Local group theory : from Frobenius to Rickard

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- Sylow, 1872

  The maximal p–subgroups of G are all conjugate under G.

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The Frobenius Category

$\mathrm{Frob}_p(G)$:

- Objects : the p–subgroups of G,
- $\mathrm{Hom}(P, Q) := \{g \in G \mid ({}^g P \subset Q)\}/C_G(P)\,.$

Note that $\mathrm{Aut}(P) = N_G(P)/C_G(P)\,.$

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Alperin's fusion theorem (1967) says essentially that $\mathrm{Frob}_p(G)$ is known as soon as the automorphisms of some of its objects are known.

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But perhaps all of $G/O_{p'}(G)$?

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(ii) H contains a Sylow p–subgroup of G, and if P is a p–subgroup of H and g is an element of G such that ${}^g P \subseteq H$, then there is $h \in H$ and $z \in C_G(P)$ such that $g = hz$.

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If the preceding conditions are satisfied, we say that H controls p–fusion in G, or that H is a control subgroup in G.

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In other words, do we have

$$G = HO_{p'}(G)\,?$$

- Frobenius theorem, 1905

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  Assume that G is p–solvable. If H controls p-fusion in G, then the inclusion induces an isomorphism $H/O_{p'}(H) \simeq G/O_{p'}(G)$.

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- Z_p^*–theorem (Glauberman, 1966)

  Assume that $H = C_G(P)$ where P is a p–subgroup of G. If H controls p-fusion in G, then the inclusion induces an isomorphism $H/O_{p'}(H) \simeq G/O_{p'}(G)$.

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  Burnside (1852–1927)

  > Assume that a Sylow p–subgroup S of G is abelian. Set $H := N_G(S)$. Then H controls p-fusion in G.

Consider the **Monster**, a finite simple group of order

$$2^{46} \cdot 3^{20} \cdot 5^{9} \cdot 7^{6} \cdot 11^{2} \cdot 13^{3} \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71 \simeq 8.10^{53}\,.$$

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and the normalizer H of one of its Sylow 11–subgroups, a group of order 72600, isomorphic to $(C_{11} \times C_{11}) \rtimes (C_5 \times \mathrm{SL}_2(5))$ (here we denote by C_m the cyclic group of order m).

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Remark : one may think of more elementary examples...

LOCAL REPRESENTATION THEORY

Let K be a finite extension of the field of p–adic numbers $\mathbb{Q}_p$ which contains the $|G|$-th roots of unity. Let $\mathcal{O}$ be the ring of integers of K over $\mathbb{Z}_p$, with maximal ideal $\mathfrak{p}$ and residue field $k := \mathcal{O}/\mathfrak{p}$.

$$\begin{tikzcd}
K & & \\
& \mathcal{O} \arrow[ul, hook] \arrow[r, two heads] & k = \mathcal{O}/\mathfrak{p} \\
\mathbb{Q}_p \arrow[uu, hook] & & \\
& \mathbb{Z}_p \arrow[ul, hook] \arrow[uu, hook] \arrow[r, two heads] & \mathbb{F}_p = \mathbb{Z}_p/p\mathbb{Z}_p \arrow[uu, hook]
\end{tikzcd}$$

Block decomposition

$$\begin{tikzcd} \mathcal{O}G \arrow[d] & = & \bigoplus & B & \text{(indecomposable algebra)} \arrow[d] \\ kG & = & \bigoplus & kB & \text{(indecomposable algebra)} \end{tikzcd}$$

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The augmentation map $\mathcal{O}G \to \mathcal{O}$ factorizes through a unique block of $\mathcal{O}G$ called *the principal block* and denoted by $B_p(G)$.

$$\begin{tikzcd}
\mathcal{O}G \arrow[r] \arrow[dr] & B_p(G) \arrow[d] \\
& \mathcal{O}
\end{tikzcd}$$

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Let us re-examine the counterexamples to factorisation coming from Burnside's theorem.

Assume that a Sylow p–subgroup S of G is abelian, let $H := N_G(S)$ be its normalizer.

Even if $G \neq H\,O_{p'}(G)$, it appears that there is some connection between the (representation theory of) $B_p(G)$ and $B_p(H)$.

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Table: Character table of $\mathfrak{A}_5$

	(1)	(2)	(3)	(5)	(5')
1	1	1	1	1	1
χ_4	4	0	1	-1	-1
χ_5	5	1	-1	0	0
χ_3	3	-1	0	$(1+\sqrt{5})/2$	$(1-\sqrt{5})/2$
χ_3'	3	-1	0	$(1-\sqrt{5})/2$	$(1+\sqrt{5})/2$

Table: Character table of $B_2(\mathfrak{A}_5)$

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Table: Character table of $\mathfrak{A}_4$

	(1)	(2)	(3)	(3')
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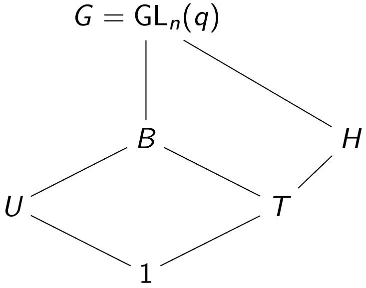
$-\alpha_3$	-3	1	0	0
$-\alpha_1$	-1	-1	$(1+\sqrt{-3})/2$	$(1-\sqrt{-3})/2$
$-\alpha_1'$	-1	-1	$(1-\sqrt{-3})/2$	$(1+\sqrt{-3})/2$

A kind of generic example :



$G = \mathrm{GL}_n(q)$

B

H

U

T

1

$$p \nmid q\,, \quad S \text{ } p\text{-Sylow}$$

$$T = S \times T_{p'}$$

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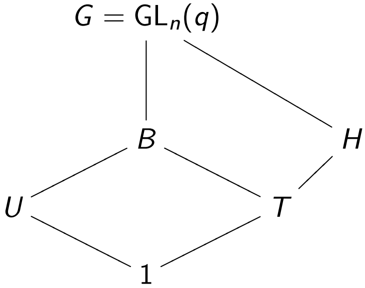

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We certainly have

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- $$M \otimes_{\mathcal{O}H} N \simeq B_p(G) \text{ as } \mathcal{O}G\text{–module–}\mathcal{O}G$$
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- Viewed as a $\mathcal{O}G$–module–$\mathcal{O}S$, we have
  $$M \simeq \mathcal{O}(G/U)\,,$$
  i.e., the functor $M \otimes_{\mathcal{O}S} ?$ is the Harish–Chandra induction.

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Which means : There is a G–equivariant collection of derived equivalences

$$\{\,\mathcal{E}(P) \,:\, \mathcal{D}^b(B_p(C_G(P)) \xrightarrow{\sim} \mathcal{D}^b(B_p(C_H(P))\,\}_{P\subseteq S}$$

compatible with Brauer morphisms.

What about the nonabelian Sylow case ?

The fact that the derived category of $B_p(G)$ is determined by $\mathrm{Frob}_p(G)$ is definitely false :

There are groups G and a subgroup H such that

- the inclusion $H \subset G$ induces an equivalence $\mathrm{Frob}_p(H) \xrightarrow{\sim} \mathrm{Frob}_p(G)$,
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But there seem to be lots of numerical similarities between $B_p(H)$ and $B_p(G)$.