the same time we construct the embeddings

$$C \overset{\beta_n}{\hookrightarrow} C^{(n)} \qquad (n \geqslant 0)$$

The ∞-cone and the embedding

$$C \overset{\eta C}{\hookrightarrow} \Lambda C$$

is the $\varinjlim_{n}$ of these constructions.

<u>Proposition</u>. $C \xrightarrow{\eta C} \Lambda C$ is the unity of a monad

$$C \xrightarrow{\eta C} \Lambda C \xleftarrow{\mu} \Lambda^2 C .$$

<u>We now describe</u> G_n, R_n,

<u>informally</u>:

G_n adds cells of dimension n.

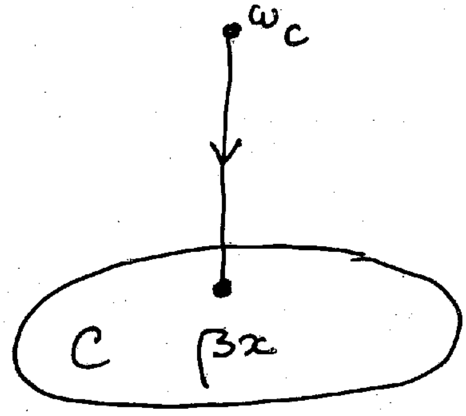
Actually, G_n is a copy of C_{n-1}.

We have a bijection

$$\lambda_n : C_{n-1} \xrightarrow{\sim} G_n$$

$$x \in C_{n-1} \longmapsto \lambda x \in G_n$$

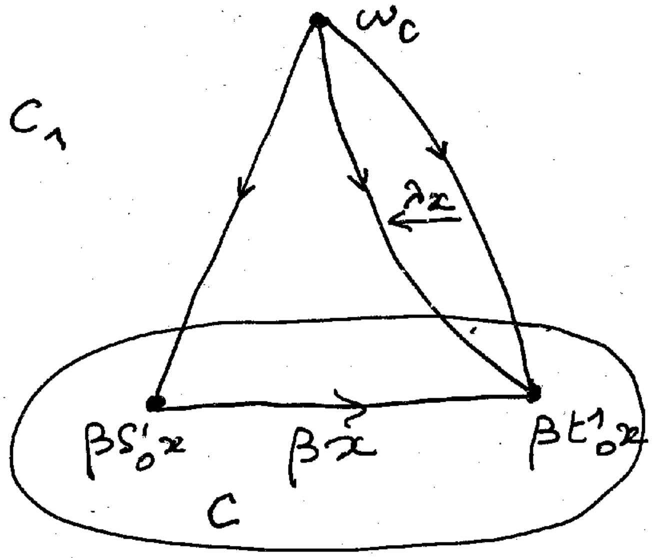
$x \in C_0$



(β is the embedding)

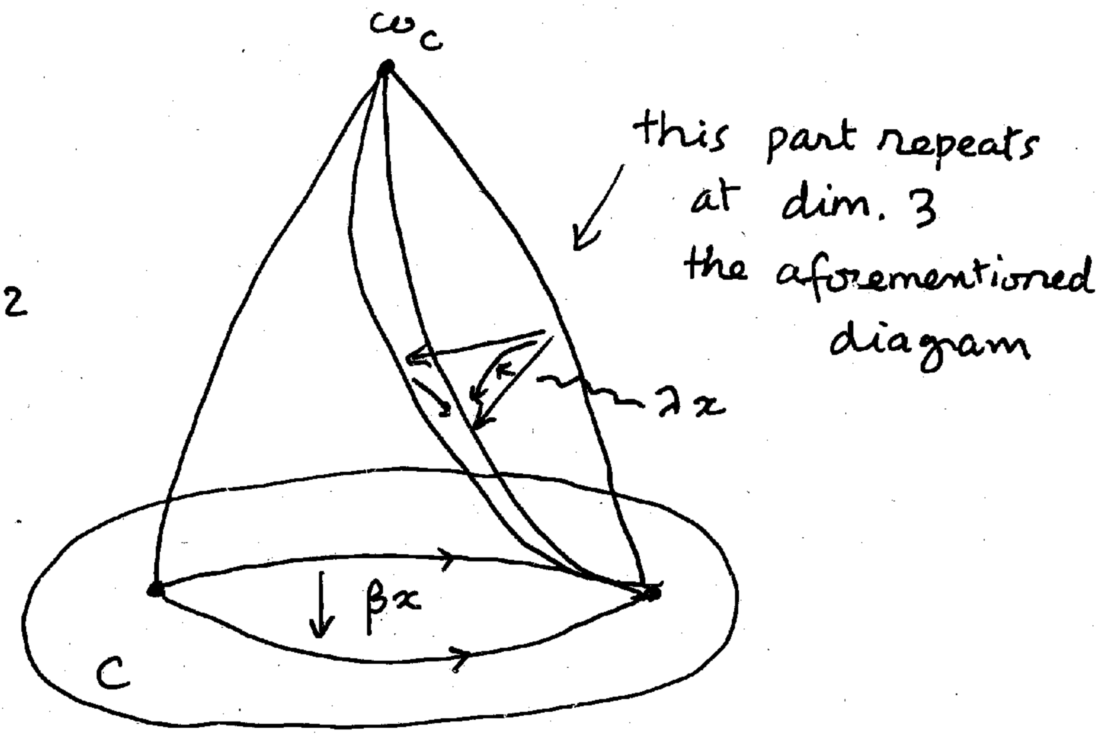
$\lambda x : \omega_C \to \beta x$

$x \in C_1$



$\lambda x : \lambda t_0^1 x \to \beta x *_0^1 \lambda s_0^1 x$.

$x \in C_2$



$$\lambda x : \lambda^2_1 x \longrightarrow \beta x *^2_0 \mathrm{id}^2_1 \lambda s^2_0 x *^2_1 \lambda s^2_1 x$$

A general formula for $x \in C_{n-1}$ and $j < n$ is

$$\boxed{\begin{aligned} s^n_j \lambda x &= \begin{cases} \omega & (j=0) \\ \lambda t^{n-1}_{j-1} x & (j \geq 1) \end{cases} \\ t^n_j \lambda x &= \beta t^{n-1}_j x *^j_0 \mathrm{id}^j_0 \lambda s^{n-1}_0 x \\ &\quad *^j_1 \mathrm{id}^j_1 \lambda s^{n-1}_1 x \\ &\quad *^j_2 \cdots \\ &\quad *^j_{j-1} \mathrm{id}^j_{j-1} \lambda s^{n-1}_{j-2} x \\ &\quad *^{n-1}_{j-1} \mathrm{id}^j_j \lambda s^{n-1}_{j-1} x . \end{aligned}}$$

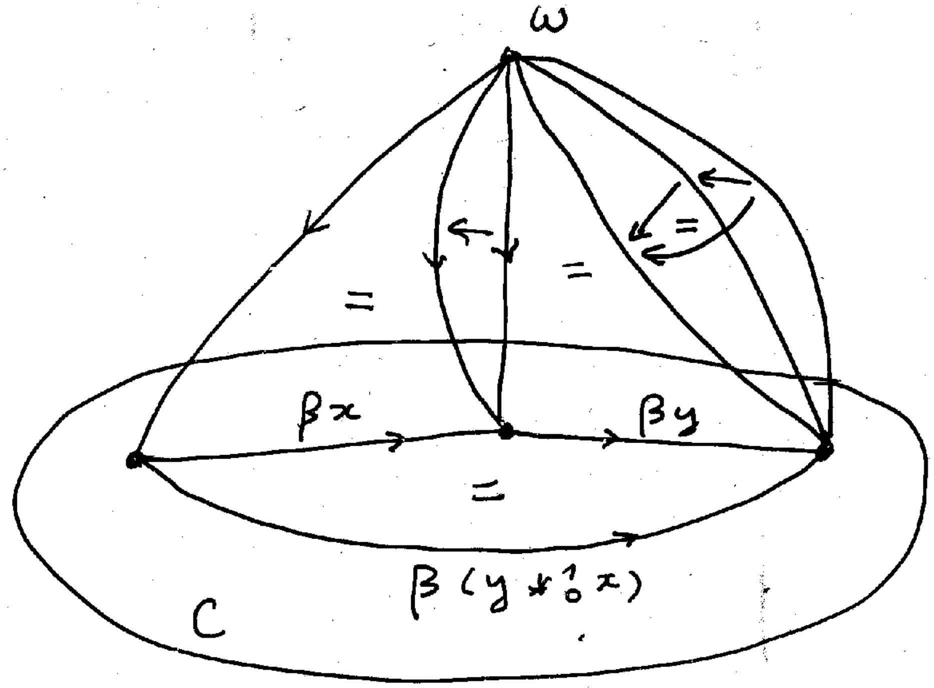
At each step, collapses ensure that these formulas hold. The collapses are described by the R_n's expressed as equalities.

for $n=0,1$ $\quad$ R_0, R_1 are empty

for $n=2$ $\quad$ R_2 is given by

$$\lambda(y *_0^1 x) = id_1^2 \beta t_1^1 y *_0^2 \lambda s_0^1 x$$

for all $x, y \in C_1$.



A general formula is,

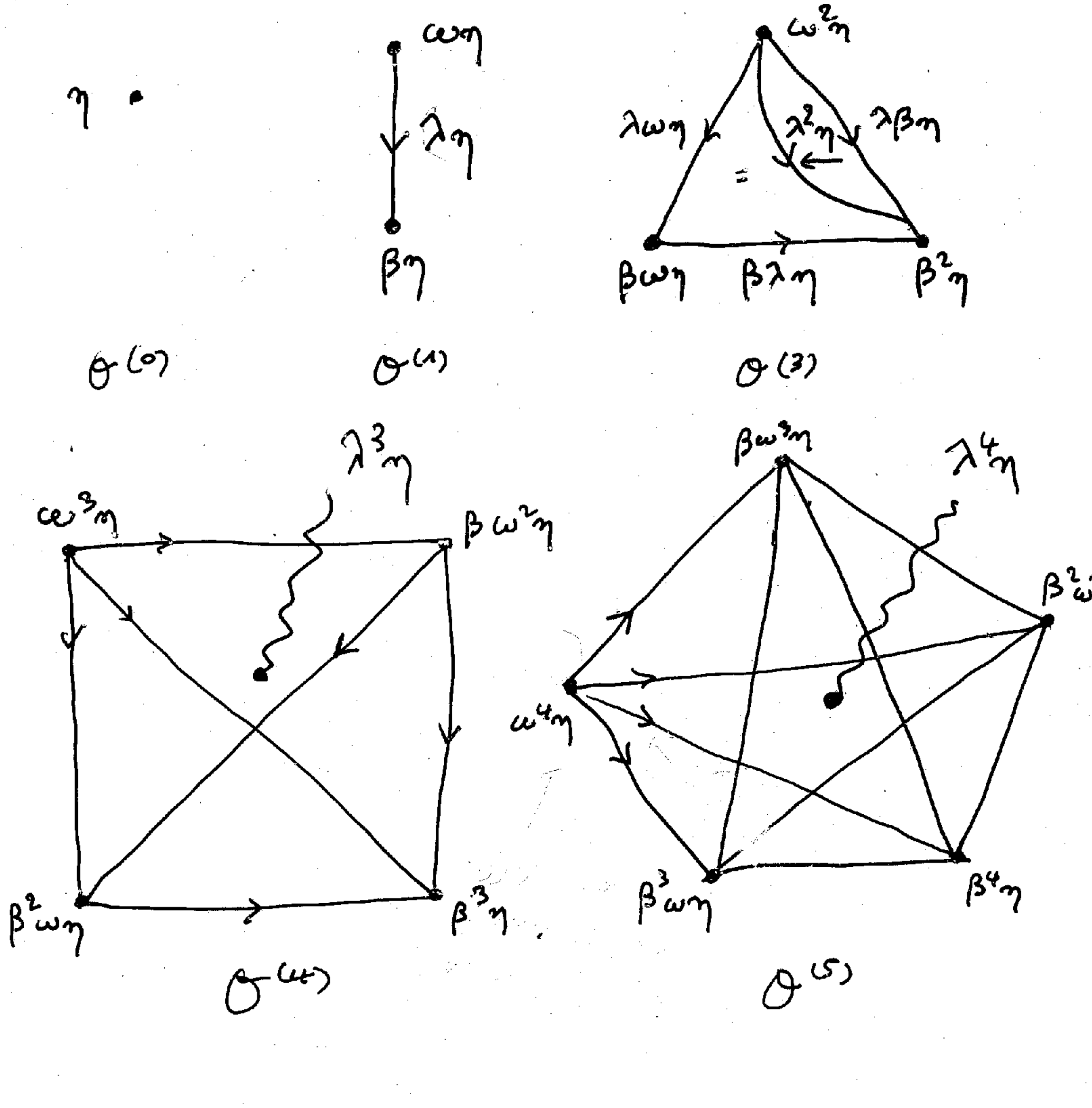
for $x, y \in C_{n-1} \times_{C_j} C_{n-1}$ and $j < n-1$,

$$
\begin{aligned}
\lambda(y *_j^{n-1} x) = \; & id^n_{j+1}\, \beta\, t^{n-1}_{j+1}\, y \\
& *^n_0 \; id^n_1\, \lambda s^{n-1}_0\, x \\
& *^n_1 \; id^n_2\, \lambda s^{n-1}_1\, x \\
& *^n_2 \; \cdots \\
& *^n_{j-1} \; id^n_{j-1}\, \lambda s^{n-1}_{j-1}\, x \\
& *^n_j \; \underbrace{\lambda x *^n_j \lambda y}
\end{aligned}
$$

We now build up a cone sequence whose elements are the orientals

$$\theta^{(n)} = \Lambda^{n+1}\phi$$

Use set $\eta = \omega\phi$, and consider ω as an operation in the sequel.



These diagrams are obtained with the following rewriting rules:

$$\eta^-_j \to \eta \qquad \eta^+_j \to \eta$$

$$(\omega x)^-_j \longrightarrow \omega x^-_j \qquad (\omega x)^+_j \to \omega x^+_j$$

$$(\beta x)^-_j \longrightarrow \beta x^-_j \qquad (\beta x)^+_j \longrightarrow \beta x^+_j$$

$$(\lambda x)^-_j \longrightarrow \lambda (x^+_{j-1})$$

$$(\lambda x)^+_j \longrightarrow \beta x^+_j +_0 \lambda x^-_0 +_1 \lambda x^- +_2 \dots +_{j-2} \lambda x^-_{j-2} +_{j-1} \lambda x^-_{j-}$$

$$\lambda(\cancel{y} +_j \cancel{x}) \to \beta y^+_{j+1} +_0 \lambda x^-_0 +_1 \lambda x^-_1 +_2 \dots +_{j-1} \lambda x^-_{j-1} +_j \lambda x +_{j+1} \lambda y$$

$\lambda(y +_j x)$

<u>Remark</u>. Notation are lightened according to the following conventions:

$$x^-_j = s^n_j x \qquad x^-_j = t^n_j x$$

$$+_j = *^n_j ,$$

and we omit identities.

Additional rules yield a more user-friendly presentation:

$$\left|\begin{array}{l} \eta = \langle 0 \rangle_0 \\ \omega \langle i \rangle_n \to \langle i \rangle_{n+1} \\ \beta \langle p_1, p_2, \dots p_j \rangle_n \to \langle p_1+1, p_2+1, \dots, p_j+1 \rangle_{n+1} \\ \lambda \langle p_1, p_2, \dots p_j \rangle_n \to \langle 0, p_1+1, p_2+1, \dots p_j+1 \rangle_{n+1} \end{array}\right.$$

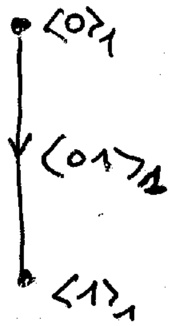
Three instance of calculus:

(1)

$$\lambda\eta \to \lambda\langle 0\rangle_0 \to \langle 0 1\rangle_1$$

$$(\lambda\eta)^-_0 \to \lambda(\eta^+_{-1}) \to \omega\eta \to \omega\langle 0\rangle_0 \to \langle 0\rangle_1$$

$$(\lambda\eta)^+_0 \to \beta\eta \to \beta\langle 0\rangle_0 \to \langle 1\rangle_1$$



(2)

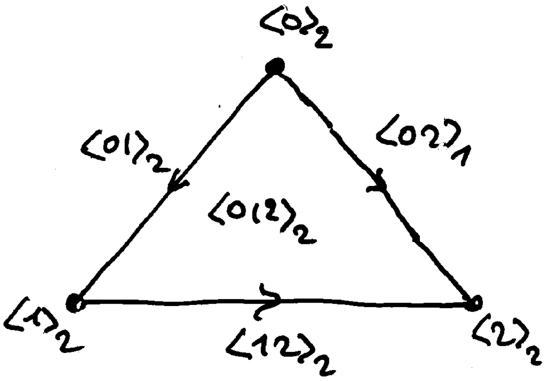
$$\lambda^2\eta \to \lambda\langle 0,1\rangle_1 \to \langle 012\rangle_2$$

$$(\lambda^2\eta)^-_1 \to \lambda((\lambda\eta)^+_0) \to \lambda\beta\eta \to \lambda\beta\langle 0\rangle_0 \xrightarrow{*} \lambda\langle 1\rangle_1 \to \langle 02\rangle_1$$

$$(\lambda^2\eta)^+_1 \to \beta\lambda\eta *_0 \lambda((\lambda\eta)^-_0) \to \beta\lambda\eta *_0 \lambda\omega\eta$$

$$\to \beta\langle 01\rangle_1 *_0 \lambda\omega\langle 0\rangle_0$$

$$\to \langle 12\rangle_2 *_0 \langle 01\rangle_2$$



(3)

$$(\lambda^3\eta)^-_2 \to \lambda((\lambda^2\eta)^+_1) \to \lambda(\beta\lambda\eta *_0 \lambda\omega\eta)$$

$$\to \beta(\beta\lambda\eta)^+_1 *_0 \lambda^2\omega\eta *_1 \lambda\beta\lambda\eta$$

$$\to \beta^2\lambda\eta *_0 \lambda^2\omega\eta *_1 \lambda\beta\lambda\eta$$

$$\xrightarrow{*} \beta^2\langle 01\rangle_1 *_0 \lambda^2\langle 0\rangle_1 *_1 \lambda\beta\langle 01\rangle_1$$

$$\xrightarrow{*} \langle 23\rangle_3 *_0 \langle 012\rangle_3 *_1 \langle 023\rangle_3 .$$

A similar work is easy for the

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