# SYMPLECTIC CONES AND TOEPLITZ OPERATORS

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In this article I give an improved description of the construction of Toeplitz operators associated to a symplectic cone, in the sense of my book with V. Guillemin [5] (cf. also [2]). I have tried to give a simpler and more geometric presentation than that of [5], exploiting more systematically the geometry of complex lagrangian manifolds, and the symbolic calculus of Fourier integral operators relevant in this question.

## 1. Fourier Integral Operators

Let $X$ be a manifold, $\Lambda$ a smooth lagrangian cone in $T^*X - \{0\}$. Hörmander ([11], [12]) defined the class $I_\Lambda$ of Lagrangian distributions associated to $\Lambda$.

If $X, Y$ are two manifolds, a Fourier integral operator $A$ of type $Y \to X$ is an operator $A$ (acting from functions on Y to generalized functions on X) whose Schwartz kernel $K$ is a Lagrangian distribution: $K \in I_\Lambda$. It is usually more significant to use the canonical relation $C$, symmetric of $\Lambda$ by the symmetry $(x, \xi, y, \eta) \to (x, \xi, y, -\eta)$ rather than the Lagrangian cone $\Lambda$ itself. A Fourier integral distribution so as a Fourier integral operator has a symbol $\sigma_A$: this is a section of the Maslov bundle $M_\Lambda$ or $M_C$, which is a line bundle on $\Lambda$, resp. $C$. Here we will only consider canonical relations contained in $(T^*X - \{0\}) \times (T^*Y - \{0\})$; the corresponding operators $A$ map $C^\infty$ to $C^\infty$, and extend continuously to distributions so that $SS\,Af \subset C(SS\,f)$ where $SS$ denotes the microsupport.

We have the following rules of symbolic calculus: if $X, Y, Z$ are three manifolds, two canonical relations $C: \ Y \to X$ and $C': \ Z \to Y$ are said to be transversal if $C \times C'$ intersects transversally $T^*X \times diag(T^*Y) \times T^*Z$. We denote $CoC'$ the image $(pr_X \times pr_Z)\,(C \times C' \cap diag(T^*Y))$ – i.e. the set of "composed" couples $(u, w) \in T^*X \times T^*Z$, such that there exists $v \in T^*Y$ with $(u, v) \in C, \ (v, w) \in C'$: this is an immersed Lagrangian if $C$ and $C'$ are transversal, because we are dealing with canonical relations to begin with. If $A$ resp. $B$ are Fourier integral operators associated with $C$ resp. $C'$, $AoB$ is a Fourier integral operator associated with $CoC'$, and we have [1]

$$\sigma_{AoB} = \sigma_A \cdot \sigma_B$$

[1] at least if the projection map $C \times C' \cap diag \to T^*(X \times Z)$ is injective - otherwise one would expect a locally finite sum of such contributions, from each "branch" of the immersed canonical relation $CoC'$.

where $\cdot$ is the natural product for Maslov bundles [2].

If we choose a frame in the symbol bundle, so that symbols are identified with numerical functions (e.g. by choosing a phase function and local coordinates), we get a relation

$$a \cdot b\,(x, \xi, z, \zeta) = Ja(x, \xi, y, \eta)b(y, \eta, z, \zeta) \quad \text{for} \quad (x, \xi, y, \eta) \in C, (y, \eta, z, \zeta) \in C'$$

where $J$ is an invertible homogeneous function on $CoC'$

To describe the Szegö kernel and Toeplitz operators we shall need Fourier integral operators with complex phases (with positive imaginary part). Their theory was developped by A. Melin et J. Sjöstrand [13], who showed that for these operators one has essentially the same symbolic calculus, provided one reinterprets suitably the real geometrical objects of the theory as complex ones. Thus the corresponding Lagrangians or canonical relations are "complex submanifolds". For real submanifolds we dispose of a dictionnary (equivalence) between submanifolds and ideals of the ring (or sheaf) of smooth functions: the ideal $I = I(V)$ of $V$ is the sheaf of functions which vanish on $V$, while the manifold $V = V(I)$ is the set of points at which all functions of $I$ vanish - this is a smooth real manifold of codimension $d$ if $I$ is locally generated by $d$ transversal real functions (i.e. with linearly independant derivatives). For a complex submanifold $V$ only the ideal $I = I(V)$ remains; it is locally generated by transversal complex functions if $V$ is "smooth". But it is still convenient to speak of $V$ as a geometric object, as if it was defined by its points. This makes no problem when we are dealing with analytic objects so that a function, or $V$, can be interpreted as a germ of function, resp. of complex submanifold (set of points), near the set of real points in some complexification. In the $C^\infty$ setting Melin and Sjöstrand still justify this geometric language representing $V$ by an equivalence class of real submanifolds in a complexification, all tangent of infinite order, and satisfying the Cauchy-Riemann equations of infinite order along the set of real points. [3].

The Melin and Sjöstrand theory is rather subtle because it only requires that the lagrangian manifolds involved be "positive" (i.e. that they may be defined by a phase function with positive imaginary part), without further transversality or non-degenerescence conditions along the set of real points (for instance the imaginary part of the defining phase function could vanish of infinite order at some points). Here the situation is simpler, and for the Fourier integral operators we need one can use a complex phase $\phi$ whose imaginary part vanishes on a smooth submanifold, and is positive and transversally elliptic ($\geq cste\ dist^2$) along this. In this case our manifold (or its defining ideal) is completely determined by its

[2] $M_{C\times C'}$ is canonically identified with $M_C \otimes M_{C'}$ (external product), and the restriction to the diagonal of the latter is also canonically identified to $(prX \times prZ)^{-1}(M_{CoC'})$ - in fact the existence of a canonical "associative" product is somehow tautological since we are dealing with symbols of operators; what is not trivial is the "functorial" identification of the Maslov bundle with the half density bundle of $\Lambda$ or $C$ twisted by the Maslov cocycle, which we will not explicitly use here

[3] in the same way the Fourier integral operators with complex phase which we will consider are characterised by the fact that their distribution kernels, for a fixed canonical relation, form a "simple holonomic" module on the algebra of pseudodifferential operators, microlocally generated by one element $K$ satisfying $n = dim\,X + dim\,Y$ relations $P_iK = 0$, where the symbols of the $P_i$ are transversal, complex-valued

jet of infinite order (Taylor series) along its real points, and the same is true for the corresponding operators (mod. smoothing operators), so that all computations essentially reduce to computations involving formal power series (whose coefficients are smooth functions on the real part).

In the sequel we will use the geometric description above for Fourier integral operator with complex phases: it is equivalently correct, and more suggestive.

## 2. Toeplitz structures

**Toeplitz operators arising from a complex structure**

Our model and starting point will be the algebra of Toeplitz operators arising from a complex structure, whose description we now recall. Let $X$ be the boundary of a smooth, compact, stricly pseudo-convex complex domain ($X$ is a $C^\infty$, compact, strictly pseudo-convex CR-manifold).

**Szegö projector -** The Szegö projector $S$ is the the orthoprojector of $L^2(X)$ on the suspace $\mathcal{O}^0(X)$ of square integrable boundary values of holomorphic functions (if the complex dimension of the domain bounded by $X$ is $\geq 2$, i.e. $dimX \geq 3$, this is equivalently the space of $L^2$ functions killed by the system $\overline{\partial}_b$ of tangential Cauchy-Riemann equations).

It is more generally useful to use the Sobolev scale: $\mathcal{O}^s(X)$ is the space of boundary values of holomorphic functions which lie in the Sobolev space $H^s(X)$ or equivalently the space of $\mathcal{H}^s$ solutions of $\overline{\partial}_b$. It follows from [6] that $S$ is continuous $H^s(X) \to \mathcal{O}^s(X)$ for $s \geq 0$, and extends continuously likewise for $s \leq 0$.

**Toeplitz operators -** If $A$ is a pseudodifferential operator on $X$, the Toeplitz operator it defines is the operator $T_A = SAS$, acting on boundary values of holomorphic functions. We will say that $T_A$ is of degree $d$ if it is continuous $\mathcal{O}^s \to \mathcal{O}^{s-d}$: this is the case if A is of degree $d$.

The Szegö projector $S$ is a Fourier integral operator with complex phase whose canonical relation is described as follows:
- let $\Sigma$ be the microsupport of boundary values of holomorphic functions. It is the half-line subbundle of $T^*X$ consisting of positive multiples of the real differential form $\frac{1}{i}\partial u|_X$ if u is any smooth defining function for our complex domain, (ie. $u = 0$, $du \neq 0$ on $X$, $u < 0$ inside the domain); $\partial u$ denotes the holomorphic part of the differential $du$. The fact that our domain is strictly pseudoconvex means that $u$ can be chosen so that $i\partial\overline{\partial}u$ is a Kähler form ($\gg 0$); this implies that $\Sigma$ is symplectic, and that $\Sigma$ is in fact one half of the set of real characteristics of $\overline{\partial}_b$.
- we denote $Z$ the complex characteristic manifold of $\overline{\partial}_b$: its hamiltonian flow defines a (complex) fibration $p: \ Z \to \Sigma$ (the projection along the bicharacteristic leaves of $\partial_b$, which in this case are the complex manifolds "*holomorphic part = constant*").

Then (as was proved in [6]), the canonical relation of $S$ is the manifold $Z \times_{diag\Sigma} \overline{Z}$ (flowout of $\overline{\partial}_b$ ($1st$ factor)$\times\partial_b$ ($2nd$ factor) from diag $\Sigma$, which is a weak geometric translation of the fact that $S$ is a projector whose kernel $S(z, w)$ of $S$ is holomorphic in $x$ and antiholomorphic in $w$.

The characteristic manifold $Z$ is involutive and "positive" along $\Sigma$, i.e. if $\overline{z}_1, \dots, \overline{z}_{n_1}$ are transversal generators of the ideal of $Z$ ((symbols of a basis of antiholomorphic tangent vector fields), and $z_1, \dots, z_{n-1}$ are the conjugates, the matrix of Poisson brackets $\frac{1}{i}\{\overline{z}_p, z_q\}$ is $\gg 0$ along $\Sigma$ (this is equivalent to strict pseudoconvexity).

**Toeplitz operators associated to a symplectic cone, general case.**

Let $X$ be a compact manifold and $\Sigma$ a symplectic subcone in $T^*X - \{0\}$. Generalizing the preceding construction, we will call *"Toeplitz structure"*, or more precisely Toeplitz structure on $X$ associated to $\Sigma$, a subspace $H_\Sigma \subset L^2(X)$ whose orthogonal projector $P_\Sigma$ is a Fourier integral operator with complex phase so that the canonical relation has for real part the graph $Id_\Sigma$ and is $\gg 0$ along $\Sigma$ (in the sense of Melin-Sjöstrand or as above). [4]

In the appendix of [5] we proved the following result:

**Theorem 1.** . *If $X$ is a compact contact manifold, and $G$ is a compact group acting on $X$, there exists an invariant Toeplitz structure.*

The proof we give here is derived from that of [5], but as announced more "geometric", taking better advantage of the geometry and algebra of Fourier integral operators. We will construct a Toeplitz projector $S$ in $L^2(X)$, and the associated Toeplitz operators, imitating as much as possible configuration of the complex case above.

**i) construction of analogues of $Z$ and of the canonical relation $C$**

let us choose a homogeneous function $q \geq 0$, transversally elliptic along $\Sigma$ (i.e. near $\Sigma$, $q(\xi) \geq \text{cst dist}(\xi, \Sigma)^2$). The hamiltonian field $H_q$ vanishes on $\Sigma$, and the eigenvalues of the transveral component of its linearization go by pure imaginary non zero opposite pairs (because it preserves both the symplectic form and $q$). It follows that there are two complex ingoing and outgoing manifolds $Z$, $\overline{Z}$ for $\frac{1}{i}H_q$ along $\Sigma$. The "hamiltonian" definition of $Z$ and the choice of signs imply that $Z$ is involutive and "positive". [5] Its characteristic flow defines a fibration $p: Z \to \Sigma$

As in the complex case above we set $C = Z \times_{diag\,\Sigma} \overline{Z}$. Obviously we have $CoC = C^* = C$.

**ii) construction of the principal symbol of the projector $S$**

Let $I_C$ denote as above the set of Fourier integral operators with (complex) canonical relation $C$: it is an involutive algebra, since $CoC = C^* = C$, and the

[4] this definition seems to depend on the choice of a smooth density on $X$ to define the $L^2$-norm; in fact it does not and in fact it is easy to see that if the projector $P$ is as required, then the same is true for the orthogonal projector $P_A$ for any norm of the form $||f||_A^2 = < Af|f >$ with $A$ a positive elliptic pseudodifferential operator on $X$.

[5] locally we may choose complex coordinates $z_j, \overline{z_j}, \dots$ so that $\overline{z_j} = 0$ are defining equations for the outgoing manifold $Z$. We have $\frac{1}{i}H_q\overline{z} = A\overline{z}$ with $A$ a smooth matrix with negative ($< 0$) eigenvalues near $\Sigma$. It follows that on $Z$ we have $\frac{1}{i}H_q\{\overline{z}, \overline{z}\} = \lambda_2 A\{\overline{z}, \overline{z}\}$ (where $\lambda_2$ is defined by $\lambda_2 A(u \wedge v) = Au \wedge v + u \wedge Av$). Since $\frac{1}{i}Hq$ is positive on $Z$ ($\frac{1}{i}H_q z = -\overline{A}z$) this implies $\{\overline{z}, \overline{z}\} = 0$ for $\overline{z} = 0$.

symbolic calculus is given by

$$\sigma_{AB}(u) = \sigma_A(p_1 u) \cdot \sigma_B(p_2 u) \quad \text{for} \quad u \in C$$

where $p_1 : C \to Z$ comes from the projection $\overline{p} : \overline{Z} \to \Sigma$ of the second factor, and $p_2 : C \to \overline{Z}$comes from the projection $p : Z \to \Sigma$ in the first factor [6].

Similarily is $P$ is a pseudodifferential operator, and $u \in C$, we have

$$\sigma_{PA}(u) = \sigma_P(p_1 u)\sigma_A(u)$$

$$\sigma_{AP}(u) = \sigma_A(u)\sigma_P(p_2 u)$$

Finally for adjoints we have

$$\sigma_{A^*}(u) = \overline{\sigma_A(su))}$$

where $s$ denotes the hermitian symmetry exchanging the two conjugate factors of $Z \times_\Sigma \overline{Z}$.

Locally we may choose a frame in the Maslov bundle (e.g. choosing local coordinates and a defining phase function). The principal symbol of an operator $A \in I_C$ is then defined by a numerical function $a$ on $C$, and symbolic calculus takes the form:

$$aob = J\ a(pu)b(\overline{p}u)$$
$$a^* = m\overline{a(su)}$$

where $J$ and $m$ are fixed homogeneous nonvanishing functions on $C$. Associativity implies

$$J(p_1 u) = J(p_2 u) = J_0$$

where $J_0(u) = J(p_1 p_2 u)$ only depends on the projection of $u$ on $\Sigma$, and we can reduce to the case $J = 1$ replacing the symbol $a$ by $aJ_0^2 J^{-1}$ (change od frame in the Maslov bundle).

For adjoints: $m$ necessarily satisfies the relation $m(u)\overline{m}(su) = 1$. If $J = 1$ the relation $(AB)^* = B^* A^*$ implies $m = m(p_1 u)m(p_2 u)$ (in particulier $m = 1$ on $\Sigma$),and again we can reduce to the case $m = 1$ (and still $J = 1$) replacing $a$ by $am^{-1/2}$ (the square root is well defined near $\Sigma$ since $m = 1$ on $\Sigma$).

Let us notice that the symbol $\sigma_{ABC}$ only depends on the restriction $\sigma_B|\Sigma$; it is equal to $\sigma_{AC}$ if $\sigma_B = 1$ on $\Sigma$. It follows that $\sigma_{AB}$ is idempotent if $\sigma_A = \sigma_B = 1$ on $\Sigma$; it is selfadjoint if $B = A^*$. It follows immediately that there exist global selfadjoint idempotent symbols.

**Remark 1 -** If $a$ and $b$ are two idempotent symbols as above we see that the ratio $\beta = b/a$ (which is a function on $C$) satisfies

$$\beta(u) = \beta(p_1 u)\beta(p_2 u) \quad \text{(idempotence)}\ ,\ \beta(u) = \overline{\beta(su)} \quad \text{(selfadjunction)}$$

It follows that there exists a unitary elliptic pseudodifferential operator $U$ with symbol $\sigma_U$ equal to 1 on $\Sigma$, such that $\beta = \sigma_U\ a\ \sigma_{U^*}$.

[6]locally near $\Sigma$ we may choose a product structure and canonical coordinates $x = (x_j)$ (real), $z = (z_k), \overline{z}_k$ (complex) so that $Z$ is defined by $\overline{z} = 0$ and $\Sigma$ by $z = \overline{z} = 0$. Then p is the projection $(z, x, 0) \to (0, x, 0)$ in $Z$); $C$ is the set of complex points $(z, x, 0\ ,\ O, x, \overline{y})$, qu'on repère par $(z, x, \overline{y}) \in Z \times_\Sigma \overline{Z}$, $p_1$ is the projection $(z, x, \overline{y}) \to (z, x, 0)$ and $p_2$ is the projection $(z, x, \overline{y}) \to (0, x, \overline{y})$

In fact this condition means $\sigma_U(p_1u)\, a\, \sigma_{U^*}(p_2u) = b(u)$. For $u \in Z$ ($p_2u \in \Sigma$) this implies $\sigma_U(p_1u) = b(p_1u)$ since $\sigma_U = 1$ on $\Sigma$ (the converse is obvious, because $\beta$ is selfadjoint and idempotent). Thus $\sigma_U$ is completly determined on $Z$. The unitarity condition for $\sigma_U$ is $\sigma_U\ \sigma_{U^*} = 1$; over complex points this writes $\sigma_{U^*}(\overline{v}) = \overline{\sigma_U(v)}$ so $\sigma_U$ is also completly determined on $\overline{Z}$ (this is compatible with the first condition since the symbol is 1 on $\Sigma = Z \cap \overline{Z}$). We may then choose a first symbol $v$ equal to what was just determined on $Z \cup \overline{Z}$ ($v(p_1u) = \beta(p_1u)$ on $Z$, $v$ equal to the inverse of $\overline{v(su)}$ on $\overline{Z}$). The symbol $vv^*$ is then equal to 1 on $Z \cup \overline{Z}$, so its square root is well defined near $\Sigma$, and the symbol $u = v(vv^*)^{-1/2}$ is suitable (note that since this symbol, which is at first defined only near $\Sigma$, is close to 1, it can be extended globally; it is then also easy to refine U as a truly unitary operator without changing its symbol).

### iii) construction of the total symbol of the projector $S$

Let $a$ be an idempotent symbol as above and $A \in I_C$ a Fourier integral operator of degree 0 with symbol $a$. Then there exists a unique idempotent $A'$ (mod. operators of degree $-\infty$) such that $A - A'$ is of degree $\leq -1$, and commutes with $A$:

$$A' = A + F(1 - 2A)$$

where $F$ is of degree -1, commutes with $A$ and $(F^2 - F)\,(1 - 4R) - R = {A'}^2 - A' = 0$.

This equation of the second degree has for unique solution among total symbols (or operator of degree $\leq -1$ mod. those of degree $-\infty$) given by the formal power series:

$$F = \frac{1}{2}(1 - (1 - 4R)^{-1/2}) = \sum \frac{(2n-1)!}{n!(n-1)!} R^n \quad \text{of} \quad R = A - A^2 \tag{2.1}$$

**Remark 2 -** in the constructions above the notion of degree we have used is that of Fourier integral operator calculus: a Fourier integral operator of degree $m$ is continuous $H^s \to H^{s-m}$ but for Fourier integral operator with complex phase the converse is not true. Here for instance if $A \in I_C$ is of degree $m$ but its symbol vanishes on $\Sigma$, $A$ is in fact continuous $H^s \to H^{s-m+1/2}$ (i.e. of degree $\leq m - \frac{1}{2}$ in the Sobolev scale). Thus the principal symbol $\sigma_A$ in the sense of Fourier integral operators contains more information than the class of $A$ mod. $L(H^s, H^{s-degA+1})$ (this distinguishes the Fourier integral operator calculus from the calculus of Hermite operators used in [5], of which Toeplitz operators are particular cases) [7]

**Remark 3 -** construction iii) is in fact completely contained in the next one. But anyway formula (2.1) shows that everything works in a formal context, where one only seeks asymptotic expansions.

### iv) construction of the projector $S$

[7]In the symbolic calculus of Hermite operators, the symbol of the Szegö projector or its symplectic analogues is an orthogonal projector associated to a smooth family of positive complex Lagrangians in the normal tangent bundle of $\Sigma$; existence of such families is very easy to prove, and corresponds to the contruction of the first order jet of the outgoing variety as above)

This is the only place where compactness comes in the picture. Let $A$ be a selfadjoint Fourier integral operator with idempotent symbol as above. If $X$ is compact, $A - A^2$ is a compact operator so the spectrum $spA$ is discrete, except for the accumulation points 0 and 1. Now if $f$ is the characteristic function of an interval $[\lambda, \infty[$, with $\lambda \in ]0, 1[$, $\lambda \notin spA$, then $\Pi = f(A)$ is a Fourier integral operator projector of the type we seek (one could as well take any function $f$ holomorphic near $SpA$, taking only the values 0 and 1, and equal to 0, resp. 1 in a neighborhood of 0, resp. 1).

This proves theorem 1 in the absolute case (i.e. when there is no group action).

### v) Toeplitz operators

We will denote $H$ (or $H_\Pi$ if there is a risk of confusion) the range of $\Pi$ in $L^2(X)$. If $P$ is a pseudodifferential operator the Toeplitz operator $T_P$ is, by definition, the "coefficient" $\Pi P \Pi$ of $P$ in $H$. Since $\Pi \in I_C$, Toeplitz operators are also exactly those Fourier integral operator $A \in I_C$ such that $A = \Pi A \Pi$, so they form an algebra.

The symbol of a Toeplitz operator is completly determined by its restriction to $\Sigma$ (cf. iii), so the symbolic calculus for Toeplitz operators lives on $\Sigma$. It is microlocally isomorphic to the calculus of pseudodifferential operators. More precisely: if $A \in I_C$ is a Toeplitz operator of degree $m$, we denote $\sigma_m(A)$ the restriction to $\Sigma$ of its symbol (rather of the ratio $\sigma A/\sigma_\Pi$; in fact the Maslov bundle has a natural trivialisation on $Sigma$, where the unit symbol is the restriction to $Sigma$ of the symbol of any projector as above). It follows from the standard calculus of Fourier integral operator that we have, for Toeplitz operators $A$, $B$ with $deg\ A = p$ , $deg\ B = q$:

$$\sigma_{p+q}(AB) = \sigma_p(A)\sigma_q(B)$$

$$\sigma_{p+q-1}([AB]) = -i\{\sigma_p(A)\ ,\ \sigma_q(B)\}$$

## 3. Invariance and conclusions

Although the constructions above depend on several choices (first of $q$ or rather $Z$, then the choice of a self-adjoint idempotent symbol), the result is somehow canonical, up to compact operators. More precisely if $\Pi$ et $\Pi'$ are two Fourier intergral projectors as above, corresponding to two canonical relations $C$, $C'$ with real part $Sigma$, it is immediate to check that $\Pi\Pi'\Pi$ (resp. $\Pi'\Pi\Pi'$) is an elliptic Toeplitz operator (its symbol is $> 0$) so $\Pi$, resp. $\Pi'$ defines a Fredholm operator $H_\Pi \to H_{\Pi'}$ (resp.$H_{\Pi'} \to H_\Pi$).

Moreover, one can improve the conjugacy result of remark 1 by successive approximations and see that any two such operators $\Pi$ et $\Pi'$ are almost conjugate: there exists a unitary pseudodifferential operator U such that $\Pi' - U\Pi U^*$ is of degree $-\infty$ (and of finite rank).

More generally let X, X' be compact manifolds, and $\Sigma \subset T^*X$ resp. $\Sigma' \subset T^*X'$ be two symplectic subcones. If $\chi$ is a symplectic isomorphism $\Sigma \to \Sigma'$, one shows that there exists a Fourier integral operator $F$ "adapted" to $\chi$, i.e. such that the canonical relation is $\gg 0$ as above, with real part the graph of $\chi$, and the symbol of $F$ is elliptic there (one can apply the same construction as in i) above, cf.[2]).

Then if $\Pi$ and $\Pi'$ are projectors defining Toeplitz structures for $\Sigma$ resp. $\Sigma'$, $\Pi' F$ defines a Fredholm operator from $H_\Pi$ to $H_{\Pi'}$. (of course the index of such an $F$ may be $\neq 0$ - in fact the index does not really make much sense since the spaces are only defined up to spaces of finite rank).

If $\Sigma$ is a symplectic cone, it can always be embedded in some cotangent bundle $T^*X$. Thus the Hilbert space $H_\Sigma$ defining a Toeplitz structure and the related algebra of Toeplitz operators are almost uniquely defined, independently of the manner in which $\Sigma$ is embedded in a cotangent bundle: if $H'_\Sigma$ is another, there is a Fredholm map ("elliptic" Fourier integral operator) $F : H_\Sigma \to H'_\Sigma$ which defines an isomorphism between the two algebras, up to ideals of oerators of finite rank.

In fact if $\Sigma$ is a symplectic cone there is a canonical embedding where one chooses for $X$ the base of $\Sigma$: the Liouville form of $\Sigma$ ($\lambda = \rho_L \omega$ with $\omega$ the symplectic form, and $\rho$ the radial vector field, infinitesimal generator of homotheties of $\Sigma$) defines an oriented contact structure on $X$, and $\Sigma$ is identified with the set of positive multiples of the contact form (considered as section of $T^*X$).

In this setting let $\chi$ be a contact automorphism of $X$, i.e. $\chi$ preserves the contact form up to a smooth positive factor, or equivalently the cotangent extension $T^*\chi$ preserves $\Sigma$. Let $U_\chi$ be the canonical unitary extension of $\chi$ to $L^2(X)$ (there is a square root of a Jacobian determinant in the transformation formulas). Then the operator $\Pi\, U_\chi$ is a Fredholm operator. In the context of Toeplitz operators, this is the analogue of a Fourier integral operator associated to the canonical transformation $T^*\chi$.

The constructions above allow the action of a compact group $G$: let $X$ be an oriented compact contact manifold, $\Sigma \subset T^*X$ the associated symplectic and $G$ a compact group of automorphisms of $X$. Then $G$ acts symplectically on $\Sigma$, and unitarily on $L^2(X)$. In the contruction above we may choose $q$ invariant (remplacing it if need be by the mean of its transforms over $G$); $Z$ and $C$ are then also invariant. We can also choose the first approximation $A$ of the projector invariant (again remplacing it if need be by the mean of its transforms over $G$). Then $\Pi = f(A^2)$ is also invariant.

If we choose $\Sigma = T^*V$ with $V$ a compact manifold, $H_\Pi$ identifies (up to a space of finite dimension) to $L^2(V)$. So what precedes gives another short proof of Weinstein's resul: any compact group $G$ of symplectomorphisms of $T^*V$ can be lifted as a group of unitary Fourier integral operator in $L^2(V)$.

However Toeplitz operators depending on a parameter may pose a problem: the formal constructions (n$^o$ i–iii) above of course still work but not the last (n$^o$ iv)

## 4. Discrete quantification

Let $\Sigma$ be a symplectic cone with compact basis. A symplectic homogeneous action of the circle group $U(1)$ on $\Sigma$ has an infinitesimal generator $\partial/\partial\theta$, which is always a hamiltonian vector field: it derives from the hamiltonian $a = < \partial/\partial\theta, \lambda >$ where $\lambda$ is the Liouville form of $\Sigma$. Such an action is elliptic if the orbit circles are all transversal to the Liouville form, i.e. if the canonical homogeneous generating hamiltonian $a$ does not vanish. In this section we suppose that we are given a free elliptic action of $U(1)$, with generating hamiltonian $a > 0$.

A typical example is described in [5]: we are given a complex cone $C \subset C^n$ which is smooth outside of the origin of $C^n$ (or equivalently a smooth complex projective manifold Z equipped with an ample line bundle). The basis $Y$ of $X$ is $C \cap the\ unit\ sphere$, and $X \subset T^*Y$ is the symplectic cone corresponding to Toeplitz operators (the half-line bundle of real "outgoing" covectors characteristic for $\overline{\partial}_b$). In this case the circle group $U(1)$ has an obvious action on Y and X ($z \to e^{i\theta}$), which obviously preserve Toeplitz operators. Moreover $A = \frac{1}{i}\partial/\partial\theta$ defines a positive elliptic Toeplitz operator whose eigenvalues are the positive ($\geq 0$) integers, and the eigenfunctions are the homogeneous holomorphic functions on C (polynomials if C is normal or if the degree is large).

In the general case, let $L$ be the level surface $a = 1$, and let $Z = L/U(1)$ be the orbit space: $L$ is a circle bundle on $Z$, with connection form the restriction of the Liouville $\lambda$. The curvature form on $Z$ is the form deduced from the structural symplectic form of $\Sigma$: it is a symplectic form with integral periods on $Z$. $Z$ does not necessarily carry a complex structure. However we may imitate the construction of the complex case above: there exists an invariant Toeplitz structure (as mentionned above this is true for any compact group action, and is essentially unique, up to a finite representation of the group). For this structure $A = \frac{1}{i}\partial/\partial\theta$ again defines a positive elliptic Toeplitz operator whose eigenvalues are integres, and positive except for a finite number of these.

In this case, as in the complex model, the algebra $\mathcal{A}$ of operators which commute with the group action (or with $A$) form an algebra, wich acts on each eigenspace $H_n = ker(A-n)$ (these are finite dimensional). This algeba gives rise to a symbolic calculus which lives on the cone $X/U(1) \sim Z \times R$, so it defines both a quantization of Z and an asymptotic operation of this on a sequence of finite dimensional spaces. Here the symbol $1/a$ (which is central as A), or equivalently the inverse $1/n$ of the eigenvalue $n$ associated to $H_n$, plays the role of the small Planck constant $h$.

An operator $P \in \mathcal{A}$ has a trace function $\tau_P(n) = Tr\ P|H_n$; modulo functions of rapid decrease of $n$ this only depends on the "total symbol" of $P$ (i.e. $P$ mod. operators of degree $-\infty$). Thus get back the canonical trace of $\mathcal{A}$, which is unique up to a constant factor $\tau_0(n)$ ($\tau_0$ an elliptic symbol)

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