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Related Semi-Classical and Toeplitz Algebras

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Abstract We investigate the relation between Toeplitz algebras and semi-classical algebras (star-products) arising for example in geometric prequantization.

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Contents

1 Introduction

Semi-classical algebras are those which appear in deformation quantization: a star product on a symplectic manifold X, as defined in [1], is an associative product $B = \sum h^n B_n$ on the set of formal power series $f = \sum h^n f_n \in C^\infty(X)((h))$, where for each n, B_n is a bidifferential operator, the leading term is the usual product: $B_0(f, g) = fg$, the leading term of the commutator law is the Poisson bracket of X: $B_1(f, g) - B_1(g, f) = \{f, g\}$, and the unit is 1 ($B_n(f, 1) = B_n(1, f) = 0$ if $n > 0$).

Toeplitz operators were introduced in [6, 8]. Modulo smoothing operators they produce a symbolic calculus locally identical to the calculus of pseudo-differential operators, and very similar to that of semi-classical algebras except that there is no longer a central h (the "small deformation parameter" h should be thought of as the inverse of the size of a large frequency).

A typical example arises from the theory of prequantization [25]. For this the data is
- a compact complex manifold X
- a complex line bundle Σ over X equipped with a hermitian metric a, strictly pseudoconvex outside of the zero section (this means that $\omega_\Sigma = i\partial\overline{\partial}a$ is a Kähler form outside of the zero section, in particular it is symplectic).

Then X is a projective manifold. The algebra of polynomial functions on Σ is $\bigoplus \Gamma(X, L^{\otimes n})$, L the dual bundle of Σ. Σ is a symplectic manifold, so as X (the pull-back of the symplectic form ω_X is $i\partial\overline{\partial}\mathrm{Log}\, a$).

The circle group $U(1)$ acts on the whole situation; we denote θ the infinitesimal generator; it is elementary that it is the hamiltonian field $\theta = H_a$.

In this situation we have an algebra of Toeplitz operators acting on the space of boundary values of holomorphic functions on the unit sphere $Y = \{a = 1\}$ of Σ (they are the operators of the form $T_P : f \mapsto SP(f)$ with P a pseudo-differential operator on Y, S the Szegö projector, i.e. the orthogonal projector onto the set of boundary values of holomorphic functions (CR functions) in $L^2(Y)$). The Toeplitz algebra $\mathcal{A}$ referred to in the title is the algebra of Toeplitz operators mod. smoothing operators; this admits a local definition and is the set of sections of a sheaf of algebras locally isomorphic to the algebra of pseudo-differential operators.

The associated semi-classical algebra (essentially the same as that introduced in [25]) is the sub-algebra $\mathcal{B} \subset \mathcal{A}$ of invariant operators. Its center is the algebra generated by θ, which is invertible (elliptic of degree 1). It has a natural structure of semi-classical algebra, setting $h = \theta^{-1}$.

In this paper we investigate generally how a Toeplitz and a semi-classical algebra can be similarly related, in presence of an action of the circle group. The answer is described more precisely in §5; it can be roughly summarized as follows : a semi-classical algebra can be embedded in a Toeplitz algebra iff its Fedosov curvature (cf. §4) is constant (the Fedosov curvature is a 2-cohomology class $r = h^{-1}\omega_X + \sum_{k\geq 0} h^k r_k$: except for the leading term $h^{-1}\omega_k$ it must not depend on h at all); a Toeplitz algebra contains a semi-classical algebra iff its Fedosov curvature is (up to an "exotic" exponent part, and the leading term ω_Σ) the pull-back of a 2-form on X.

The paper is organized as follows: the definitions and classification results for star-algebras are recalled, in §1-2.

In §3 we have grouped some definitions and facts concerning subprincipal symbols and involutions, which are immediately pertinent to the classification and, although easy, do not seem to appear elsewhere.

In [18] Fedosov introduced a very elegant description of semi-classical algebras, where the classifying invariant is the cohomology class of a formal closed 2-form on X: $R = \frac{1}{h}\omega_X + \sum h^n r_n$, the curvature of a "Fedosov connection". Fedosov's method can be adapted with trivial modifications to classify Toeplitz algebras, and in the situation above it is interesting to compare the two invariants. This is done in §4.

The relation between Toeplitz and semi-classical algebras is investigated in §5, which also contains a related result about compact group actions: any action of a compact group on the symplectic basis X resp. Σ lifts to the semi-classical algebra $\mathcal{B}$ over X resp. Toeplitz $\mathcal{A}$ over Σ; the lifting is unique up to isomorphism.

2 Notations

2.1 Conic Manifolds

In this paper we only consider star-products on real manifolds. Star products live on cones; we first recall the notations of [2, 4].

Definition 2.1 *A real cone is a C^∞ principal bundle Σ with group $\mathbf{R}_+^\times$. The basis is $B\Sigma = \Sigma/\mathbf{R}_+^\times$.*

Example 2.2 If D_ω is the long line, $\Sigma = T^*D_\omega$–(the zero section) is a real non trivial cone. However in this article we will always assume that manifolds are paracompact : a real cone Σ over a paracompact basis $B\Sigma$ is always trivial, isomorphic to $B\Sigma \times \mathbf{R}_+^\times$, but the product structure is not part of the data.

Definition 2.3 *(i) We denote $\mathcal{O}(m)$ the sheaf on $B\Sigma$ of homogeneous functions of degree m of Σ, $\mathcal{O} = \bigoplus \mathcal{O}(n)$.*

(ii) We denote $\widehat{\mathcal{O}}$ the sheaf on $B\Sigma$ of formal symbols :

$$f \in \widehat{\mathcal{O}} \quad \text{if} \quad f = \sum_{m \leq m_0} f_m \quad \text{with} \quad f_m \in \mathcal{O}(m), \quad m \quad \text{an integer, } m \to -\infty \tag{2.1}$$

(f is an "asymptotic expansion" for $\xi \to \infty$ on Σ).

(iii) For any integer $k \geq 1$ we denote $\widehat{\mathcal{D}}_k$ the sheaf (on $B\Sigma$) of formal k-differential operators : $P(f_1, \ldots, f_k) = \sum_{m \leq m_0} P_m(f_1, \ldots, f_k)$ with P_m a k-linear differential operator homogeneous of degree m with respect to homotheties, m an integer, $m \to -\infty$. For $k = 1$ we just write $\widehat{\mathcal{D}}$.

Locally we may choose homogeneous local coordinates x_j on Σ. Then $P_m(f_1, \ldots, f_k)$ is a sum of homogeneous monomials

$$a_\alpha(x)\, \partial_1^{\alpha_1} \ldots \partial_n^{\alpha_n}$$

i.e. a_α is homogeneous of degree $m + \sum \alpha_k \mathrm{deg}\,(x_k)$. Two cases will be useful:

1) $x_1, \ldots, x_{n-1}$ are homogeneous of degree 0 and are local coordinates on the basis $B\Sigma$, x_n is homogeneous of degree 1 or -1;

2) the x_j are all of degree $\frac{1}{2}$. There is no restriction on the order of P_m.

Below "degree" will always refer to the degree w.r. to homotheties; thus if $P \in \widehat{\mathcal{D}}_k$ each term P_m of degree m is of finite order, although the resulting infinite sum P may be of infinite order.

We will denote

$$\widehat{\mathcal{D}}^{\times} \subset \widehat{\mathcal{D}} \tag{2.2}$$

the sheaf of invertible formal differential operators : $P = \sum P_k \in \widehat{\mathcal{D}}^{\times}$ is invertible iff its leading term $\sigma(P) = P_{m_0}$ is invertible, i.e. P_{m_0} is of order 0, the multiplication by a nonvanishing function homogeneous of degree m_0.

Definition 2.4 *We denote by $\widehat{\mathcal{D}}_0^{\times}$ the subsheaf of those invertible P such that $P(1) = 1$, i.e. P is of degree 0, $P_0 = 1, P_m(1) = 0$ if $m < 0$.*

In some instances it will be convenient to use the complex extension

$$\Sigma_c = \Sigma \times_{R_+^{\times}} \mathbf{C}^{\times} \tag{2.3}$$

to which homogeneous functions of integral degree $f \in \mathcal{O}(n)$ or formal series $f \in \widehat{\mathcal{O}}$ obviously extend.

2.2 Star Algebras

Definition 2.5 *A star product on $B\Sigma$ is a unitary associative algebra law on $\widehat{\mathcal{O}}$*

$$B = \sum B_m \in \mathcal{D}_2 : \widehat{\mathcal{O}} \otimes \widehat{\mathcal{O}} \to \widehat{\mathcal{O}} \tag{2.4}$$

for which the unit is $1 \in \widehat{\mathcal{O}}$, $B_0 = 1$ (i.e. $B_0(f,g) = fg$). B is a deformation of the usual product.

A star-algebra on $B\Sigma$ is a sheaf of unitary associative algebras on $B\Sigma$, locally isomorphic to $\widehat{\mathcal{O}}$ equipped with a star-product, where the structural patching sheaf of groups is $\widehat{\mathcal{D}}_0^{\times}$.

Thus locally, in some frame, $f * g = \sum B_k(f,g)$ with B_k a bidifferential operator homogeneous of degree $-k \to -\infty$, $B_0 = 1$.

Since any $P \in \widehat{\mathcal{D}}_0^{\times}$ respects the natural filtration of $\mathcal{O}$ and the leading term ($P = 1$ mod. terms of lower degree), a star-algebra $\mathcal{A}$ is equipped with a natural filtration and a canonical "symbol isomorphism" $\sigma : \operatorname{gr} \mathcal{A} \to \mathcal{O}$. This conic framework for star products was described in [2, 4]. The distinction between star-products and star-algebras is not really essential on real manifolds, where a star-algebra is always isomorphic to $\widehat{\mathcal{O}}$ equipped with a star-product because a "total symbol" i.e. a global isomorphism $P : \mathcal{A} \xrightarrow{\sim} \widehat{\mathcal{O}}$ with $P \in \widehat{\mathcal{D}}$ locally always exists (locally, by definition, and on a real manifolds these can be patched using a partition of unity); as mentioned in [2] this is no longer true over holomorphic manifolds: however, the most usual star-algebras such as the algebra of pseudo-differential or semi-classical pseudo differential operators on a manifold are not equipped with a canonical total symbol, so for functorial manipulations they are star-algebras rather than star-products.

In this paper we will be concerned by the two main cases : Toeplitz algebras and semi-classical algebras, and the relation between them.

Let us recall that a deformation quantization or star product on a symplectic manifold X is a unitary associative product law $f * g = \sum h^n B_n(f,g)$ on the space of formal series $f = \sum h^n f_n \in C^{\infty}(X)[[h]]$, where each B_n is a bilinear differential operator on X, $B_0(f,g) = fg$, the unit is 1 (B_n has no "constant term" if $n > 0$), and the leading term in the commutator law is given by the Poisson bracket of X: $B_1(f,g) - B_1(g,f) = \{f,g\}$. Existence of such star-products was proved in [15].

We will refer to these as "semi-classical" star-products or star-algebras. The corresponding cone is $X \times \mathbf{R}_+$ equipped with the Poisson bracket hc_X, c_X the Poisson bracket of X; the canonical coordinate of $\mathbf{R}_+$ (homogeneous of degree 1, corresponding to a large frequency) is $\frac{1}{h}$; Σ is odd-dimensional. The "Planck constant" h (or its inverse $a = h^{-1}$) is central and part of the data, i.e. for these we only consider automorphisms which preserve h.

Toeplitz algebras (or Toeplitz star-products) are those associated to a symplectic cone Σ, e.g. the algebra of pseudo-differential operators (mod. smoothing operators) on a manifold X: $\Sigma = T^*X-\{\text{its zero section}\}$, equipped with its canonical symplectic structure, or the algebra of Toeplitz operators (mod. smoothing operators) of [8]; Σ is even dimensional, there is no longer a central formal "Planck constant" h, and the center is reduced to constants. The existence of a canonical Toeplitz algebra on a symplectic (even-dimensional) cone was proved in [8].

2.3 Models

The Moyal-Weyl star product gives typical examples of star products : let V be a graded vector space and $c = \sum c_{ij}\partial_i\partial_j \in V \otimes V$ a 2-tensor of degree ≤ -1. Then

$$f * g = \exp \frac{1}{2} c(\partial_\xi, \partial_\eta)\, f(\xi)g(\eta) \mid_{\eta=\xi} \tag{2.5}$$

is a star-product on V. The corresponding Poisson bracket is the top-order part (of degree -1) of the antisymmetrization of c:

$$\{f,g\}(\xi) = \frac{1}{2}(c_{-1}(\partial_\xi, \partial_\eta) - c_{-1}(\partial_\eta, \partial_\xi))\, f(\xi)g(\eta) \mid_{\eta=\xi} \tag{2.6}$$

Toeplitz algebras are locally isomorphic to any such model for which the Poisson bracket is symplectic. The most usual is the differential model: V has a set of "conjugate" coordinates: x_k of degree 0, ξ_k of degree 1, the Poisson bracket is

$$\{f,g\} = \sum \partial_{\xi_k} f \partial_{x_k} g \;-\; \partial_{x_k} f \partial_{\xi_k} g$$

and the star product is given by the Leibnitz rule:

$$f * g = \sum \frac{1}{\alpha} \partial_\xi^{(\alpha)} f \partial_x^{(\alpha)} g$$

We will rather use the more symmetric Weyl calculus: all coordinates are homogeneous of degree $\frac{1}{2}$, c is an antisymmetric bidifferential operator with constant coefficients so it is homogeneous of degree -1, and the star-product is given by formula (2.5) above. Here, and on several occasions below, we use the ξ_j coordinate functions (or other functions of fractional degree) as intermediate tools, but the elements of $\widehat{\mathcal{O}}$ are sum of monomials of integral degree.

For semi-classical star products the canonical local model is the following: X is a vector space (coordinates x_k of degree 0) equipped with a symplectic Poisson bracket $c = \sum c_{jk}\partial_j \wedge \partial_k$. For the star-product there is an extra central formal variable h:

$$f * h(\xi) = \exp \frac{h}{2} c(\partial_\xi, \partial_\eta) f(\xi)g(\eta)|_{\eta=\xi} \tag{2.7}$$

(h is part of the data and we only consider isomorphisms preserving it).

It is known (and easy) that all Toeplitz algebras (resp. semi-classical star-algebras) are locally isomorphic. Global existence of a star-algebra over a symplectic manifold X, resp. symplectic cone Σ a star-algebra was proved in ([15, 8]).

3 Automorphisms and classification

If $\mathcal{A}$ is a star-algebra, an automorphism U of $\mathcal{A}$ is by definition a formal differential operator $U \in \widehat{\mathcal{D}}$ which preserves the star-product and symbols. Thus $U = 1$ mod. operators of negative degree so $\operatorname{Log} U$ is well defined; it is a derivation of degree ≤ -1. Also the fractional powers $U^s (s \in \mathbf{C})$ are well defined. For a semi-classical algebra $\mathcal{B}$ we require U to preserve h; the set of such automorphisms is denoted $\operatorname{Aut}_h \mathcal{B}$.

An inner automorphisms $U = \operatorname{Ad} P$, i.e. $Uf = PfP^{-1}$ with P of degree 0; inner automorphisms define a sheaf $\operatorname{Int} \mathcal{A} \sim \mathcal{A}_0^\times$ mod. its center, whose sections are locally inner automorphisms. (Inner automorphisms $\operatorname{Ad} P$ with P of integral degree $\neq 0$ also exist, but for these $(\operatorname{Ad} P)^s$ is not, even locally, an inner automorphism). If $U = \operatorname{Ad} P$ is an inner automorphism, $D = \operatorname{Log} U$ is locally an inner derivation $D = \operatorname{ad} p$ with $p = \operatorname{Log} P$, P of degree 0.

From [2] we recall the following facts:

3.1 Semi-classical algebras

Lemma 3.1 *Automorphisms of semi-classical algebras are locally inner automorphisms.*

If $\mathcal{B}$ is a semi-classical algebra the sheaf $\operatorname{Aut} \mathcal{B}$ of automorphisms is isomorphic to the sheaf of invertible sections of degree 0, mod. its center: $\mathcal{B}_0^\times / \mathbf{C}[[h]]^\times$.

We will rather use the sheaf $\widetilde{\mathcal{B}}_0^\times$ whose sections are pairs φ, f with $f \in \mathcal{B}_0$ invertible, $\varphi \in \mathcal{O}(0); e^\varphi = \sigma(f)$ (equivalently - the sheaf of logarithms): thus we have an exact sequence

$$0 \to \mathbf{C}[[h]] \to \widetilde{\mathcal{B}}_0^\times \to \operatorname{Aut} \mathcal{B} \to 0 \tag{3.1}$$

Although $\widetilde{\mathcal{B}}_0^\times$ is not commutative, it has partitions of unity (it is a "soft" sheaf) so the canonical map $H^1(X, \operatorname{Aut} \mathcal{B}) \to H^2(X, \mathbf{C}[[h]])$ is an isomorphism:

Proposition 3.2 *The set h-Alg(X) of equivalence classes of semi-classical algebras over X is in 1-1 correspondence with $H^1(X, \operatorname{Aut} \mathcal{B}) \simeq H^2(X, \mathbf{C}[[h]])$.*

(more accurately there is a free transitive action of $H^2(X, \mathbf{C}[[h]])$.)

The automorphisms of a semi-classical algebra $\mathcal{B}$ preserve h (by definition). For further use in §5, it will be useful to note the following fact about automorphisms of $\mathcal{B}$ as a star-algebra: such an automorphism induces an automorphism of the center $\mathbf{C}((h))$, and its restriction is thus completely determined by $U(h) = \sum_1^\infty a_k h^k$ with $a_1 \neq 0$.

Lemma 3.3 *The restriction map* $\operatorname{Aut} \mathcal{B} \to \operatorname{Aut} \mathbf{C}[[h]]$ *is onto (there is an exact sequence* $0 \to \operatorname{Aut}_h \mathcal{B} \to \operatorname{Aut} \mathcal{B} \to \operatorname{Aut} \mathbf{C}[[h]] \to 0$*).*

Equivalently we have, for derivations of degree 0, an exact sequence:

$$0 \to \operatorname{Der}_h \mathcal{B} \to \operatorname{Der} \mathcal{B} \to \operatorname{Der} \mathbf{C}[[h]].$$

This is immediate for the local model, so also for global derivations since the sheaves *Der* of derivations of $\mathcal{B}$ are "soft" (they have partitions of unity, and $H^1(X, \operatorname{Der}_h(\mathcal{B})) = 0$).

3.2 Toeplitz algebras

Lemma 3.4 *Any automorphism of a Toeplitz algebra* $\mathcal{A}$ *is locally of the form* $U = (\operatorname{Ad} P)^s \operatorname{Ad} Q$ *where* P *is elliptic of degree* 1*,* Q *elliptic of degree* 0*.*

Definition 3.5 *We denote* $\underline{\omega}$ *the sheaf (on* $B\Sigma$*) of differential 1-forms on* Σ *which are closed, homogeneous of degree* 0*. This is canonically isomorphic, via the Poisson bracket* c_Σ*, to the sheaf of symplectic vector fields on* Σ*, homogeneous of degree* -1*.*

We denote $\mathcal{A}_-^\times \subset \mathcal{A}^\times$ *the set of invertible elements of symbol* 1*.*

If U is an automorphism, $D = \operatorname{Log} U$ is a well defined derivation of degree ≤ -1, whose leading term $\delta = \sigma(U)$ is a symplectic derivation, homogeneous of degree -1, and defines a section $\sigma(U) \in \underline{\omega}$ (its "symbol"). If $U = (\operatorname{Ad} P)^s \operatorname{Ad} Q$ we have $\sigma(U) = s\frac{dp}{p} + \frac{dq}{q}$ (with p, q the symbols of P, Q).

Then $\sigma(U) = 0$ means that there exists $Q \in \mathcal{A}_-^\times$ such that $U = \operatorname{Ad} Q$. Q is unique since the center of a Toeplitz algebra is $\mathbf{C}$.

Thus we have an exact sequence of sheafs

$$0 \to \mathcal{A}_-^\times \to \operatorname{Aut} \mathcal{A} \to \underline{\omega} \to 0 \tag{3.2}$$

Here again $\mathcal{A}_-^\times$, although not commutative, is soft so the canonical map $H^1(B\Sigma, \operatorname{Aut} \mathcal{A}) \to H^1(B\Sigma, \underline{\omega})$ is one to one:

Proposition 3.6 *The set T-Alg*$(B\Sigma)$ *of equivalence classes of Toeplitz algebras over* $B\Sigma$ *is isomorphic to* $H^1(B\Sigma, \operatorname{Aut} \mathcal{A}) \simeq H^1(B\Sigma, \underline{\omega})$*.*

The classifying isomorphisms above depend on the choice of an initial star-algebra as base-point. In section 4 we recall the analysis of Fedosov, which provides a canonical base-point.

3.3 The exponent Character

Let Σ be a symplectic cone and $\mathcal{A}$ a Toeplitz algebra on Σ. Denote $Is_{\mathcal{A}}$ the groupoïd of isomorphisms of $\mathcal{A}$, over the groupoïd Sp_Σ of germs of homogeneous symplectic maps of Σ: an element of $\mathrm{Is}_{x' \leftarrow x}$ is an isomorphism $\mathcal{A}_x \to \mathcal{A}'_x$ over a germ of homogeneous symplectic map $\Sigma_x \to \Sigma_{x'}$. Obviously its Lie algebra is the sheaf $\mathcal{G}$ of derivations of degree 0 of $\mathcal{A}$, over the Lie algebra of homogeneous symplectic vector fields. This contains the algebra $\mathcal{G}_1$ of locally inner derivations, which is invariant hence locally integrable.

Choosing a base point $x \in \Sigma$ we get a homomorphism $\delta_{\mathcal{A}} : \pi_1(\Sigma, x) \to \operatorname{Der} \mathcal{A}_x / \operatorname{Int} \mathcal{A}_x = \mathbf{C}$, which to a loop γ assigns the exponent of U_1 if $U_t \in Is_{\gamma(t) \leftarrow x}$ is any smooth family of germs of isomorphisms such that $U^{-1}\frac{dU}{dt}$ is an inner automorphism, $U_0 = \mathrm{Id}$. (If $\mathcal{A}$ is involutive (see below in §3), we may choose $U(t)$ so that it respects the involution, which implies $\delta_{\mathcal{A}} = 0$, and in particular that the exponent of U_1 vanishes; so in general $\delta_{\mathcal{A}} \in H^1(B\Sigma, \mathbf{C})$ is the exponent part of the Fedosov class of $\mathcal{A}$ - see below §4.)

Proposition 3.7 *Let* D *be the infinitesimal generator of an action of the circle group* $U(1)$ *on* $\mathcal{A}$ *(with symbol* H_a*, the generator of the corresponding action on* Σ*). Then the exponent of* D *is* $2\pi\delta_{\mathcal{A}}(\gamma)$ *if* γ *is any orbit.*

In particular the exponent vanishes if the orbit homology class in $H_1(\Sigma, \mathbf{C})$ vanishes, e.g. if there is a fixed point (Σ connected), or if the circle action on the "sphere" $\{a = 1\}$ defines a principal bundle with compact basis and symplectic curvature (in the second case the orbit defines a torsion element of $\pi_1(B\Sigma)$ so its homology class in $H_1(B\Sigma, \mathbf{C})$ vanishes).

4 Subprincipal Symbol and Involutions

4.1 Subprincipal Symbol

Let $\mathcal{A}$ be a Toeplitz resp. semi-classical algebra.

Definition 4.1 *A subprincipal symbol on $\mathcal{A}$ is a differential operator of degree -1 (for all m, $a \in \mathcal{A}_m \mapsto \mathrm{sub}_m(a) \in \mathcal{O}(m-1)$) such that*

(i) if $a \in \mathcal{A}_m$, $\mathrm{sub}_m(a) = \sigma_{m-1}(a)$ if $\sigma_m(a) = 0$; $\sigma_m(a) = \mathrm{sub}_m(a) = 0$ imply $a \in \mathcal{A}_{m-2}$

(ii) if a, b are of degree m, m', we have

$$\mathrm{sub}\,(a * b) = \mathrm{sub}\,(a)\sigma(b) + \sigma(a)\mathrm{sub}\,(b) + \frac{1}{2}\{\sigma(a), \sigma(b)\} \in \mathcal{O}(m+m'-1)$$

(iii) if a, b are of degree m, m', we have

$$\mathrm{sub}_{m+m'-2}[a,b] = \{\mathrm{sub}\,(a), \sigma(b)\} + \{\sigma(a), \mathrm{sub}\,(b)\} \in \mathcal{O}(m+m'-2)$$

For example on the standard Weyl algebra (Toeplitz or semi-classical), the canonical subprincipal symbol of $a = \sum a_{m-k}$ is the second term a_{m-1}. If $a(x,D) = a_m(x,D) + a_{m-1}(x,D) + \dots$ is a pseudodifferential operator on $\mathbf{R}^n$ ($D = \frac{1}{i}\frac{\partial}{\partial x}$) its subprincipal symbol is $a_{m-1} - \frac{1}{2i}\sum \frac{\partial^2 a_m}{\partial x_k \partial \xi_k}$.

Proposition 4.2 *1) An automorphism U of $\mathcal{A}$ preserves a principal symbol iff it is of degree ≤ -2.*

2) Two subprincipal symbols are conjugate.

1) is obvious. Let us prove 2): let $\mathcal{A}$ be a Toeplitz or semi-classical algebra equipped with a subprincipal symbol sub. In view of (i), (ii) any other subprincipal symbol is of the form $\mathrm{sub}'a = \mathrm{sub}\,a + \delta(a_m)$ with δ a differential operator, homogeneous of degree -1. (ii) implies $\delta(ab) = \delta(a_m)b_m + a_m\delta(b_m)$ i.e. δ is a derivation. Then (iii) implies that $\delta(\{a,b\} = \{\delta a, b\} + \{a, \delta b\}$ i.e. δ is symplectic. Then (3.2) δ is the leading term of a star derivation D and we have $\mathrm{sub}'(a) = \mathrm{sub}\,(e^D a)$. Thus two subprincipal symbols are locally conjugate - in fact globally because the sheaf of automorphisms which preserve a given subprincipal symbol is "soft".

Using partitions of unity it is easy to see that a map satisfying (i), (ii) exists on any semi-classical or Toeplitz algebra (on a real paracompact manifold). But such a map is not sufficient for global computations and does not deserve the name "subprincipal symbol". Subprincipal symbols, satisfying (iii), do not exist on all semi-classical or Toeplitz algebras; in particular they do not always exist on the canonical Toeplitz algebra constructed in [8] (cf. §3.3 below).

4.2 Involutions

An anti-involution on an algebra $\mathcal{A}$ is a linear operator J such that $J^2 = \mathrm{Id}$, $J(ab) = J(b)J(a)$.

On a star-algebra $\mathcal{A}$ we only consider anti-involutions which preserve symbols of degree 0. Since $J([a,b)] = -[Ja, Jb]$ this implies

$$\mathrm{gr}\,J(\sum f_n) = \sum (-1)^n f_n$$

the sublying geometric map is $\xi \mapsto -\xi$ on the cone $\Sigma_c = \Sigma \times_{R_+^\times} \mathbf{C}^\times$; the induced map on the basis $B\Sigma$ is still the identity map. For an anti-involution on a semi-classical algebra we further require $J(h) = -h$.

For example, on the model semi-classical algebra, there is a canonical involution: $a(x,h) \mapsto a(x,-h)$. On the algebra of differential operators on $\mathbf{R}^n$ there is a canonical involution $P(x,\partial_x) = \sum a_\alpha(x)\partial^\alpha \mapsto^t P = \sum(-1)^\alpha \partial^\alpha\, a_\alpha(x)$.

Anticipating on §4, let us note that on the Weyl model for Toeplitz algebras $\widehat{W}$ described below, the canonical involution is $a(x,\xi) \mapsto a(ix,i\xi)$ when using canonical local coordinates x_k, ξ_k of degree $\frac{1}{2}$ (recall that a is sum of functions of integral degree, i.e. of even degree w.r. to x,ξ so the i factor does not really appear - otherwise the corresponding geometric map should be interpreted as a map on the double cover $\sqrt{(T\Sigma)_c}$).

Proposition 4.3 *(i) Any two anti-involutions on $\mathcal{A}$ are (globally) conjugate.*
(ii) If $\mathcal{A}$ has an anti-automorphism, it also has an anti-involution.
(iii) An automorphism U which preserves an anti-involution J is of the form $\mathrm{Ad}\, P$ *with* $P \in \mathcal{A}_-^\times, JP = P^{-1}$*; in particular its symbol is* 0.

Proof : (i) if J, J' are two anti-involutions, $U = J'J$ is an automorphism and $J' = UJ = JU^{-1}$, so $J' = U^{1/2}JU^{-1/2}$.

(ii) If A is an anti-automorphism, A^2 is an automorphism which commutes with A, so $J = (A^2)^{-1/2}A$ is an anti-involution.

(iii) Let J be an anti-involution. If U is an automorphism we set $U^J = JUJ$.

In the Toeplitz case, we obviously have $\sigma(U^J) = -\sigma(U)$. In particular if $U^J = U$, we have $\sigma(U) = 0$, so $U = \mathrm{Ad}\, P$ for some $P \in \mathcal{A}_-^\times$.

Now if P is elliptic we have $J(PJfP^{-1}) = (JP)^{-1}fJP$ i.e. $(\mathrm{Ad}\, P)^J = \mathrm{Ad}\,(JP^{-1})$, so $\mathrm{Ad}\, P$ commutes with J iff $JP = cP^{-1}$, c central. If $P \in \mathcal{A}_-^\times$ we have necessarily $c = 1$.

In the semi-classical case U is locally an inner automorphism $\mathrm{Ad}\, P$ with P of degree 0. The condition $U^J = U$ implies that $c = P\,JP$ is central i.e. an invertible power series $c(h)$; this is even because $Jc = JP\,P = c$. Replacing P by $P' = c^{-\frac{1}{2}}P$, we get $U = \mathrm{Ad}\, P'$ with again $P' \in \mathcal{A}_-^\times, JP' = P'^{-1}$.

A Toeplitz or semi-classical algebra with involution has a subprincipal symbol: $\mathrm{sub}_m(a) = \frac{1}{2}\sigma_{m-1}(a - (-1)^m Ja)$. In fact our models have both an involution and a subprincipal symbol, and the group of automorphisms which preserve the involution is contained in the group of automorphisms which preserve sub.

On a symplectic cone there is (up to isomorphism) exactly one Toeplitz algebra possessing an anti-involution. As we will see below this is the "base-point" in Fedosov's construction, but it is not necessarily the "canonical" algebra constructed in [8] (§3.3).

For semi-classical algebras the situation is slightly more complicated. We will see below that the base-point $\mathcal{A}_0$ constructed by Fedosov is involutive. Then the set of equivalence classes of semi-classical algebras is isomorphic to $H^1(X, \mathrm{Aut}^J(\mathcal{A}_0) \simeq H^2(X, \mathbf{C}^{\mathrm{odd}}[[h]])$.

4.3 Example

If Σ is a real symplectic cone, we constructed in [8] a canonical algebra of operators acting on a suitable subspace $H \subset L^2(B\Sigma)$, whose symbol star-algebra $\mathcal{A}_0$ we have already referred to above.

For example let $X = B\Sigma$ be the boundary of a strictly pseudo-convex complex domain Ω, H is the space of boundary values of holomorphic functions on Ω, and $\mathcal{A}_0$ is a star completion of the algebra of differential operators on Ω. Now the algebra which obviously has an anti-involution (hence also a subprincipal symbol) is the algebra $\mathcal{A}_{\frac{1}{2}}$ of twisted differential operators acting on half-forms (half forms do not always exist, but the twisted algebra always does: it is constructed by twisting $\mathcal{A}_0$ by the cocycle $\mathrm{Ad}\,(\mathrm{Jacobian})^{\frac{1}{2}}$). The symbol of this twistor is an element of $H^1(X, \underline{\omega_0}) \subset H^1(X, \underline{\omega})$, where $\underline{\omega_0} \subset \underline{\omega}$ is the sheaf of 1-forms on X (contained in the sheaf of closed 1-forms homogeneous of degree 0 on Σ). Its image in $H^2(X, \mathbf{C})$ is $\frac{1}{2}\mathrm{ch}\,\Omega$ (the Chern class of the sheaf of holomorphic differential forms on Ω).

In this case the "canonical" Toeplitz algebra has a subprincipal symbol iff this Chern class vanishes ($\mathrm{ch}\,\Omega = 0$ on X).

Here is a more specific example: let Ω be the unit ball in a complex line bundle L over a complex manifold Y (since Ω is strictly pseudo-convex, Y is projective and L' is ample). Then the condition $\mathrm{ch}\,\Omega_\Omega = 0$ on $X = \partial\Omega$ means that $\mathrm{ch}\,L$ is proportional to $\mathrm{ch}\,L$ on Y. This is always true if $\dim Y = 1$, but no longer necessarily true if $\dim Y \geq 2$ e.g. if Y is the abelian variety $\mathbf{C}^2/\mathbf{Z}^4$.

The Toeplitz algebra $\mathcal{A}_0$ constructed in [8] is not always involutive, but it acts on a Hilbert space - or more accurately on the scale of Hilbert spaces $\mathcal{O}^s$ of boundary values of holomorphic functions which lie in the Sobolev space H^s (more generally the image of H^s by the Toeplitz projector π_Σ).

It is natural to ask which Toeplitz algebras act similarily on a Hilbert space. We will require that this action be microlocally isomorphic to the action above (or microlocally isomorphic to the action of pseudo-differential operators on the scale of Sobolev spaces - in fact it necessarily is):

Let $\mathcal{A}$ be a Toeplitz algebra on Σ acting on a scale of Hilbert spaces as indicated: there is a covering Y_i of the basis Y and over each Y_i an isomorphism $U_i : H \to H_0$ respecting the scale. Then $U_{ij} = U_i U_j^{-1}$ is a Toeplitz operator of degree 0 (it must respect the scale); the difference cocycle for $\mathcal{A}_0$ is $u_{ij} = \mathrm{Ad}\,U_{ij}$; its "exponent" vanishes (there is no exotic component), and its image in $H^2(Y, \mathbf{C}^\times)$ also must vanish. We may conclude:

Proposition 4.4 *$\mathcal{A}$ acts on a scale of Hilbert spaces as above iff the Chern class $c(u) \in H^2(Y, \mathbf{C})$ of the difference cocycle $u = (u_{ij})$ defined by comparison with $\mathcal{A}_0$ is integral.*

Remark 4.5 This does not mean that the cohomology class of the Fedosov curvature defined below is integral, but that the difference with the class of $\mathcal{A}_0$ is integral; this is half-integral and usualy not better.

Remark 4.6 If $\mathcal{A}$ is exotic, it may still happen that $c(u)$ is integral, i.e. the cocycle u_{ij} is a "coboundary" of the form $U_i U_j^{-1}$. Then $\mathcal{A}$ will act on a distribution space, locally isomorphic to $\bigcup H^s$. It does not preserve the scale, but it will still preserve a "Sobolev space" of variable order such as used by Višik and Eskin [30].

5 Fedosov Connections

In [18], Fedosov gave a very elegant and geometric description for the existence and classification of semi-classical algebras: a semi-classical star algebra $\mathcal{B}$ over a real symplectic manifold X can be embedded as the subalgebra of flat sections of a suitable "Fedosov connection" $\mathrm{ad}\,\nabla$ in a "universal" star algebra $\widehat{W}$ over the tangent bundle TX (the definitions are recalled below); the curvature of ∇ is a closed central 2-form with leading term $h^{-1}\omega_X$, whose cohomology class determines $\mathcal{B}$ up to isomorphism; in particular this construction singles out a base-point (the algebra whose curvature is exactly $h^{-1}\omega_X$). This description can easily be adapted to Toeplitz algebras (more generally to Poisson cones of constant rank [4]). We will need this adapted description, and recall here how it works. In fact we give a direct proof of the embedding theorem, with complements which will be useful in §5.

5.1 Valuations and Relative Tangent Algebra $\widehat{W}$

Let Σ be a Poisson cone. On the tangent bundle $T\Sigma$ each fiber $T_x\Sigma$ inherits of a Poisson bracket c_x with constant coefficients.

We will denote W the sheaf generated by homogeneous polynomials on $T\Sigma$: if we choose homogeneous local coordinates $x = (x_1, \dots, x_n)$ on Σ and $\xi = (\xi_1, \dots, \xi_n)$ denote the corresponding tangent coordinates, a section of W is locally a finite sum of homogeneous monomials $a_\alpha(x)\xi^\alpha$. The degree of a homogeneous monomial is

$$\deg a_\alpha\, \xi^\alpha = \deg a_\alpha + \sum \alpha_k \deg \xi_k \tag{5.1}$$

The degree valuation is

$$p(f) = \inf(-\deg f_\alpha) \quad \text{if } f \text{ is sum of homogeneous monomials } \ f_\alpha \tag{5.2}$$

We define the weight (or order) of a homogeneous monomial $a_\alpha(x)\xi^\alpha$ as

$$w(a_\alpha(x)\xi^\alpha) = -\deg a_\alpha + \frac{|\alpha|}{2} \tag{5.3}$$

The order valuation is

$$w(f) = \inf w(f_\alpha) \quad \text{if } f \text{ is sum of homogeneous monomials } \ f_\alpha \tag{5.4}$$

Thus W is equipped with two valuations $p \le w$.

The Poisson brackets c_x define fiberwise a Weyl product on W:

$$f * g\,(x,\xi) = \exp \frac{1}{2} c_x(\partial_\xi, \partial_\eta)\ f(x,\xi)\, g(x,\eta)\ |_{\eta=\xi} \tag{5.5}$$

Since we will be dealing simultaneously with several product laws, we will adopt the following notations, to avoid confusions if need be:

$$* \quad \text{or} \quad *_W,\ *_A \quad \text{for the star-product,} \quad \times \ \text{(or no sign) for the usual product} \tag{5.6}$$

Both p and w are valuations for this product, i.e.

$$p(f * g) \ge p(f) + p(g), \quad w(f * g) \ge w(f) + w(g) \tag{5.7}$$

The graded algebra $\mathrm{gr}_p W = \bigoplus_k (W_{p\ge k}/W_{p>k})$ is commutative (its product is the usual product since $p([f,g]) \ge p(f) + p(g) + 1$). The graded algebra $\mathrm{gr}_w W$ is not commutative; in fact the star-product $*$ is homogeneous for w, i.e. if f, g are sums of monomials of pure weight k resp. k', then $f * g$ and $[f,g]$ are sums of monomials of pure weight $k + k'$; the whole situation is analogous to the situation for the symbolic calculus of Hermite operators (cf. [7, 22]).

Definition 5.1 *We denote $\widehat{W}_\Sigma$ (or $\widehat{W}$ if there is no confusion) the completion of W for the valuation w.*

Note that $\widehat{W}$ is a set of formal functions on $T\Sigma$, equipped with a star-product.

$\widehat{W}$ is obviously functorial, i.e. if $u : \Sigma \to \Sigma'$ is a morphism of Poisson cones (a smooth homogeneous map $\Sigma \to \Sigma'$ compatible with the Poisson brackets, i.e. $u_* c_\Sigma = c_{\Sigma'}$ or $\{f \circ u, g \circ u\} = \{f, g\} \circ u$ for any smooth functions f, g on Σ') then $u^* : \widehat{W}_{\Sigma'} \to \widehat{W}_\Sigma$ ($f \mapsto f \circ Tu$) is a homomorphism af algebras, both for the star-product and the ordinary product.

In the semi-classical case (X a symplectic manifold), the relevant cone is $\Gamma = X \times \mathbf{R}_+^\times$ equipped with the Poisson bracket hc_X with $h = a^{-1}, a$ the canonical coordinate of $\mathbf{R}_+^\times$. We have $T\Gamma = TX \times T\mathbf{R}_+^\times$; the center of $\widehat{W}_\Gamma$ is the commutative algebra, completion of $\mathbf{C}[a, \alpha]$ for the order valuation w, where (a, α) are the canonical coordinates of $T\mathbf{R}_+^\times$ ($a = h^{-1}, \alpha$ the corresponding tangent coordinate; $w(a) = -1, w(\alpha) = -\frac{1}{2}$).

For semi-classical algebras on X, we will rather use, as Fedosov, the completion $\widehat{W}_h = \widehat{W}_h(X)$ for the valuation w of the sheaf on X of algebras, whose sections are polynomials $f = \sum f_{k,\alpha}(x) h^k \xi^\alpha$; (the valuations p, w are defined as above, ($p(h) = 1$, functions on X are homogeneous of degree 0 ($p = 0$)). $\widehat{W}_h$ is canonically a quotient of $\widehat{W}_\Gamma$: $a \mapsto h^{-1}, \alpha \mapsto 0$.

5.2 Automorphisms and Derivations of $\widehat{W}$

Let Σ be a symplectic cone (resp. X a symplectic manifold). In this section we examine derivations and automorphisms of $\widehat{W}_\Sigma$ (resp. $\widehat{W}_h$). We only consider automorphisms U such that $w(U - \mathrm{Id}) > 0$.

Any derivation or automorphism preserves the center of $\widehat{W}$ (i.e. $\widehat{\mathcal{O}}_\Sigma$, resp. $\widehat{\mathcal{O}}_X((h))$). We will say that D resp. U is an $\mathcal{O}$-derivation (resp. automorphism) if it fixes the center (i.e. D or $U - \mathrm{Id}$ vanishes on the center).

Proposition 5.2 *All $\mathcal{O}$-derivations, resp. automorphisms are inner derivations (resp. automorphisms). More precisely, if D is an $\mathcal{O}$-derivation, resp. U an $\mathcal{O}$-automorphism, there exists a unique section d resp. $u \in \widehat{W}$ such that $d = 0$, resp. $u = 1$ for $\xi = 0$, and $D = \operatorname{ad} d$ resp. $U = \operatorname{Ad} u$ (i.e. $Df = [d, f]$, $Uf = u * f * u^{-1}$.*

Proof : this is obvious on the local model (Σ or X a graded vector space), where all $\widehat{\mathcal{O}}$-derivations (resp. automorphisms) are inner, because $\operatorname{ad} d = \operatorname{ad}(d - d(x, 0)), \operatorname{Ad} U = \operatorname{Ad} \frac{U}{U(x,0)}$. The uniqueness property just reflects the fact that the center is $\widehat{\mathcal{O}}_S$ resp. $\mathcal{O}_X((h))$. The global statement follows (uniqueness ensures that objects constructed on a covering patch together).

Note that if U is an $\widehat{\mathcal{O}}$-automorphism ($w(U - \mathrm{Id}) > 0$), $D = \operatorname{Log} U$ is well defined, $w(D) > 0$, so $D = \operatorname{ad} d$ with some d such that $w(d) > 0$ and $U = \operatorname{Ad} u$ with $u = e^d$; however $d(x, 0) = 0$ is not equivalent to $e^d(x, 0) = 1$ (the set of sections which vanish for $\xi = 0$ is not an ideal for $*$).

An immediate consequence of this proposition is that the sheaf (on $Y = B\Sigma$ or X) of $\widehat{\mathcal{O}}$-derivations or $\widehat{\mathcal{O}}$-automorphisms is "soft" (has partitions of unity).

5.3 Embeddings

Let $\mathcal{A}$ be a Toeplitz algebra over Σ (resp. $\mathcal{B}$ a semi-classical algebra over X). We will say that a homomorphism of algebras $\mathcal{A} \to \widehat{W}$ (resp. $\mathcal{B} \to \widehat{W}_h$) is a good embedding if it respects w

($w(u) \geq 0$ i.e. $w(uf) \geq p(f)$) and locally, in any set of homogeneous coordinates x_j we have $u(x_j) = x_j + \xi_j + r_j$ with $w(r_j) \geq p(x_j) + 1$ (in the semi-classical case we require $u(h) = h$). The second condition does not depend on the choice of homogeneous coordinates.

Theorem 5.3 *1) For any Toeplitz algebra $\mathcal{A}$ (resp. semi-classical algebra $\mathcal{B}$) there exists a good embedding $\mathcal{A} \to \widehat{W}_\Sigma$ (resp. $\mathcal{B} \to \widehat{W}_h$).*

2) If u_1, u_2 are two good embeddings there exists a unique $\widehat{\mathcal{O}}$-automorphism U such that $u_2 = Uu_1$ (two good embeddings are conjugate).

Proof : locally the theorem is immediate: if (x_j) is a set of local homogeneous symplectic coordinates ($\{x_i, x_j\} = c_{ij} = \text{constant}$), a model good embedding is $f(x) \mapsto uf = f(x + \xi)$. As mentioned it is often convenient to choose the x_j, ξ_j homogeneous of degree $\frac{1}{2}$; we may then use them only as intermediate tools since the monomials of WH must be homogeneous of integral degree.

If v is another good embedding, $v(x_j) = x_j + \eta_j$, we have $[\eta_i, \eta_j] = [\xi_i, \xi_j] = c_{ij}$ because the x_j are central in $\widehat{W}$, so by the universal property of the Weyl algebra there exists a unique algebra homomorphism U such that $U\xi_j = \eta_j$ ($Uf = f$ if f is central). Since $w(\eta_j - \xi_j) > w(\xi_j)$, U respects the valuation w and its unique continuous extension to $\widehat{W}$ is bijective.

This unique conjugacy statement is obviously also global.

In the general case there exists an open covering Σ_i of Σ (resp. X_i of X) and for each i a good embedding $u_i : \mathcal{A} \to \widehat{W}$ over Σ_i (resp. ...). For each pair (i, j) there is a unique $\widehat{\mathcal{O}}$-automorphism U_{ij} over $\Sigma_i \cap \Sigma_j$ such that $u_i = U_{ij}u_j$, and this is a cocycle ($U_{ij}U_{jk} = U_{ik}$ because $U_{ij}U_{jk}u_k = U_{ik}u_k = u_i$). Since the sheaf of $\widehat{\mathcal{O}}$-automorphisms (or derivations) is soft, (U_{ij}) is a coboundary: $U_iU_{ij} = U_j$ for a suitable family (U_i) so that the U_iu_i patch together to produce a global good embedding.

Remark 5.4 Another way to state this result is the following: the completion of $\mathcal{O} \otimes \mathcal{A}$ is isomorphic to $\widehat{W}$; the valuation used to define the completion is the unique valuation such that $w(f \otimes g) = p(f) + p(g)$, $w(1 \otimes f - f \otimes 1) = p(f) + \frac{1}{2}$ if $1 \otimes f - f \otimes 1 \neq 0$ i.e. f is not a constant.

Remark 5.5 In the above construction we did not require that the embedding u preserve the homogeneity valuation p. It is easy to show that there also exists a good embedding u which preserves p (see §4.5 below). When this is the case we have $\sigma(uf) = \sigma(f) \circ \overline{u}$ where $\overline{u}$ is a smooth homogeneous map $T\Sigma \to \Sigma$ compatible with the Poisson brackets, and tangent to the map $(x, \xi) \mapsto x + \xi$ along the zero-section of $T\Sigma$. It is still true that two such embeddings u_1, u_2 are conjugate: $u_2 = Uu_1$ with U an $\mathcal{O}$-automorphism ($w(U - \text{Id}) > 0$), but U cannot preserve the homogeneity valuation p if the geometric maps $\overline{u_1}, \overline{u_2}$ are not equal.

5.4 Vector Fields with Coefficients in $\widehat{W}$

To describe the other derivations or automorphisms, it is convenient to choose a homogeneous symplectic connection ∇^s on Σ (resp. X). We can choose it homogeneous of degree 0 (with respect to homotheties) and torsionless, but any symplectic connection with coefficients in $\widehat{\mathcal{O}}_0 \otimes sp$ (degree ≤ 0) will do. ∇^s extends canonically to $\widehat{W}$, and if V is a vector field on Σ (homogeneous or with coefficients in $\widehat{\mathcal{O}}$), ∇^s_V is a derivation, both for the star-product and the usual product, (it respects both valuations p, w if V is homogeneous of degree 0).

We identify the Lie algebra of sections of $sp(T\Sigma)$ with the sub-Lie algebra W_2 of $\widehat{W}_\Sigma$ of homogeneous polynomials of degree 2 with respect to ξ: $\gamma = \sum \gamma_{jk}\xi_j\xi_k$ (acting by ad : $\mathrm{ad}\,\gamma f = [\gamma, f]$). Thus locally, if we choose homogeneous symplectic coordinates x_j ($\{x_i, x_j\} = c_{ij} = \mathrm{cst}$), we have

$$\nabla^s f(x,\xi) = \sum dx_j \Big(\frac{\partial f}{\partial x_j} + \sum[\gamma_{ijk}\xi_j\xi_k, f]\Big) \tag{5.8}$$

with $\gamma = \sum dx_i\, \gamma_{ijk}(x)\xi_j\xi_k$ homogeneous of degree 1 ($\mathrm{ad}\,\gamma$ of degree 0).

If D is a derivation of $\widehat{W}_\Sigma$, its restriction to the center is a derivation of $\widehat{\mathcal{O}}_\Sigma$ i.e. a vector field V on Σ with coefficients in $\widehat{\mathcal{O}}$, so we have $D = \nabla^s_V + \mathrm{ad}\, d$ for some sections $d \in \widehat{W}$.

It is useful to introduce, as Fedosov, "derivations with coefficients in $\widehat{W}$". These are well defined because the symplectic Lie algebra *sp* is identified with the subalgebra $W_2 \subset \widehat{W}$. They form a Lie algebra Vect_W which is an extension of the Lie algebra $\mathrm{Der}_{\widehat{W}}$ of derivations of $\widehat{W}$. Its sections can be written $\nabla^s_V + d$, where the rule for changes of coordinates for ∇^s_V is the rule for vectors with coefficients in the Lie algebra $sp(T\Sigma) = W_2 \subset \widehat{W}$. The bracket is

$$[\nabla^s_{V_1} + d_1\,,\, \nabla^s_{V_2} + d_2] = [\nabla^s_{V_1}, \nabla^s_{V_2}] + \nabla^s_{V_1}(d_2) - \nabla^s_{V_2}(d_1) + [d_1, d_2] \tag{5.9}$$

where the first bracket is given by the rule for connections with coefficients in $sp = W_2$:

$$[\nabla^s_{V_1}, \nabla^s_{V_2}] - \nabla^s_{[V_1,V_2]} = R^s(V_1 \wedge V_2) \in W_2 \tag{5.10}$$

If V is a derivation with coefficients in $\widehat{W}$ we denote $\mathrm{ad}\,V$ the derivation it defines. We have $\mathrm{ad}\,V = 0$ if and only if V is central (of the form $\nabla^s_0 + f$, $f \in \widehat{\mathcal{O}}$), and any derivation $D \in \mathrm{Der}\,\widehat{W}$ has a unique representative $D = \mathrm{ad}\,(\nabla^s_V + f)$ with $f(x,0) = 0$.

5.5 Fedosov Connections

Let $\widehat{\Omega}_\Sigma$ (resp $\widehat{\Omega}_X$) denote the exterior algebra of differential forms on Σ (resp. X) with coefficients in $\widehat{\mathcal{O}}$ (resp $\mathcal{O}((h))$).

As for derivations we define connections with coefficients in $\widehat{W}$: such a connection is of the form $\nabla = \nabla^s + \gamma$ with γ a 1-form with coefficients in $\widehat{W}$. It acts on forms with coefficients in $\widehat{W}$ by

$$\mathrm{ad}\,\nabla f = \nabla^s(f) + [\gamma, f]$$

The curvature is $R = \nabla^2$, identified with a 2-form with coefficients in $\widehat{W}$. We have $(\mathrm{ad}\,\nabla)^2 = \mathrm{ad}\,R$ so $\mathrm{ad}\,\nabla$ is flat (integrable) if and only if the curvature R is central, i.e. a 2-form with coefficients in $\widehat{\mathcal{O}}$ (independent of ξ). We will then say, as Fedosov, that ∇ is abelian.

Definition 5.6 *1) Let (x_j) be local coordinates on Σ, (ξ_j) the corresponding tangent coordinates. We denote $\xi_j^\vee$ the dual linear fiber coordinates on T_Σ, such that $[\xi_i^\vee, \xi_j] = \delta_{ij}$.*

2) We denote τ the 1-form with coefficients in $\widehat{W}$, dual to the canonical form of Σ with coefficients in $T\Sigma$ (resp. TX):

$$\tau = \sum dx_j \xi_j^\vee \quad \textit{(resp.} \quad \tau = \frac{1}{h}\sum dx_j\xi_j^\vee) \tag{5.11}$$

τ is the unique 1-form with linear coefficients (in ξ) such that for any differential form f with coefficients in $\widehat{W}$:

$$[\tau, f] = \sum dx_j \frac{\partial f}{\partial \xi_j} \tag{5.12}$$

We have $p(\tau) = -1, w(\tau) = -\frac{1}{2}$ (τ is homogeneous of degree 1 and its coefficients vanish of order 1 on the zero section). Also $\tau^2 = \frac{1}{2}[\tau, \tau]$ is the symplectic form:

$$\tau^2 = \omega_\Sigma \quad (\text{resp. } h^{-1}\omega_X) \tag{5.13}$$

Definition 5.7 *A Fedosov connection is a connection ∇ with coefficients in $\widehat{W}$ of the form*

$$\nabla = \nabla^s - \tau + \gamma \quad \textit{with} \quad w(\gamma) \geq 0.$$

Theorem 5.8 *Let $\mathcal{A}$ be a Toeplitz algebra on Σ (resp. $\mathcal{B}$ a semi-classical algebra on X). Then*
1) For any good embedding $u : \mathcal{A} \to \widehat{W}$ (resp. ...) there exists a unique abelian Fedosov connection ∇ such that $\nabla u = 0$. This is unique up to a central form.
2) Conversely if ∇ is an abelian Fedosov connection, $\ker \nabla \subset \widehat{W}$ *is a Toeplitz (resp. semi-classical) algebra.*

1) Locally, let us choose symplectic coordinates (e.g. homogeneous of degree $\frac{1}{2}$). The standard local embedding $u_0 : f(x) \mapsto u_0 f = f(x+\xi)$ is killed by $\nabla^0 = d_x - \tau$, and obviously this is the only Fedosov connection which kills it, with coefficients vanishing on the zero-section $\{\xi = 0\}$. If u is another good embedding, it is conjugate to u_0: $u = \operatorname{Ad} U^{-1} u_0$ so it is killed by the Fedosov connection $\nabla = \nabla_0 + \gamma$ with $\gamma = U^{-1}\nabla_0(U)$; we have $w(\gamma) > 0$ since $w(U - \mathrm{Id}\,) > 0$. Note that γ may not vanish for $\xi = 0$, but we can replace it by $\gamma - \gamma(x, 0)$.

2) The converse is immediate: if ∇ is an abelian Fedosov connection, the map $\tilde{f} \in \ker \operatorname{ad} \nabla \mapsto \tilde{f}(x, 0)$ is clearly one to one (for each $f \in \widehat{\mathcal{O}}$ there exists a unique $\tilde{f} \in \widehat{W}$ such $\nabla \tilde{f} = 0, \tilde{f}(x, 0) = f(x)$ because ∇ is "transversal" to the zero-section; further since ∇ is a Fedosov connection, we have locally $w(\tilde{f} - f(x+\xi) \geq w(f+1))$.

(The proofs for the semi-classical case are the same).

5.6 Fedosov curvature

Fedosov connections provide a one to one correspondence between Toeplitz or semi-classical algebras and 2-forms on Σ (resp. X):

Theorem 5.9 (Fedosov) *1) Any closed 2-form $R = \omega_\Sigma + r$ on Σ with coefficients in $\mathcal{O}$ (resp. $\omega_X + r$)..) is the curvature of an abelian Fedosov connection.*
2) Two abelian connections have the same curvature if and only if they are conjugate. They define isomorphic algebras if and only if their curvatures differ by an exact form.

Proof : 1) For the sake of completeness, we repeat the proof of Fedosov's, by successive approximations, improving the weight w of the error: if ∇ is a Fedosov connection, the leading term (term of lowest weight w) of $\operatorname{ad}\nabla$ is

$$\operatorname{ad}\tau : f \mapsto \sum dx_j \frac{\partial f}{\partial \xi_j} \quad (\text{of weight } -\frac{1}{2})$$

If $\nabla^2 = R + \rho_N$ with $w(\rho_N) = \frac{N}{2} \geq 0$, one looks for a 1-form γ_{N+1} such that $w(\nabla + \gamma_{N+1})^2 - R) \geq \frac{N+1}{2}$, i.e.

$$[\tau\, ,\, \gamma_{N+1}] = \rho_N + (w \geq N) \tag{5.14}$$

Let I_ξ denote the interior product: $I_\xi f = \xi_j \partial_{x_j} \lrcorner f$ (the usual product, not the star-product of $\widehat{W}$). Then if $f = \sum dx_{i_1} ... dx_{ik} f_\alpha(x)\, \xi^\alpha$ we have

$$I_\xi \mathrm{ad}\, \tau + \mathrm{ad}\, \tau I_\xi f = L_\xi f = \sum (|\alpha| + k) dx_{i_1} ... dx_{ik} f_\alpha\, \xi^\alpha$$

Since we have $w([\tau, \rho_N]) > \frac{N-1}{2}$, equation 5.14 has a solution (in fact it has a unique solution of pure weight $\frac{N+1}{2}$ killed by I_ξ).

By successive approximations we get an exact solution - in fact there is a unique and canonical solution γ such that $I_\xi \gamma = 0$ (which implies $[I_\xi, \nabla] = 0$).

Remark 5.10 The denomination "canonical" is abusive, because it depends in fact on the choice of a torsionless homogeneous symplectic connection ∇^s on Σ to begin with. In ambiguous cases we will refer to "the canonical Fedosov connection associated with ∇^s".

2) Obviously if $\nabla_1 = U \nabla U^{-1}$ and $R = \nabla^2$ is central, we have $\nabla_1^2 = URU^{-1} = R$. The converse (i.e. the existence of U such that $\nabla_1 = U \nabla U^{-1}$ if ∇ and ∇_1 have the same central curvature) is proved by successive approximation as above.

Note that $\ker \mathrm{ad}\, \nabla = \ker \mathrm{ad}\, \nabla_1$ means that $\nabla_1 = \nabla + \gamma$ with γ central, $w(\gamma) \geq 0$; this implies $R_1 = R + d\gamma$, hence the last assertion of 2).

Remark 5.11 The canonical solution is in fact of weight $p(\mathrm{ad}\, \nabla) \geq 0$ ($p(\nabla) \geq -1$), so that the corresponding embedding u is associated to a geometrical map $\underline{u} : T\Sigma \to \Sigma$, preserving Poisson brackets and tangent to the map $(x, \xi) \mapsto x + \xi$.

Remark 5.12 If $\mathcal{A}$ is a Toeplitz or semi-classical algebra, an embedding u defines a total symbol $\sigma_U(f) = uf(x, 0)$ which is an isomorphism $\mathcal{A} \to \widehat{\mathcal{O}}$ (locally in $\widehat{\mathcal{D}}_0^\times$), and a star-product B_u ($\sigma_u(f *_{\scriptscriptstyle A} g) = B_u(\sigma_u f, \sigma_u g)$). If we replace u by Uu, U an automorphism of $\widehat{W}$, σ_u is replaced by $P_{U,u} \sigma_u$ for some well defined asymptotic operator $P_{U,u} \in \widehat{\mathcal{D}}_0^\times$ ($P_{UV,u} = P_{U,Vu} P_{V,u}$), and B_u is replaced by $P_{U,u} B_u$ ($(PB)(f,g) = P\, B(P^{-1} f, P^{-1} g)$).

The set of transition operators $P_{U,u}$ is a strict subset of $\widehat{\mathcal{D}}_0^\times$; for example we have $P_{U,u} = 1$ if U is even, i.e. $U = 1 + \sum u_\alpha(x) \xi^\alpha$, $\alpha > 0$ even). Likewise the set of all total symbols σ_u is a strict subset of the set $\widehat{\mathcal{D}}_0^\times \sigma_u$ of all possible total symbols equivalent to σ_u, and the set of all corresponding star-products B_u is a strict subset of the set of all star-products equivalent to a given one; for instance one always has $B_u(f, g) = fg + \frac{1}{2}\{g, g\} + \cdots$. I will not describe further here this special class of star-products. In fact I do not know if it has been distinguished before.

5.7 Base-point

Fedosov's construction gives a canonical base-point in the set of star-algebras, viz. the algebra corresponding to a connection with "trivial curvature" $R = \omega_\Sigma$ (resp. $R = h^{-1} \omega_X$).

Note that the Weyl algebra $\widehat{W}$ has a canonical involution (and a sub-principal symbol, fiberwise). The algebra $\mathcal{A}_\nabla$ is a sub-involutive algebra of $\widehat{W}$ if ∇ respects the involution. If this is the case the curvature R also respects the involution, i.e. it is odd (an odd power series $h^{-1} \omega_X + \sum h^{2k+1} \omega_k$ in the semi-classical case, $R \sim 0$ in the Toeplitz case). Conversely if R is odd, Fedosov's construction obviously yields a connection ∇ which respects the involution. Thus the base-point is involutive; in the Toeplitz case, it is the unique involutive algebra; in the semi-classical case we have seen that there are many other nontrivial involutive algebras.

For further use let us also note the following result: let G_h denote the group of symbol preserving automorphisms of $\mathbf{C}((h))$ (i.e. automorphisms U of $\mathbf{C}((h))$ with its standard commutative algebra structure, such that $w(U(h) - h) > 0$). G_h obviously acts on the set of semi-classical algebras on X (replacing h by $U(h)$).

Proposition 5.13 *If X is compact, the action of G_h on h-Alg(X) is free*

On any symplectic manifold X, a symbol-preserving isomorphism $\tilde{U}$ preserves the center, hence induces an automorphism $U \in G_h$. Recall that the local model for semi-classical algebras is the algebra $\mathcal{B}_0$ of (jets along $\{\xi = 0\}$ of) pseudodifferential operators on $\mathbf{R}^n \times \mathbf{R}$ which commute with $h^{-1} = \frac{\partial}{\partial t}$ ($X = T^*\mathbf{R}^n$, $\xi = h\frac{\partial}{\partial x}$). Clearly any $U \in G_h$ lifts to $\operatorname{Aut}\mathcal{B}_0$ (e.g. $\tilde{U}$ such that $\tilde{U}(x) = x, \tilde{U}(\frac{\partial}{\partial x}) = \frac{\partial}{\partial x}$, $\tilde{U}(t)$ such that $[U(\frac{\partial}{\partial t}), \tilde{U}(t)] = 1$).

The proposition states that if X is compact, an isomorphism which preserves symbols, but not necessarily h, in fact fixes h.

Proof : if $U \in G_h$, it acts on $\widehat{W}_h$. If R is the Fedosov curvature of a semi-classical algebra $\mathcal{B}$, the curvature of $U(\mathcal{B})$ is $U(R)$. Now the leading term of R is $h^{-1}\omega_X$. If $U \neq 0$ the leading term of $U - 1$ is $c\,h^k\frac{\partial}{\partial h}$ for some integer $k \geq 2$ and constant $c \neq 0$, so that the leading term of $U(R) - R$ is $-c\,h^{k-2}\omega_X$; this is $\neq 0$ if X is compact (ω_X is not a coboundary).

Therefore if $U \in G_h$ and $U\mathcal{B}$ is isomorphic to $\mathcal{B}$, then $U(h) = h$ i.e. $U = \mathrm{Id}$. Equivalently if $\tilde{U}$ is a symbol-preserving isomorphism of $\mathcal{B}$, it fixes the center ($U(h) = h$).

6 Related Toeplitz and Semi-classical Algebras

6.1 Homomorphisms between Star Algebras

If $\mathcal{A}, \mathcal{A}'$ are two star-algebras over cones Σ, Σ' there is an obvious notion of homomorphism $U : \mathcal{A} \to \mathcal{A}'$ (preserving the filtrations); the symbol map is $f \mapsto u^*f = f \circ u$ where u is a smooth homogeneous map $\Sigma' \to \Sigma$ preserving the Poisson brackets, i.e. $u^*\{f, g\} = \{u^*f, u^*g\}$.

For instance the algebra of semi-classical pseudo-differential $P(x, h\partial, h)$ on a manifold V is isomorphic to the algebra of germs of pseudo-differential operators $P(x, t, \partial_x, \partial_t)$ on $X \times \mathbf{R}$ which commute with ∂_t (i.e. do not depend on t), near the line-bundle $\{\xi = 0\}$: the map takes h to ∂_t^{-1}. It is immediate that any "1-codimensional" embedding of a semi-classical algebra to a Toeplitz is locally isomorphic to the embedding above.

Using this example it is easy to embed a semi-classical algebra $\mathcal{A}_X$ over a symplectic manifold X in a Toeplitz algebra $\mathcal{A}_\Sigma$ where Σ is a disjoint union of pieces as above. However without more information on the projection $B\Sigma \to X$ this only gives very poor information: to reconstruct $\mathcal{A}_X$ from $\mathcal{A}_\Sigma$ one would need to know how various components patch together, which requires further non trivial information.

Definition 6.1 *We will say that a semi-classical algebra $\mathcal{B}$ on X is "related" to $\mathcal{A}$ if there exists an injective homomorphism $\mathcal{B} \to \mathcal{A}$, where the corresponding projection $Y \to X$ is a principal circle bundle (this definition will be slightly refined below).*

A typical example is as in the introduction: Σ is a holomorphic line bundle over a projective manifold X, Y is the unit sphere bundle for a strictly pseudo-convex hermitian norm a.[1]

[1] the definition of homotheties must be modified to make $\omega_\Sigma = \frac{i}{2}\partial\overline{\partial}a$ homogeneous of degree 1, i.e. $t.z = \sqrt{t}z$ for $t > 0$.

Let $u : \mathcal{B} \to \mathcal{A}$ be a relating homomorphism, $\overline{u} : U \to X$ (or $\Sigma \to \Gamma = X \times \mathbf{R}_+$) the corresponding map. The symbol of the operator $A \in \mathcal{A}$ corresponding to h^{-1} is $a = h^{-1}$. The integral curves of the hamiltonian field H_a are the fibers, which are closed circles, and since a is homogeneous of degree 1 its period is locally constant, constant if X is connected.

Proposition 6.2 *If $\mathcal{A}$ and $\mathcal{B}$ are related, the symplectic cone Σ is equipped with a homogeneous symplectic action of the circle group $U(1)$.*

Replacing h by $h\times$constant, we may as well suppose (and will do so in the rest of this paragraph) that the infinitesimal generator is H_a, with period 2π.

We determine below which pairs $\mathcal{A}, \mathcal{B}$ of Toeplitz and semi-classical algebras can thus be related, but first investigate how the circle group or more generally a compact group acts on a Toeplitz (or semi-classical) algebra.

6.2 Action of a Compact Group

Let $\mathcal{A}$ be a Toeplitz algebra on Σ (resp. an h-algebra on X) as above.

Theorem 6.3 *Let G be a compact group acting on Σ by symplectic homogeneous isomorphisms. We suppose $g^*\mathcal{A} \sim \mathcal{A}$ for all $g \in G$, i.e. $g^*R_A - R_A$ is exact, with R_A the Fedosov curvature of $\mathcal{A}$ (this is always true if G is connected). Then*
1. *the action of G lifts to $\mathcal{A}$.*
2. *Any two liftings are conjugate through an automorphism of $\mathcal{A}$.*

This follows from the fact that $\operatorname{Aut}\mathcal{A}$ is a complete filtered group, and $\operatorname{gr}\mathcal{A}$ is a G-vector space, so since G is compact any continuous cocycle is a coboundary. The first assertion is also seen using Fedosov connections:

1. If G acts on Σ (resp. X), its action extends functorially to $\widehat{W}$ resp. $\widehat{W}_h$. By hypothesis the cohomology class of $\mathcal{A}$ is invariant, so it has an invariant representative R. Then Fedosov's construction yields an invariant connection ∇ (we first choose ∇^s invariant so the starting point $\nabla^s - \tau$ is invariant), and G acts on $\ker\nabla \sim \mathcal{A}$.

2. Likewise if $U_0(g)$, denoted below $f \mapsto gf \in \mathcal{A}$, and $U_1(g) = U_g$ are two liftings, we set

$$u_g = U_g\, g^{-1} \in \operatorname{Aut}\mathcal{A} \quad \text{so that we have} \quad u_{gh} = u_g\, g u_h g^{-1}$$

This means that $\sigma(u_g)$ is a 1-G-cocycle with coefficients in $\operatorname{gr}\operatorname{Aut}\mathcal{A}$ (i.e. we have $\sigma(g \cdot u_h) - \sigma(u_{gh}) + \sigma(u_g) = 0$), so it is of the form $\sigma(u_g) = g.\sigma(v) - \sigma(v)$ (the cohomology vanishes in positive degree since G is compact). By successive approximations we get $v \in \operatorname{Aut}\mathcal{A}$ such that $u_g = v -^{-1} gvg^{-1}$ i.e. $U_g = v^{-1}gV$.

6.3 Circle Action

We suppose now that G is the circle group $G = U(1)$. The action of G on Σ has an infinitesimal generator θ, which is a symplectic vector field homogeneous of degree 0, and a generating function a which is a homogeneous function of degree 1 :

$$\theta = H_a, \quad \text{with} \quad a = I_\theta(\lambda_\Sigma) \tag{6.1}$$

where $I_\theta = \sum \theta_j L_{\frac{\partial}{\partial x_j}}$ denotes the interior product by θ, λ_Σ the Liouville form of Σ.

We will also denote

$$L_\theta \quad \text{the vector field with coefficients in } \widehat{W} \text{ lifting } \theta \tag{6.2}$$

Let $\mathcal{A}$ be a Toeplitz algebra on Σ. As noted above (theorem 6.3) the circle group action lifts to $\mathcal{A}$. The infinitesimal generator D of this action is well defined; it is unique up to conjugation by an automorphism of $\mathcal{A}$,[2] and there exists an equivariant embedding $u : \mathcal{A} \to \widehat{W}$, e.g. we choose an invariant representative of the curvature R: the corresponding canonical Fedosov connection ∇ is then invariant:

$$[L_\theta, \nabla] = 0 \tag{6.3}$$

We denote $\widetilde{\mathcal{A}} \subset \widehat{W}$ the image of $\mathcal{A}$, and set

$$\nabla_\theta = [I_\theta, \nabla], \qquad R_\theta = [I_\theta, R] \tag{6.4}$$

here and everywhere else $[,]$ is the superbracket: $[f,g] = (-1)^{\overline{f}\overline{g}}(f*g - g*f)$, where $\overline{f}, \overline{g}$ denotes the degree of f resp. g as differential forms. ∇_θ is a vector field with coefficients in $\widehat{W}$, and we have

$$L_\theta = \nabla_\theta + \alpha \quad (\alpha \in \widehat{W}) \tag{6.5}$$

We have $\alpha(x,0) = 0$ if ∇ "vanishes" on the zero section, in particular if ∇ is the canonical connection associated to R.

We have $R_\theta = [I_\theta, \nabla^2] = [[I_\theta, \nabla], \nabla]$. Since $[I_\theta, \nabla] = \nabla_\theta = L_\theta - b$ and ∇ is invariant, i.e. $[L_\theta, \nabla] = 0$, we have

$$R_\theta = [L_\theta - \alpha, \nabla] = \nabla(\alpha) \tag{6.6}$$

Lemma 6.4 *Notations being as above, the infinitesimal generator D of the action of $U(1)$ on $\mathcal{A}$ is an inner derivation if and only if R_θ is exact (as a form on Σ with coefficients in $\widehat{\mathcal{O}}$).*

Proof : $\operatorname{ad} L_\theta$ coincides with $\operatorname{ad}\alpha$ on $\widetilde{\mathcal{A}}$. In other words the $\widehat{\mathcal{O}}$-derivation of $\widehat{W}$ corresponding to D is $\operatorname{ad}\alpha$. If D is an inner derivation there exists $b \in \widetilde{\mathcal{A}}$ ($\nabla b = 0$) such that $\alpha - b$ is central, so $R_\theta = \nabla\alpha = d(\alpha - b)$ is exact.

6.4 Elliptic Circle Action

Definition 6.5 *An action on Σ of the circle group $U(1)$ is elliptic if its generating function a is > 0 (there is an obviously symmetric case $a < 0$).*

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[2]Here is an alternate proof of this: we first choose a derivation D_0 of degree 0 in $\mathcal{A}$ such that $\sigma(D_0) = \partial_\theta$, for instance $D_0 = \operatorname{ad} A^0$ where $A^0 \in \mathcal{A}$ is any element with symbol a. Then for $t \in \mathbf{R}$, $\exp tD_0$ is a well defined group of isomorphisms above $\exp t\theta$. In particular $e^{2\pi D_0}$ is an automorphism of $\mathcal{A}$ (over $\exp 2\pi\theta = \mathrm{Id}$): it is of the form $\exp 2\pi\delta$ with δ a derivation of degree -1 which commutes with D_0. Then $D = D_0 - \delta$ is the infinitesimal generator of an action of $U(1)$, lifting the action on Σ ($U_t = \exp tD$).

If $D_1 = D + \delta_1$ with δ_1 a derivation of degree -1), we see by successive approximations that D_1 is conjugate to $D + \delta_2$ where δ_2 commutes with D (if $U = 1 + u$ with u of degree $-N$, $U^{-1}D_1U = [D,u] + v$ with v of degree $-N-1$, and if the Fourier series of u is $u = \sum u_k$, we have $[D,u] = \sum iku_k$). Then if $\exp 2\pi D_1 = \mathrm{Id}$ we have $\exp 2\pi\delta_2 = \mathrm{Id}$ i.e. $\delta/2 = 0$ since δ_2 is of degree < 0.

From now on we suppose that Σ is equipped with a free elliptic action of $U(1)$, with generating function $a > 0$ ($\theta = H_a, a \in \mathcal{O}_\Sigma(1)$). We will use a as privileged radial coordinate.

The basis of Σ is identified with the"unit sphere"

$$Y = \{a = 1\} \subset \Sigma \sim \Sigma/\mathbf{R}_+^\times \tag{6.7}$$

Y is a principal $U(1)$-bundle, with basis X and connection form λ_Y:

$$X = Y/U(1) \ , \quad \lambda_Y = \lambda_\Sigma | Y \tag{6.8}$$

where as above λ_Σ denotes the Liouville form λ_Σ of Σ; we will also denote $\lambda_Y = \frac{\lambda_\Sigma}{a}$ the pull-back to Σ.

The basis $X = Y/U(1)$ is a symplectic manifold with symplectic form

$$\omega_X \quad \text{such that} \quad p^*(\omega_X) = d\lambda_Y \tag{6.9}$$

The $U(1)$ action lifts canonically to $\widehat{W}_\Sigma$, and as above L_θ denotes the vector field with coefficients in $\widehat{W}_\Sigma$ defined by the infinitesimal generator θ.

Let $\Gamma = \Sigma/U(1)$. We have $T\Gamma = T\Sigma/TU(1)$, where the tangent generator is (obviously) the $\sum \theta_j \frac{\partial}{\partial \xi_j}$; it corresponds to a vector field τ_θ with coefficients in $\widehat{W}$:

$$\tau_\theta = I_\theta \cdot \tau \tag{6.10}$$

where $I_\theta = \sum \theta_j I(\frac{\partial}{\partial x_j})$ is the extension to $\Omega \otimes \widehat{W}$ of the interior product by θ, and τ is the canonical 1-form of $\widehat{W}$ (def. 5.6).

Lemma 6.6 *1) $\widehat{W}_\Gamma$ is identified with the subalgebra of $\widehat{W}_\Sigma$ ov sections invariant by $TU(1)$, i.e. killed both by L_θ and* $\operatorname{ad}\tau_\theta$.
2) The Weyl algebra $\widehat{W}_h(X)$ is identified with the quotient $\widehat{W}_\Gamma \,/\, (\tau_\theta = 0)$

(if $U(1)$ acts by translations $(x_1, \ldots, x_n) \mapsto (x_1 + u, x_2, \ldots, x_n)$, which is always true in a suitable set of local coordinates, then $TU(1)$ acts by $(x, \xi) \mapsto (x_1 + u, \ldots, x_n, \xi_1 + v, \ldots, \xi_n)$).

Let now $\mathcal{A}$ be a Toeplitz algebra on Σ, equipped with an extension of the action of $U(1)$ (we have seen this exists, and is unique up to conjugation).

Up to equivalence, $\mathcal{A}$ is determined by the cohomology class of its Fedosov curvature R. We may choose R invariant, so as the Fedosov connection (if we start with an invariant torsionless symplectic connection, the canonical construction of §4.6 produces an invariant connection). There is a corresponding equivariant embedding, which we will adjust further suitably below.

The curvature is $R = \omega_\Sigma + r$, with r closed, of weight $w(r) \geq 0$. As any closed 2-form on Σ, r is cohomologous to an invariant form homogeneous of degree 0, so we may suppose:

$$r = \mu_X + \lambda_Y \nu_X + (\gamma_X + c\lambda_Y)\frac{da}{a} \tag{6.11}$$

where μ_X, ν_X, γ_X are the pull-backs of 2 and 1-forms on X.

Since $dr = 0$, c is a constant, and $d\gamma_X + c\omega_X = 0$, $d\nu_X = 0$, $d\mu_X + \omega_X \nu_X = 0$.

We necessarily have $c = 0$ if ω_Σ is not exact (on any component of X); this is always the case if X is compact.

Since $I_\theta \omega_\Sigma = -da$ we have, with the notations above:

$$R_\theta = I_\theta R = -da + \nu_X + c\frac{da}{a} \tag{6.12}$$

We have set $L_\theta = \nabla_\theta + \alpha$. Since $\nabla_\theta = [I_\theta, \nabla$ and $[L\theta, \nabla] = 0$ we get $R_\theta = [I_\theta, \nabla^2] = [\nabla_\theta, \nabla] = -[\alpha, \nabla$, i.e.

$$R_\theta = \nabla(\alpha) \tag{6.13}$$

Le us further note that the leading term of α is $\tau_\theta = I_\theta(\tau)$.

Lemma 6.7 *There exists an invertible section U of $\widehat{W}$ such that $U\alpha U^{-1} = \tau_\theta$.*

This is immediate by successive approximations as above: if $w(\alpha_n) \geq \frac{n}{2}$ there exists u such that $w(u) \geq \frac{n+1}{2}$, $[\tau_\theta, u] = \sum \theta_j \frac{\partial}{\partial \xi_j} = \alpha_n$ (there is a unique solution such that $u(,\xi) = 0$ if ξ belongs to the hyperplane orthogonal to the Liouville form λ, which is transversal to $\operatorname{ad}\tau_\theta$). Then $(1+u)^{-1}\tau_\theta(1+u) = \tau_\theta + \alpha_n + (w \geq \frac{n+1}{2})$.

Let $\mathcal{B} = \mathcal{A}^{U(1)} \subset \mathcal{A}$ be the invariant subalgebra. The image $\widetilde{\mathcal{B}} \subset \widetilde{\mathcal{A}}$ is

$$\widetilde{\mathcal{B}} = \ker L_\theta \cap \widetilde{\mathcal{A}} = \ker \nabla \cap \ker \operatorname{ad} \alpha \tag{6.14}$$

It is also contained in $\widehat{W}_\Gamma$ if $\alpha = \tau_\theta$.

Lemma 6.8 *$\mathcal{B}$ possesses a non-trivial central element iff the component ν_X of the Fedosov curvature of $\mathcal{A}$ vanishes. Then $\mathcal{B}$ possesses a structure of related semi-classical algebra.*

Proof : Let $B \neq 0$ be a central element of $\mathcal{B}$, of degree $k \neq 0$. The symbol of B is $c\,a^k$ for some constant $c \neq 0$, and replacing B by $(\frac{B}{c})^{\frac{1}{k}} \in \mathcal{B}$ we may suppose $\sigma(B) = a = h^{-1}$. Then the infinitesimal generator is necessarily of the form

$$D = \operatorname{ad}(B'), \quad \text{with} \quad B' = B + c \operatorname{Log} B + \sum_1^\infty c_k B^{-k}$$

With the notations above, $\alpha' = \alpha - \tilde{B}'$ is central, and $T_\theta = \nabla\alpha = d\alpha'$. The only term of B' not in $\mathcal{B}$ is $c\text{Log}\,B$ with differential $c\frac{da}{a}$+exact, so $R_\theta - c\frac{da}{a}$ is exact, i.e. $\nu_X \sim 0$ and c is the coefficient of $\frac{da}{a}$ which appears in (6.11),(6.12).

6.5 Related Toeplitz and Semi-classical Algebras

We may now finish the analysis of related Toeplitz and semi-classical algebras. We will suppose the basis X compact (or more generally that ω_X is not exact, on each component of X). We leave as an exercise the case where ω_X is exact so the constant c above can be $\neq 0$.

We refine definition 6.1 as follows:

Definition 6.9 *We will say that a relating homomorphism $\mathcal{B} \to \mathcal{A}$ is good (or that we have a good relation) if the operator A corresponding to h^{-1} is an infinitesimal generator of $U(1)$*

Theorem 6.10 *We suppose X compact. 1) Let $\mathcal{A}$ be a Toeplitz algebra over Σ with elliptic action of $U(1)$ as above, and Fedosov curvature R_A, and let $\mathcal{B}$ be the invariant subalgebra. Then $\mathcal{B}$ possesses a structure of semi-classical algebra if and only if $\nu_X = I_\theta R \sim 0$. In this case the infinitesimal generator of the action of $U(1)$ on $\mathcal{A}$ is an inner derivation* $\operatorname{ad} A$ *and there exists a unique semiclassical structure on $\mathcal{B}$ such that $h = f(A) = A^{-1} + \ldots$ for any formal series $f(T) = T^{-1} + \sum_{k>1} f_k T^{-k}$. It is well-related to $\mathcal{A}$ iff $h^{-1} = A + c$, $c \in \mathbf{C}$ a constant.*

2) Let $\mathcal{B}$ be a semiclassical algebra over $X = Y/U(1)$ as above. Then $\mathcal{B}$ has a related Toeplitz algebra if and only if its Fedosov curvature R_B is constant mod. ω_X, i.e. of the form $R_B = h^{-1}\omega_X + constant + \varphi(h)\omega_X$. It is well-related $\mathcal{A}$ iff $R_B = h^{-1}\omega_X + \mu$, μ a constant 2-form on X (independent of h).

The Fedosov curvature of $\mathcal{A}$ is then $R_A = p^*R_B + \frac{da}{a}p^*\gamma_X$ for some $\gamma_X \in H^1(X)$. In particular there is a unique (up to isomorphism) "non exotic" such $\mathcal{A}$ ($\gamma_X = 0$).

Proof : Let $\mathcal{A}$ be a Toeplitz algebra on Σ, and choose R, ∇ as above, so $L_\theta = \nabla_t h + \alpha, \alpha = \tau_\theta$. Since X is compact we have $c = 0$ anyway. If $\mathcal{A}$ contains a semi-classical algebra, we have $\nu_X \sim 0$ so we may as well suppose $\nu_X = 0, R = \omega_\Sigma + \mu_X + \gamma_X \frac{da}{a}$.

This being so we have $R_\theta = \nabla\alpha = -da$ so $\nabla(a + \alpha) = 0$. The infinitesimal generator is $\operatorname{ad} A$ where A is the element with total symbol a ($\tilde{A} = a + \alpha$). Moreover with this choice $\tilde{A}$ is a polynomial of degree 1 with respect to ξ so $\tilde{A} * f = \tilde{A}f + \frac{1}{2}\{\tilde{A}, f\}$. It follows in particular that $\tilde{A} * f = \tilde{A}f$ (or $a * f = af$ in $\mathcal{A}$ for the star product and total symbol defined by this choice of embedding) if A commutes with f, and we get a well related semiclassical algebra by setting $h = A^{-1}$.

If V is a vector field on X, we denote $\tilde{V}$ the unique vector field on Σ which projects on V, such that $I_\theta\tilde{V} = I_\rho\tilde{V} = 0$, with ρ the generator of homotheties (i.e. $\tilde{V}$ is orthogonal to the conic rays and fiber circles and projects on V - in particular it is homogeneous of degree 0 and rotation invariant). With our choice of embedding ($\tilde{D} = \operatorname{ad}\alpha, \alpha = \tau_\theta$), $\nabla_{\tilde{V}}$ obviously preserves $\Omega_\Gamma = \ker L_\theta \cap \ker \operatorname{ad}\alpha$, and kills a and α so goes down to Ω_h. The Fedosov connection ∇^B of $\mathcal{B}$ is then obtained by restriction:

$$\nabla^B_V = \text{ the derivation of } \Omega_h \text{ defined by } \nabla_{\tilde{V}}|_{\Omega_h} \tag{6.15}$$

The curvature R^B is the induced by R: $R^B = \mu_X$. Note that it does not depend on γ_X, and changing γ_X, which is equivalent to twisting $\mathcal{A}$ by a cocycle $A^{s_{ij}}$, (s_{ij}) a cocycle on X with coefficients in $\mathbf{C}$, does not change $\mathcal{B}$, the commutator of A. Among the Toeplitz algebras related to $\mathcal{B}$ the only one which is not "exotic" has curvature μ_X.

Any other related semi-classical structure is obtained by replacing h by $\varphi(h)$ for some formal series $\varphi(h) = h + \sum_2^\infty \varphi_k h^k$; as seen in Prop.5.13, the corresponding semi-classical algebras are pairwise non-isomorphic. Among them, are well-related to $\mathcal{A}$ those for which $\varphi(h) = \frac{h}{1+\gamma h}$ ($\varphi(h)^{-1} = h^{-1} + \gamma$).

Remark 6.11 If ν_X is not cohomologous to 0, the invariant subalgebra $\widetilde{\mathcal{B}}$ is not a semiclassical algebra, because its center is not big enough. However its lifting to the universal cover $\widetilde{X}$ is; $\widetilde{\mathcal{B}}$ itself is obtained by gluing together local models of semi-classical algebras with isomorphisms which preserve symbols, but do not fix h ($h \mapsto U(h) = h + \sum_{k\geq 2} u_k h^k$).

Example 6.12 In the canonical model e have $SS = \mathbf{C}^n - 0$, $a = |z|^2$. The symplectic form is $\omega_\Sigma = i\partial\overline{\partial}a = i\sum dz_j d\overline{z}j$ (twice the usual one - the coordinates $z_j, \overline{z}_j$ are homogeneous of degree $\frac{1}{2}$ so a is of degree 1); we have $\{\overline{z}_p, z_q\} = i\delta_{pq}$ (the Poisson bracket is $c = i\sum \partial_{\overline{z}_j} \wedge \partial_{z_j}$.

The infinitesimal generator of homotheties is $\rho = \frac{1}{2}\sum z_j \frac{\partial}{\partial \bar z_j} + z_j \frac{\partial}{\partial \bar z_j}$. Rotations are the usual ones: $z \mapsto e^{it}z$, ith infinitesimal generator $\theta = i\sum z_j \frac{\partial}{\partial \bar z_j} - z_j \frac{\partial}{\partial \bar z_j} = H_a$.

$\mathcal{A}$ is the Weyl algebra, with product $f * g = \exp \frac{1}{2}c(\partial_x, \partial_y)\, f(x)g(y)|_{y=x}$. The action of $U(1)$ lifts to $\mathcal{A}$, with generator $\operatorname{ad} a$. The canonical embedding $\mathcal{A} \to \widehat{W}$ is $f \mapsto f(x+\xi)$, the corresponding connection is $\nabla = d - \tau$, $\tau = i(dz_j\overline{\zeta}_j - d\bar z_j \zeta_j)$.

The invariant subalgebra $\mathcal{B}$ is a semiclassical algebra over $X = P_{n-1}(\mathbf{C})$, ith $h = a^{-1}$.

Note that in this case, with the notations above, we have $a + \alpha = |z+\zeta|^2 \neq \tau_\theta$ (we still have to modify ∇ to get the connection used above).

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