A construction of a nerve for the ∞-catégories

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Introduction.

R.Street has given ins [3] a construction of a nerve for the ∞-categories [1]. In this work we suggest a new construction for a nerve which uses the classical techniques of homological algebra. Although the identity beetwen the two constructions remains to be proved, we have verified that they coincide up to dimension 6 (*i.e.* the ones which are explicit in [3]).

more precisely, Street construts a cosimplicial object in the category of (little) ∞-categories, a functor $\mathcal{O} : \mathbf{\Delta} \longrightarrow \infty$-**Cat**, from which he deduces immediately a functor nerve $N : \infty$-**Cat** $\longrightarrow$ **Simpl**. We propose a new construction $\mathcal{O}' : \mathbf{\Delta} \longrightarrow \infty$-**Cat** of such an objet.

the starting point of our construction uses a result of D.Bourn [1] which establishes a functorial equivalence between the category **Compl** of chain complexes of abelian groups and the category of ∞-category objects in **Ab** [2]. In fact we only use the functor B : **Compl** $\longrightarrow \infty$-**Cat** deduced from the previous equivalence forgetting the abelian groups structures on the ∞-catégories(this construction will be racalled in section 1). From each objet $[n]$ of Δ (where $n \in \mathbf{N}$), the ∞- category $\tilde{\Delta}^n = \mathcal{O}'([n])$ which reduce to an n-category, is called, as in [3], the n-oriental.

Here is an informal and abreviated description (forgetting in particular the degenereted cells) of the orientals of little dimensions.

- For $n = 0$: $\qquad 0 \qquad$ (one object)
- For $n = 1$:

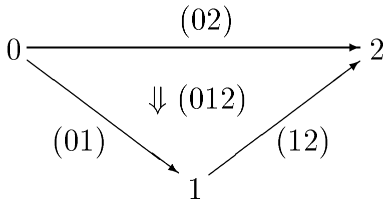
$$0 \xrightarrow{(01)} 1$$

[1] In fact, Street uses the more general notion of ω-category (which admits, infinite dimensional cells), but here we shall restrict ourselves to ∞-categories, and we shall adopt a terminology similar to the one, that can be found for instance in [2].

[2] Result which is similar to a previous one from Dold-Kan establishing an equivalence between **Compl** and **Simpl**(**Ab**)

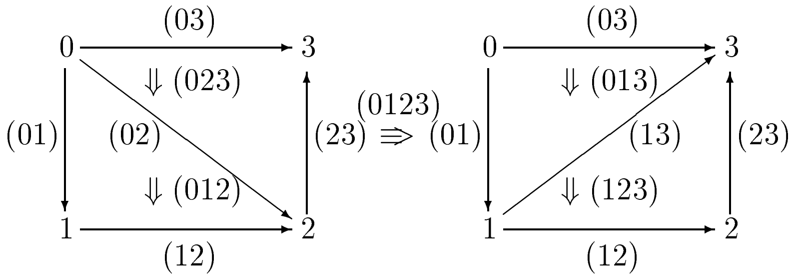
(two objets,one arrow)

- For $n = 2$:



(three objects,three arrows and one 2-cell : $(02) \longrightarrow (12) \circ (01)$)

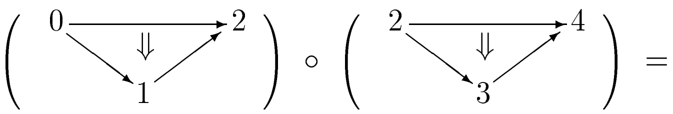
- For $n = 3$: It is a 3-cell into a tetraedra:

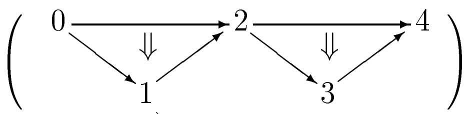


where parts of the picture has been split in two parts to make the 3-cell (0123) (see notations in the text).

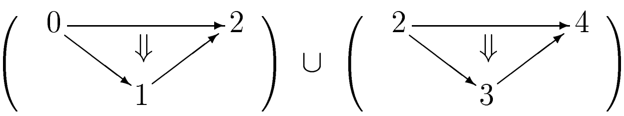
Let us illustrate this, again in an informal way, whith the exemple of a 2-cell composition in $\tilde{\Delta}^n$ $(n \geq 4)$ an essential difference between the techniques used in [3] and the one that will be developped in this paper.

The composition:

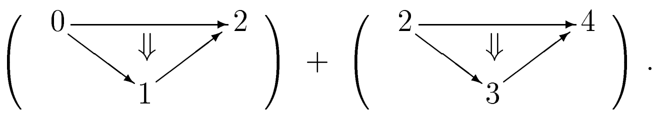




is interpreted in [3] set theoretically (or geometrically) as a union:



whereas we shall here interpret it algebraically by a sum in an abelian group:



1 Construction of the orientals $\tilde{\Delta}^n$.

the construction is made in several steps:

1^{st} step: $\mathbf{\Delta}$ (resp. $\mathbf{\Delta}'$) will note the following category:

- The objets of $\mathbf{\Delta}$ (resp. $\mathbf{\Delta}'$) are the orderd sets $[n] = (\{0, 1, \ldots n\}, \leq)$, for all $n \in \mathbf{N}$, ($\leq$ is the natural ordering on the integer).
- the morphisms of $\mathbf{\Delta}$ (resp. $\mathbf{\Delta}'$) are the order preserving functions (resp. strictly order preserving function) $\mathbf{s} : [n] \longrightarrow [m]$ $(n, m \in \mathbf{N})$. It will be often convenient to denote such a fonction simply by $(s_0, s_1, \ldots, s_n)$, where $s_i = \mathbf{s}(i)$ for $0 \leq i \leq n$. We have then: $0 \leq s_0 \leq s_1 \leq \ldots \leq s_n \leq m$ (resp. $0 \leq s_0 < s_1 < \ldots < s_n \leq m$).

e call *simplicial set* (resp. *présimplicial set*) an objet of the category $\mathbf{Simpl} = \mathbf{Ens}^{\mathbf{\Delta}^{op}}$ (resp. $\mathbf{Simpl}' = \mathbf{Ens}^{\mathbf{\Delta}'^{op}}$), that is to say a functor of the form $S : \mathbf{\Delta}^{op} \longrightarrow \mathbf{Ens}$ (resp. $S : \mathbf{\Delta}'^{op} \longrightarrow \mathbf{Ens}$). In particular, if $Y : \mathbf{\Delta}' \longrightarrow \mathbf{Simpl}'$ is the Yoneda embeding, we have, for all $n \in \mathbf{N}$ a présimplicial set $\Delta'^n = Y([n])$. Let us denote by Δ'^n_p the set $\Delta'^n([p]) = \mathrm{Hom}_{\mathbf{\Delta}'}([p], [n])$ of the $\mathbf{s} = (s_0, s_1, \ldots, s_p)$ such that $0 \leq s_0 < s_1 < \ldots < s_p \leq n$.

2^d step: we denote **Compl** the category of the chain complexes of de abelian groups. An objet of **Compl** is then a diagram of the form:

$$A_0 \xleftarrow{\partial_1} A_1 \xleftarrow{\partial_2} A_2 \longleftarrow \cdots A_{n-1} \xleftarrow{\partial_n} A_n \longleftarrow \cdots$$

where $\partial_n \partial_{n+1} = 0$ for all $n \in \mathbf{N}$.

We have a functor $Z : \mathbf{Simpl}' \longrightarrow \mathbf{Compl}$ which associates to any presimplicial set S the following complex $Z(S)$:

$$\mathbf{Z}S_0 \xleftarrow{\partial_1} \mathbf{Z}S_1 \xleftarrow{\partial_2} \mathbf{Z}S_2 \longleftarrow \cdots \mathbf{Z}S_{n-1} \xleftarrow{\partial_n} \mathbf{Z}S_n \longleftarrow \cdots$$

where $S_n = S([n])$ for all $n \in \mathbf{N}$, where $\mathbf{Z}S_n$ is the free abelien groupe generated by S_n and where ∂_n is the lineare application defined, for each element $\mathbf{s}$ of the canonical bases of $\mathbf{Z}S_n$, by:

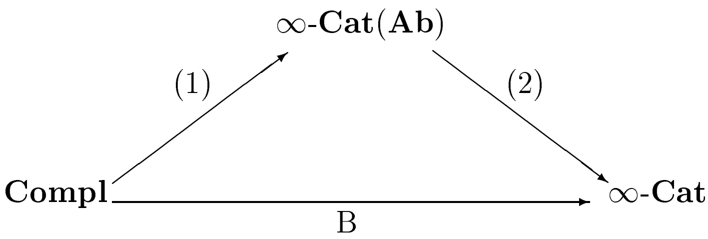
$$\partial_n(\mathbf{s}) = \sum_{i=0}^{n} (-1)^i d_i(\mathbf{s}).$$

In this formula $d_i : S_n \longrightarrow S_{n-1}$ is the application $d^i = S(\delta_i)$ where $\delta_i : [n-1] \longrightarrow [n]$ is defined by:

$$\delta_i(j) = \begin{cases} j & \text{si } 0 \leq j < i \\ j+1 & \text{si } i \leq j \leq n-1 \end{cases}$$

We shall have to consider exclusively the case of the complexes $K^n = Z(\Delta^n)$. Note that K^n is of dimension n, in the sense that for all $n < p$ we have $K^n_p = 0$ and $K^n_n \neq 0$.

3^{nd} step: The main point in our construction is the description of the following composite functor B:



where (1) is the correspondance defined by Bourn [1] and (2) is the underlying functor, induced by: **Ab** $\longrightarrow$ **Ens**.

Precisely, let A be a chain complex

$$A_0 \xleftarrow{\partial_1} A_1 \xleftarrow{\partial_2} A_2 \longleftarrow \cdots A_{n-1} \xleftarrow{\partial_n} A_n \longleftarrow \cdots$$

we associate to it the following ∞-category (actually an ∞-groupoïde) $B(A)$:

$$B(A)_0 \underset{\text{cod}}{\overset{\text{dom}}{\leftleftarrows}} B(A)_1 \underset{\text{cod}}{\overset{\text{dom}}{\leftleftarrows}} B(A)_2 \leftleftarrows \cdots B(A)_n \underset{\text{cod}}{\overset{\text{dom}}{\leftleftarrows}} B(A)_{n+1} \leftleftarrows \cdots$$

(the previous diagram is only the ∞-graphe underlying the ∞- category $B(A)$. See Voir [2]) where:

$$B(A)_p = A_0 \times A_1 \times \cdots \times A_p.$$

and where

1. the domains et codomains are defined for all $0 \leq q < p$ by:

$$\begin{aligned} \text{dom}^p_q(a_0, a_1, \ldots, a_p) &= (a_0, a_1, \ldots, a_q) \\ \text{cod}^p_q(a_0, a_1, \ldots, a_p) &= (a_0, a_1, \ldots, a_q + \partial_{q+1}(a_{q+1})) \end{aligned}$$

2. the compositions by:

$$(a'_0, a'_1, \ldots, a'_p) \circ^p_q (a_0, a_1, \ldots, a_p) = \\ (a_0, a_1, \ldots, a_q, a'_{q+1} + a_{q+1}, \ldots, a'_p + a_p)$$

iff

$$\text{dom}^p_q(a'_0, a'_1, \ldots, a'_p) = \text{cod}^p_q(a_0, a_1, \ldots, a_p)$$

3. the identities par:

$$\mathrm{id}^p_q(a_0, a_1, \ldots, a_q) = (a_0, a_1, \ldots, a_q, 0, \ldots, 0)$$

In particulier in that way we obtain the ∞-catégories $B(K^n)$. In fact they are n-catégories, more precisely: for $p > n$ a p-cell is always an identity.

4^{th} step: The last ingredient essential to the construction of the orientals is the "polarisation" of the elements of a free abelian group $\mathbf{Z}X$.

If $x = \sum_i n_i x_i$ (where $x_i \in X$ for all i) is an element of $\mathbf{Z}X$ we denote x^- the element $\sum_i (n_i^-) x_i$ where the notation n^- for an integer n is defined by:

$$n^- = \begin{cases} -n & \text{si } n \leq 0 \\ 0 & \text{si } n > 0 \end{cases}$$

We can now define, for all $0 \leq p$ the application

$$\mathrm{cell}^n_p : \Delta'^n_p \longrightarrow B(K^n)_p$$

setting, for each simplex $\mathbf{s}$:

$$\mathrm{cell}^n_p(\mathbf{s}) = (c_0, c_1, \ldots, c_p)$$

où $c_p = \mathbf{s}$ and $c_i = (\partial(c_{i+1}))^-, (0 \leq i \leq p)$. Finally we obtain, for all $n \in \mathbf{N}$, the ∞-catégory $\tilde{\Delta}^n$, defined as a sub-∞-catégory of $B(K^n)$ generated by the cells of the form $\mathrm{cell}^n_p(\mathbf{s})$, for all $\mathbf{s} \in \Delta'^n_p$, $0 \leq p \leq n$. Evidently $\tilde{\Delta}^n$ is an n-catégorie.

2 Property of orientals.

In an ∞-catégorie C we shall say that a n-cell x is *closed* if $\mathrm{dom}^n_{n-1}(x) = \mathrm{cod}^n_{n-1}(x)$ and that x is degenerated if there exists an $(n-1)$-cell y such that $x = \mathrm{id}^n_{n-1}(y)$. An ∞-catégory is said to be *loop free* if any closed cell is degenerated. C is said to be *n-non-degenerated* if there exists non degenerated n-cells.

proposition. 1 *The n-catégory $\tilde{\Delta}^n$ is n-non-degenerated and loop free.*

Proof: It is clear that the n-cell $\mathrm{cell}^n_n(I_n)$, Where $I_n = (0, 1, \ldots, n)$ is the unique n-simplex of Δ'^n, is non degenerated and then that $\tilde{\Delta}^n$ is non degenerated.

Let us prove now that $\tilde{\Delta}^n$ is loop free. First let us consider the linear map $\pi_p : K^n_p \longrightarrow \mathbf{Z}$ defined on the generators $\mathbf{s} = (s_0, s_1, \ldots, s_p) \in \Delta'^n_p$ of the group $K^n_p = \mathbf{Z}\Delta'^n_p$ by $\pi_p(\mathbf{s}) = s_0 + s_1 + \cdots + s_p$ (sum in $\mathbf{Z}$). Then let us

define the function $\bar{\pi} : \tilde{\Delta}^n_p \longrightarrow \mathbf{Z}$ by setting $\bar{\pi}_p(\mathbf{c}) = \pi_p(c_p)$ for all p-cellule $\mathbf{c} = (c_0, c_1, \ldots, c_p)$ of $\tilde{\Delta}^n_p$. The proposition will then follow from the following property:
If $\mathbf{c} \in \tilde{\Delta}^n_p$ $(0 \leq p \leq n)$ is non degenerate we have:

$$\bar{\pi}_{p-1}(\mathrm{dom}^p_{p-1}(\mathbf{c})) < \bar{\pi}_{p-1}(\mathrm{cod}^p_{p-1}(\mathbf{c})).$$

Indeed, from this fact, we deduce immediately that $\mathbf{c}$ is not a loop and hence that $\tilde{\Delta}^n$ has no loop. By induction we can reduce this to the case where $\mathbf{c} = \mathrm{cell}^n_p(\mathbf{s})$ is a generator p-cell, thanks to the formula:

$$\begin{aligned} \bar{\pi}_{p-1}(\mathrm{cod}^p_{p-1}(\mathbf{c}' \circ^p_q \mathbf{c})) - \bar{\pi}_{p-1}(\mathrm{dom}^p_{p-1}(\mathbf{c}' \circ^p_q \mathbf{c})) = \\ [\ \bar{\pi}_{p-1}(\mathrm{cod}^p_{p-1}(\mathbf{c}')) - \bar{\pi}_{p-1}(\mathrm{dom}^p_{p-1}(\mathbf{c}'))\ ] \\ +[\ \bar{\pi}_{p-1}(\mathrm{cod}^p_{p-1}(\mathbf{c})) - \bar{\pi}_{p-1}(\mathrm{dom}^p_{p-1}(\mathbf{c}))\ ] \end{aligned}$$

for all $\mathbf{c}, \mathbf{c}' \in \tilde{\Delta}^n_p$ such that $\mathrm{dom}^p_q(\mathbf{c}') = \mathrm{cod}^p_q(\mathbf{c})$ $(0 \leq q < p)$. Then if $\mathbf{s} \in {\Delta'}^n_p$, we have:

$$\bar{\pi}_{p-1}(\mathrm{cod}^p_{p-1}(\mathrm{cell}^n_p(\mathbf{s}))) - \bar{\pi}_{p-1}(\mathrm{dom}^p_{p-1}(\mathrm{cell}^n_p(\mathbf{s}))) = \pi_{p-1}(\partial \mathbf{s})$$

In order to prove that $\pi_{p-1}(\partial \mathbf{s}) > 0$ we distinguish two cases:

- $p = 2k\ :\ \pi_{p-1}(\partial \mathbf{s}) = \sum_{i=0}^{k-1}(s_{2i+1} - s_{2i})\ + (\pi_p(s) - s_{2n}) > 0$
- $p = 2k+1\ :\ \pi_{p-1}(\partial \mathbf{s}) = \sum_{i=0}^{k-1}(s_{2i+1} - s_{2i}) > 0,$

this finish the proof. □

An n-catégorie C is said *finite* if for all $0 \leq p \leq n$ the set C_p is finite. To say that C is *finitely generated* means that there exist a sequence of finite subsets $E_p \subset C_p$ (for all $0 \leq p \leq n$) for which the only sub-n-category C' of C satisfying $E_p \subset C'_p$ (for all $0 \leq p \leq n$) is C itself $(C' = C)$.

proposition. 2 . *A loop free and finitely generated n-catégory is finite. In particular the n-categories $\tilde{\Delta}^n$ are finite.*

preuve: By induction it can be reduced to this case: $E_p = C_p$ for all $0 \leq p \leq n-1$; and then to the case where the sub-(n-1)-category $C^{(n-1)}$ underlying C (that is to say obtained by forgetting the non degenerated n-cells of C) is finite. In order to provel this,it will be convenient to introduce some new definitions. We shall set $E = E_p$ and we shall call elementary E-factorisation of an n-cell x a system of the form:

$$(a, (u_1, v_1), \ldots, (u_{n-1}, v_{n-1})) \in E \times (C_1 \times C_1) \times \cdots \times (C_{n-1} \times C_{n-1})$$

such that:

$$x = \bar{v}_{n-1} \circ_{n-2} (\ldots (\bar{v}_2 \circ_1 ((\bar{v}_1 \circ_0 a \circ_0 \bar{u}_1) \circ_1 \bar{u}_2) \ldots) \circ_{n-2} \bar{u}_{n-1}$$

where we suppose that the compositions exist and where we write, in an abreviated way $\circ_p$ instead of $\circ_p^n$ and $\bar{w}$ instead of $\mathrm{id}_p^n(w)$ for $w \in C_p$ with $0 \leq p < n$. We shall say that a n-cell x is *E-elementary* if it admits such a factorisation. We easily prove the following stability property:

If x is an E-elementary n-cell, if $w \in C_p$ for a $0 \leq p < n$ and if the composition $\bar{w} \circ_q x$ (resp. $x \circ_q \bar{w}$) exists for a $0 \leq q < p$, then this composition is an E-elementary n-cell.

From the hypothesis we dedduce that the elementary E-factorisations are in finite number . Therefore the number of E-elementary n-cells is also finite. Since by hypothesis the n-cells in E generate C, we have the same resultfor the E-elementary n-cells, and it only remains to prove, in order to conclude that the E-elementary n-cells generate, by composition, a finite number of these n-cells.

Let us first remark that one can always reduce to some $(n-1)$-compositions by multiple inductions. precisely the Godement's rule allows us for instance to write a q-composition (with $q < n-1$) as:

$$x' \circ_q x = (\overline{\mathrm{cod}_{q+1}(x')} \circ_q x) \circ_{q+1} (x' \circ_q \overline{\mathrm{dom}_{q+1}(x)}).$$

Let us consider the E-irreductibles n-paths, that is to say the finite sequences $(x_1, x_2, \ldots, x_k)$ of E-elementary n-cells such that the composite $x = x_1 \circ_{n-1} x_2 \circ_{n-1} \cdots \circ_{n-1} x_k$ is defined and such that for all $1 \leq i < j \leq k$ the composite $x_{ij} = x_i \circ_{n-1} x_{i+1} \circ_{n-1} \cdots \circ_{n-1} x_j$ is non degenerated. The number of these E-irreductible n-paths necessarily finite since the number of E-elementary n-cells it finite and the integer $k+1$ is smaller than the number of elements of C_{n-1}. This is true because if $(x_1, x_2, \ldots, x_k)$ is such a E-elementary n-path, the $(n-1)$-cells

$$y_0 = \mathrm{cod}_{n-1}(x_1), \ y_i = \mathrm{dom}_{n-1}(x_i) \ \ (1 \leq i \leq k)$$

are necessarily distinct by hypothesis. It then follows from proposition 1 that $\tilde{\Delta}^n$ is finite. □

3 Construction of the cosimplicial object $\mathcal{O}'$.

We are going to begin with the construction of a functor $\mathcal{O}' : \mathbf{\Delta}' \longrightarrow \infty\text{-}\mathbf{Cat}$ which will be extended further to $\mathbf{\Delta}$.

We define $\mathcal{O}'$ in the following way:

- On the objets: $\mathcal{O}'([n]) = \tilde{\Delta}^n$, for all $n \in \mathbf{N}$.
- On the morphisms: For any stricly order preserving $f : [n] \longrightarrow [m]$ we

define the functor $\mathcal{O}'(f) : \tilde{\Delta}^n \longrightarrow \tilde{\Delta}^m$ by factorisation:

$$\begin{array}{ccc} \tilde{\Delta}^n & \dashrightarrow & \tilde{\Delta}^m \\ \downarrow & & \downarrow \\ B(K^n) & \xrightarrow[B(Z(Y(f)))]{} & B(K^m) \end{array}$$

where $Y : \mathbf{\Delta}' \longrightarrow \mathbf{Simpl}'$ is the Yoneda embedding and where vertical arrows are inclusions. The factorisation means that $B(Z(Y(f)))$, simply denoted by f in the sequel, transforms any generator cell of $\tilde{\Delta}^n$ into an n-cell of $\tilde{\Delta}^m$. More precisely we establish the formula $f(\mathrm{cell}_p^n(\mathbf{s})) = \mathrm{cell}_p^n(f_p(\mathbf{s}))$ for all $\mathbf{s} \in \Delta'^n_p$. For this, it is sufficient to check that $f(c^-) = f(c)^-$ for all $c \in K_p^n$ and to use the fact that $Z(Y(f))$ is a chain complex morphism.

We are going to prove now that the functor $\mathcal{O}' : \mathbf{\Delta}' \longrightarrow \infty$-$\mathbf{Cat}$ extends to the whole $\mathbf{\Delta}$. The construction of this extention is less obvious and needs to be done in stages.

First we are going to characterise in proposition 3 below the expression of different terms in the generator cells of $\tilde{\Delta}^n$. We shall use the notations:

- $I_n = (0, 1, \ldots, n)$ is the unique n-simplex of Δ'^n.
- $< i_1, i_2, \ldots, i_k >$ for $0 \leq i_1 < i_2 < \ldots < i_k \leq n$, is a notation for the $(n-k)$-simplex of Δ'^n obtained from I_n by removing $i_1, i_2, \ldots, i_k$.
- $(\partial^-(c)) = (\partial(c))^-$ for all $c \in K_p^n$.
- $\sum S$ is an abreviation for $\sum_{s \in S} s$.

proposition. 3 *For all $1 \leq k \leq n$, we have:*

$$(\partial^-)^k(I_n) = \sum\{< i_1, i_2, \ldots, i_k > \mid\ i_1 \equiv 1, i_2 \equiv 2, \ldots, i_k \equiv k \pmod 2\}.$$

Proof: It is done by induction on k.
First for $k = 1$ it amounts to the definition:

$$\partial^-(I_n) = \sum\{< i > | i \equiv 1\},$$

where we have simply denoted $\equiv$ the congruence modulo 2. Next let us suppose the formula true for k and let us prove it for $k+1$. We shall rely on the formula:

$$\partial(< i_1, i_2, \ldots, i_k >) =$$

$$
\begin{aligned}
& \sum_{j=0}^{i_1-1}(-1)^j < j, i_1, \ldots, i_k > \\
+ & \sum_{l=1}^{k-1}\sum_{j=i_l-1}^{i_{l+1}-1}(-1)^{j-l} < i_1, \ldots, i_l, j, i_{l+1}, \ldots, i_k > \\
+ & \sum_{j=i_k+1}^{n}(-1)^{j-k} < i_1, \ldots, i_k, j >
\end{aligned}
$$

for all $< i_1, i_2, \ldots, i_k >$ such that $i_1 \equiv 1, i_2 \equiv 2 \ldots, i_k \equiv k$.

This formula allows us to calculate the coefficients of the terms in $< j_1, j_2, \ldots, j_{k+1} >$ which appear in the calculation of $\partial((\partial^-)^k I_n)$, and then of $(\partial^-)^n_{k+1}$. For a simplex $< j_1, j_2, \ldots, j_{k+1} >$ we can evaluate the contribution of each of $< i_1, i_2, \ldots, i_k >$, $i_1 \equiv 1, i_2 \equiv 2 \ldots, i_k \equiv k$, by making a reasoning by case in the following way:

- If $j_1 \equiv 0$, $< j_1, j_2, \ldots, j_{k+1} >$ can only appear in a derivation under the form $< j, i_1, \ldots, i_k >$ with $i_1 \equiv 1, i_2 \equiv 2, \ldots, i_k \equiv k$. This is a consequence of the inductive hypothesis on the modulo 2 constraint. Its coefficient is egal to $(-1)^j = 1$ and then $< j_1, j_2, \ldots, j_k >$ does not appear in $(\partial^-)^{k+1} I_n$.

- If $j_1 \equiv 1, j_2 \equiv 2 \ldots j_{k+1} \equiv k+1$, the only appearance of $< j_1, j_2, \ldots, j_{k+1} >$ is under the form $< i_1, \ldots, i_k, j >$ whith $i_1 \equiv 1, i_2 \equiv 2 \ldots i_k \equiv k$. Its coefficient ist egal to $(-1)^{j+k} = (-1)^{k+1+k} = -1$. then this term appear once and only once in $(\partial^-)^{k+1} I_n$.

- If $j_1 \equiv 1, j_2 \equiv 2 \ldots j_l \equiv l$ and $j_{l+1} \equiv l$ for a certain l, $1 \leq l \leq k$, $< j_1, j_2, \ldots, j_{k+1} >$ appears exactly two times which correspond to the derivations with respect to $< i_1, i_2, \ldots, i_k >$ and $< i'_1, i'_2, \ldots, i'_k >$. Precisely:

  $$
  \begin{aligned}
  < j_1, j_2, \ldots, j_k > &= < i_1, \ldots, i_{l-1}, i_l, j, i_{l+1}, \ldots, i_k > \\
  < j_1, j_2, \ldots, j_{k+1} > &= < i'_1, \ldots, i'_{l-1}, i'_l, j', i'_{l+1}, \ldots, i'_k >,
  \end{aligned}
  $$

  Hence with $j_1 = i_1 = i'_1, \ldots, j_{l-1} = i_{l-1} = i'_{l-1}$, $j_l = i_l = j'$, $j_{l+1} = j = i'_l$, $j_{l+2} = i_{l+1} = i'_{l+1}, \ldots, j_k = i_k = i'_k$. The coefficients are respectively equal to $(-1)^{j-l} = (-1)^{2l} = 1$ and $(-1)^{j'-(l-1)} = -1$ and hence cancel each other out. It follows that $< j_1, j_2, \ldots, j_{k+1} >$ appears neither in $\partial((\partial^-)^k(I_n))$ nor in $(\partial^-)^{k+1}(I_n)$. It is clear that by examining these different cases, we establish the required formula. □

Let us return to the construction of a embedding of $\mathcal{O}'$ to $\mathbf{\Delta}$. To each order preserving function (not necessarily strict) $f : [n] \longrightarrow [m]$ and to each integer $p \in \mathbf{N}$ we can associate a linear map $f_p : K^n_p \longrightarrow K^m_p$ by setting for each $\mathbf{s} = (s_0, s_1, \ldots, s_p) \in \Delta'^n_p$:

$$f_p(\mathbf{s}) = [f(s_0), f(s_1), \ldots, f(s_p)]$$

where the brackets of the second membre is defined for all $0 \le t_0 \le t_1 \le \ldots \le t_p \le m$ by:

$$[t_0, t_1, \ldots, t_p] = \begin{cases} (t_0, t_1, \ldots, t_p) & \text{si} \quad t_0 < t_1 < \ldots < t_p \\ 0 & \text{sinon.} \end{cases}$$

We verify without difficulties that this defines first a homomorphism of complexes $K^f : K^n \longrightarrow K^m$ and next a functor by setting: $K([n]) = K^n$, $K(f) = K^f$. So we obtain a composite functor:

$$\mathbf{\Delta} \xrightarrow{K} \mathbf{Compl} \xrightarrow{B} \infty\text{-}\mathbf{Cat}$$

from which we will obtain $\mathcal{O}'$ as a sub-functor.

For this purpose we are going to show that for all $f : [n] \longrightarrow [m]$ $\mathbf{\Delta}$, the functor $B(K^f) : B(K^n) \longrightarrow B(K^m)$ transforms the generator cells of $\tilde{\Delta}^n$ into generator cells of $\tilde{\Delta}^m$. This property has already been proved in the case where f is strictly increasing, let us prove it now for the surjections which are of the form $f : [n+1] \longrightarrow [n]$ and defined by:

$$f(i) = \begin{cases} i & \text{si} \quad 0 \le i \le j \\ i-1 & \text{si} \quad j < i \le n+1 \end{cases}$$

and where the integer j such that $0 \le j \le n+1$ is fixed.

proposition. 4 *f being the surjection defined above (and hence j being fixed), we have:*

$$f(\mathrm{cell}(I_{n+1})) = \mathrm{id}^{n+1}_n(\mathrm{cell}(I_n))$$

where we have written simply f instead of $B(K^f)$ in the first membre.

Proof: It suffices to verify that: $f_{n+1}(I_{n+1}) = 0$ and $f_{n+1-k}((\partial^-)^k(I_{n+1})) = (\partial^-)^{k-1}(I_n)$ for all $1 \le k \le n+1$. taking into account the characterisation of $(\partial^-)^k(I_{n+1})$ given in the proposition 3, it remains to prove that $f_{n+1-k} : K^{n+1}_{n+1-k} \longrightarrow K^n_{n+1-k}$ satisfies $f_{n+1-k}(\sum J) = \sum J'$ where:

$$J = \{< i_1, i_2, \ldots, i_k > \mid \ i_1 \equiv 1, \ldots, i_k \equiv k \pmod 2 \\ \text{et} \quad 0 \le i_1 < i_2 < \ldots < i_k \le n+1\}$$

$$J' = \{< i'_1, i'_2, \ldots, i'_{k-1} > \mid \ i'_1 \equiv 1, \ldots, i'_{k-1} \equiv k-1 \pmod 2 \\ \text{et} \quad 0 \le i'_1 < i'_2 < \ldots < i'_{k-1} \le n\}$$

More precisely we are going to prove that f transforms any $< i_1, i_2, \ldots, i_k >$ into 0 or into $< i'_1, i'_2, \ldots, i'_{k-1} >$ and that, conversely, $< i'_1, i'_2, \ldots, i'_{k-1} >$ is obtained of in one and only one way in this manner. Let us then consider a $< i_1, i_2, \ldots, i_k >$ according to the position of j and $j+1$ with respect to $i_1, i_2, \ldots, i_k$. We have different cases:

- If $j, j+1 \notin \{i_1, i_2, \ldots, i_k\}$, we have:

$$f(< i_1, i_2, \ldots, i_k >) = 0 \quad \text{because} \quad f(j) = f(j+1)$$

- If $j = i_l$ and $j+1 \notin \{i_1, i_2, \ldots, i_k\}$ or if $j+1 = i_l$ and $j \notin \{i_1, i_2, \ldots, i_k\}$, we have:

$$f(< i_1, i_2, \ldots, i_k >) =< i_1, \ldots, i_{l-1}, i_{l+1} - 1, \ldots, i_k - 1 >$$

  and if we put $< i'_1, i'_2, \ldots, i'_{k-1} >=< i_1, \ldots, i_{l-1}, i_{l+1} - 1, \ldots, i_{k-1} >$ we have indeed that $i'_h \equiv h$ for all $1 \leq h \leq k-1$ since $i_{h+1} - 1 \equiv i_h \equiv h$.

- If $j = i_l$ and $j+1 = i_{l+1}$, we have:

$$f(< i_1, i_2, \ldots, i_k >) =< i_1, \ldots, i_l, i_{l+2} - 1, \ldots, i_k - 1 >$$

  and if we put $< i'_1, i'_2, \ldots, i'_{k-1} >=< i_1, \ldots, i_l, i_{l+2} - 1, \ldots, i_k - 1 >$ we have indeed that $i'_h \equiv h$ as above.

Let us now consider a $< i'_1, i'_2, \ldots, i'_{k-1} >$ according to the parity of the position of j with respect to $i'_1, i'_2, \ldots, i'_{k-1}$. We have the different cases:

- If $j \notin \{i'_1, i'_2, \ldots, i'_{k-1}\}$:

$$f(< i_1, i_2, \ldots, i_k >) =< i'_1, i'_2, \ldots, i'_{k-1} >$$

  admits the solution, according to the cases:

  - If $i'_l < j < i'_{l+1}$ and $j \equiv l+1$,

$$< i_1, i_2, \ldots, i_k >=< i'_1, \ldots, i'_l, j, i'_{l+1} + 1, \ldots, i'_{k-1} + 1 >$$

  - If $i'_l < j < i'_{l+1}$ et $j \equiv l$,

$$< i_1, i_2, \ldots, i_k >=< i'_1, \ldots, i'_l, j+1, i'_{l+1} + 1, \ldots, i'_{k-1} + 1 >$$

  (The raisoning is analogous in the bordering cases $j < i'_1$ and $i'_{k-1} < j$)

- If $j = i'_l$:

$$f(< i_1, i_2, \ldots, i_k >) =< i'_1, i'_2, \ldots, i'_k >$$

  admits as a solution:

$$< i_1, i_2, \ldots, i_k >=< i'_1, \ldots, i'_{l-1}, j, j+1, i'_{l+1} + 1, \ldots, i'_{k-1} + 1 > .$$

and this solutions are evidently unique. This finish the proof. □

The previous result can be generalized in the following way:

proposition. 5 *Under the same hypothesis on f in the proposition 4, for each $\mathbf{s} \in \Delta_p^n$ we have, according to cases:*

1. *If $f(\mathbf{s}(i)) < f(\mathbf{s}(i+1))$ for all $0 \leq i \leq p-1$, we have:*

$$f(\text{cell}(\mathbf{s})) = \text{cell}(f_p(\mathbf{s})).$$

2. *If $f(\mathbf{s}(i_0)) < f(\mathbf{s}(i_0)+1)$ for one $0 \leq i_0 \leq p-1$, we have:*

$$f(\text{cell}(\mathbf{s})) = \text{id}(\text{cell}(\mathbf{s}'))$$

where $\mathbf{s}' : [p-1] \longrightarrow [n]$ is defined by:

$$\mathbf{s}'(i) = \begin{cases} f(\mathbf{s}(i)) & si\ i \leq i_0 \\ f(\mathbf{s}(i)) - 1 & si\ i_0 + 1 \leq i. \end{cases}$$

(or else $\mathbf{s}' = (\ f(\mathbf{s}(0)), \ldots, f(\widehat{\mathbf{s}(i_0)}), \ldots, f(\mathbf{s}(p))\ )$.)

Proof: $\mathbf{s}$ being strictly increasing, let us first note that:

$$f_p(\mathbf{s}) = f_p(s_p(I_p)) = (f \circ \mathbf{s})_p(I_p),$$

$$\text{cell}(\mathbf{s}) = \text{cell}(s_p(I_p)) = \mathbf{s}(\text{cell}(I_p)).$$

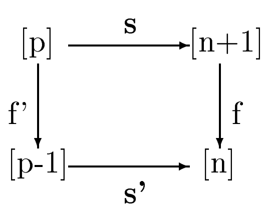
1. Under the first hypothesis the function $\mathbf{s}' = f \circ \mathbf{s}$ is strictly is strictly increasing, hence:

$$f(\text{cell}(\mathbf{s})) = f(\mathbf{s}(\text{cell}(I_p))) = \mathbf{s}'(\text{cell}(\text{I}_\text{p})) = \text{cell}(\mathbf{s}'(I_p)) = \text{cell}(f_p(\mathbf{s}))$$

2. Under the second hypothesis let us denote $f' : [p] \longrightarrow [p-1]$ the function defined by:

$$f'(i) = \begin{cases} i & \text{si } i \leq i_0 \\ i-1 & \text{si } i > i_0. \end{cases}$$

(Let us note that the integer i_0, if it exists, is unique). Then the the following diagram commutes:



where $\mathbf{s}, \mathbf{s}'$ are strictly increasing and f, f' are surjective. Then:

$$f(\mathrm{cell}(\mathbf{s})) = (f \circ \mathbf{s})(\mathrm{cell}(I_p)) = (\mathbf{s}' \circ f')(\mathrm{cell}(I_p)) = \mathbf{s}'(f'(\mathrm{cell}(I_p))).$$

According to the proposition 4 on a

$$f'(\mathrm{cell}(I_p)) = \mathrm{id}(\mathrm{cell}(I_{p-1}))$$

then:

$$\mathbf{s}'(f'(\mathrm{cell}(I_p))) = \mathbf{s}'(\mathrm{id}(\mathrm{cell}(\mathrm{I_{p-1}}))) = \\ \mathrm{id}(\mathbf{s}'(\mathrm{cell}(I_{p-1}))) = \mathrm{id}(\mathrm{cell}(\mathbf{s}'_{p-1}(I_{p-1}))) = \mathrm{id}(\mathrm{cell}(\mathbf{s}')).$$

□

Hence we have proved that for each increasing and surjective function $f : [n+1] \longrightarrow [n]$, the ∞-functor $f : B(K^{n+1}) \longrightarrow B(K^n)$ transforms the generator cells of $\tilde{\Delta}^{n+1}$ into generator cells of $\tilde{\Delta}^n$. from which the factorisation follows.

Finally, since any increasing function $f : [n] \longrightarrow [m]$ is composite of increasing injections and surjections previously described,we can deduice that the ∞-foncteur $f : B(K^n) \longrightarrow B(K^m)$ factorises itself into a sub-functor denoted $\mathcal{O}'(f) : \tilde{\Delta}^n \longrightarrow \tilde{\Delta}^m$. Moreover the fonctoriality of the composition:

$$\mathbf{\Delta} \xrightarrow{K} \mathbf{Compl} \xrightarrow{B} \infty\text{-}\mathbf{Cat}$$

implies the funtoriality of $\mathcal{O}' : \mathbf{\Delta} \longrightarrow \infty\text{-}\mathbf{Cat}$. This functor is evidently an embedding of the one already defined $\mathcal{O}' : \mathbf{\Delta}' \longrightarrow \infty\text{-}\mathbf{Cat}$.

Conclusion.

From this cosimplicial object $\mathcal{O}' : \mathbf{\Delta} \longrightarrow \infty\text{-}\mathbf{Cat}$ the functor nerve $N : \infty\text{-}\mathbf{Cat} \longrightarrow \mathbf{Simpl}$ is simply obtained by setting:

$$N(C)([n]) = \mathrm{Hom}_{\infty\text{-}\mathbf{Cat}}(\mathcal{O}'([n]), C)$$

for each ∞-catégorie C, for each $[n]$ de $\mathbf{\Delta}$ and by extending this formula in an obvious way to morphisms. In this way, we can calculate the cohomology - ou rather *an* cohomology - of an ∞-catégorie C by reducing it to the cohomology of the simplicial set $N(C)$.

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