Formal Norms and Star-Exponentials

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Abstract. We recall in a more natural manner the description of the formal norm for pseudo-differential operators introduced by us in 1967 and show how this can be used to simplify a recent construction of star-exponentials.

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1. Introduction

In [2] G. Dito and P. Schapira constructed an algebra of germs of infinite order analytic "$\hbar$-pseudo-differential operators" depending on a parameter t, containing the exponentials $\exp \frac{i}{\hbar} tP$. Here we show how the formal norm of pseudo-differential operators (ψDOs) of [1] can be used to give a short explanation of their construction. The formal norm was introduced in [1] and was used to define and handle analytic pseudo-differential operators, whose theory was considerably expanded in [3]. It can be useful in many other problems concerning analytic pseudo-differential operators. I have repeated its description here, in a simplified and more natural manner; the application to star-exponentials is made in §3.

2. Formal norms

2.1. $\hbar$-PSEUDO-DIFFERENTIAL OPERATORS

We first recall the formal pseudo-differential calculus that is used. The problem we are dealing with is essentially local, and we only do this on $\mathbb{R}^n$; we will not be concerned here with global problems, or the problem of switching between equivalent star-products: for ψDOs these are well understood and do not usually involve recursive computations for which the formal norm is useful.

Let U be an open set of $T^*\mathbb{R}^n \simeq \mathbb{R}^{2n}$, resp. of $T^*\mathbb{C}^n \simeq \mathbb{C}^{2n}$. A formal $\hbar$-ΨDO on U of degree 0 is a formal series

$$f = \sum_{j=0}^{\infty} \hbar^j f_j(x, \xi)$$

where the f_j are smooth functions on U, resp. holomorphic.

The star-product is defined, by the Leibniz formula, as[1]

$$f * g = \sum_\alpha \frac{(-i\hbar)^{-|\alpha|}}{\alpha!} \partial_\xi^\alpha f \, \partial_x^\alpha g = L_f g, \quad \text{with} \quad L_f = \sum_\alpha \frac{i^{-|\alpha|}}{\alpha!} \partial_\xi^\alpha f \, (\hbar \partial_x)^\alpha \tag{1}$$

For formal operators we introduce a first formal norm:

$$N(f) = \sum_{j,k \geq 0} f_j^k \, \frac{T^{2j+k}}{(j+k)!} \tag{2}$$

where f_j^k denotes norm of the k-th derivative of f_j as a k-linear (symmetric) form:

$$f_j^k(x) = \sup_{|y_j| \leq 1} |f_j^{(k)}(y_1, \ldots, y_k)| \quad \text{at the point } x \in U$$

(this is a the continuous function on U).

Also useful, and coherent with the notations for "ordinary" ψDOs, are formal $\hbar$-ΨDOs of degree d, of the form $h^{-d} f$ (f as above, d an integer, or any complex number). It is immediate to adapt the formal norm to these.

By definition f is "analytic" (or "convergent") if its formal norm $N(f)$ is a convergent series (this meaning that for any compact $K \subset U$ there exist $A, R > 0$ such that for $x \in K$ its sum is $\leq A$ when $0 \leq T < R$).

The following result follows immediately from standard inequalities on binomial coefficients:

THEOREM 1. *For the norm $N(f)$ we have the inequalities*

$$N(fg) \ll N(f)N(g), \; N(\hbar \partial_i f) \ll T N(f). \tag{3}$$

$\ll$ means that the coefficient of T^k in the left-hand side is $\leq$ the same in the right-hand side. In the first inequality fg denotes the ordinary (commutative) product. The first inequality would work as well for many other denominators, e.g. the product $j!k!$; the denominator $(j+k)!$ was chosen for the second.

We now define a second formal norm:

$$N'(f) = N(L_f) = \sum N(\frac{1}{\alpha!} \partial_\xi^\alpha f) \, T^{|\alpha|} \tag{4}$$

PROPOSITION 2. *We have $N(f) \ll N'(f) \ll N(f)((1+n)T)$*

[1] the i in the formula is the i of the Fourier transformation. In essentially algebraic formulas as here, it could be omitted.

The first inequality is obvious; for the second we note that we have

$$N'(f) \ll \sum_{j,k,\alpha \geq 0} f_j^{k+|\alpha|} \frac{T^{2j+k+|\alpha|}}{(j+k+|\alpha|)!} \frac{(j+k+|\alpha|)!}{(j+k)!\,\alpha!}$$

and

$$\sum_{|\alpha| \leq m} \frac{(j+m)!}{(j+m-|\alpha|)!\,\alpha!} \leq (1+n)^{j+m}.$$

Thus f is analytic iff $N(f)$ or $N'(f)$ is convergent.

THEOREM 3. *For any formal ψDOs f, g we have*

$$N(f * g) \ll N'(f) N(g) \tag{5}$$

This follows immediately from Theorem 1. N, N' are not quite multiplicative norms, but just as good, e.g. if P is a polynomial, we have $N(P(f)) \ll |P|(N'(f))$ (where $|P|$ is obtained from P by replacing the coefficients by their norms). In [1] we constructed a multiplicative norm N'', with rather elaborate coefficients, and still not perfect because it does not satisfy $N''(1) = 1$.

2.2. PSEUDO-DIFFERENTIAL OPERATORS

The formal norm was originally defined for "ordinary" analytic pseudo-differential operators. Analytic pseudo-differential operators are constructed so that they preserve locally analyticity, an analytic wave front sets. Here we just recall how formal norms works for ΨDOs on open sets of $T^*\mathbb{R}^n$, and will not expand the theory further. The corresponding exponentials are the $\exp itP$ with $\deg \mathrm{P} = 1$, representing germs of Fourier integral transformations.

Locally (on $\mathbb{R}^n$) the formal norm for "ordinary" ψDO can be reduced to the preceding one. We recall how this works (it also works the other way round[2]): let $U \subset T^*\mathbb{R}^n$ be an open conical set (i.e. $(x, \xi \in U \Rightarrow (x, \lambda\xi) \in U$ if $\lambda > 0$). A "classical" ΨDO is a formal series $\sum f_j(x, \xi)$ where for each j, f_i is a smooth function homogeneous of degree $j \to -\infty$ [3]. The composition law (star-product) is given again by the Leibniz rule (1).

To a formal ΨDO as above we may associate the $\hbar$-symbol $f_\hbar = \sum \hbar^{-j} f_j$ (this preserves the $*$-product). From there the formal norms and analyticity are defined in the same manner, and Theorem 3 holds in the same manner.

[2] there are many "natural" manners of relating $\hbar$ΨDO's and "classical" ΨDO's (see e.g.[4]). As usual the local theory is easy, the global is not. But anyway one always gets the same analytic theory.

[3] one usually supposes that U does not meet the zero section $\xi = 0$ – if it does f_j must be polynomial in ξ and $j > 0$.

3. star-exponentials

Formal series in t of the form

$$F=\sum_{m\geq 0}(\frac{t}{\hbar})^m f_m(t), \tag{6}$$

where the $f_m=\sum_{k\geq 0} f_{m,k}t^k$ are formal series in t with coefficients $\hbar$-ΨDOs of degree 0, obviously form a ring (we get a slightly larger ring by allowing a factor h^{-k} in front; in the complex case we require that the f_m be holomorphic in t).

By definition such an object F is "convergent" (or analytic) if the formal series (in T,t)

$$\widetilde{N}(F)=\sum_{m\geq 0} m!N(f_m)\,T^m \tag{7}$$

is convergent ($N(f_m)$ denotes the formal series $\sum_{k\geq 0}N(f_{m,k})t^k$)

THEOREM 4. *Convergent series as above form a ring.*

Indeed if F,G are convergent, we get

$$\widetilde{N}(F*G)=\sum_{p,q\geq 0}(p+q)!N(f_p*G_q)\,T^{p+q}\ll\sum_{p,q\geq 0}\frac{(p+q)!}{p!\,q!}(p!N'(f_p)\,T^p)(q!N(g_q)\,T^q)$$

$$\ll(\widetilde{N}'(F)\widetilde{N}(G))(2T)\qquad(\text{because }\frac{(p+q)!}{p!\,q!}\leq 2^{p+q})$$

For example if f is of degree 0, $\exp\frac{i}{\hbar}tf$ is convergent in the above sense; it represents a germ of one parameter group of FIO transformations. Translating this for "usual" ψDOs is straightforward.

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