

FROM THE ATOMIC HYPOTHESIS TO MICROLOCAL ANALYSIS
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Contents

1	INTRODUCTION	1
2	THE SCHRÖDINGER EQUATION AND SEMICLASSICAL ANALYSIS	3
2.1	Schrödinger equation	3
2.2	WKB Asymptotics	4
2.3	Newtonian and Hamiltonian Mechanics	6
2.4	Caustic	7
2.5	Feynman Integral	8
2.6	Stationary Phase.	9
2.7	\hbar Fourier Integral Operator	11
3	HIGH FREQUENCY ASYMPTOTICS AND MICROLOCAL ANALYSIS	12
3.1	Differential Operators	12
3.2	Microsupport	13
3.3	Pseudo-differential Operators	14
3.4	Symbolic calculus, principal Symbol	15
3.5	Symplectic geometry	16
3.6	Fourier Integral Distributions, Fourier Integral Operators	18
3.7	Models, propagation of singularities	19
3.8	Eigenvalues of elliptic operators	20
3.9	Miscellaneous	21
3.9.1	Microlocal regularity	21
3.9.2	Weyl Calculus	21
3.9.3	Analytic Pseudo-differential Operators and Analytic Wave Front set	22
3.9.4	Gevrey Classes	23
3.9.5	Uncertainty principle	24
3.9.6	Carleman estimates	24

1 INTRODUCTION

The question of using waves or particles to describes physical phenomenons like the propagation of fluids, gas, electricity and light has been a central issue of science since the beginning of scientific times. Descartes developed geometric optics, which is best explained by a corpuscular vision of light; Huygens developed wave analysis and was already quite aware of a duality particle/wave. Ideas became more

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precise in the 19th century; Boltzmann favored atoms while Mach, slightly earlier, was the leader of the vigorous school which used what we call at present waves. Both points of view become eventually totally entangled in the 20th century with the advent of quantum physics.

A fundamental idea of quantum physics is that small objects behave both as waves and particles, and that physical observables do not commute and cannot be measured simultaneously (Heisenberg). In Schrödinger's approach, an elementary particle is described by its wave amplitude $\varphi(x, t)$, which is a complex valued function of a space variable $x \in \mathbb{R}^3$ (depending on time t). Equivalently this is represented by the Fourier transform:

$$\hat{\varphi}(\xi) = \int_{\mathbb{R}^n} e^{-ix \cdot \xi} \varphi(x) d^n x, \tag{1.1}$$

a function of an impulsion variable $\xi \in \mathbb{R}^3$ (still depending on time).

The original function is given back by the Fourier reciprocity formula (inverse Fourier transform):

$$\varphi(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} \hat{\varphi}(\xi) d^n \xi. \tag{1.2}$$

In this description, "particles" - corresponding to the case where $\varphi(x)$ vanishes outside a small set, well localized in space, can sometimes be viewed as limits of waves, or conversely, but Heisenberg's incertitude principle, which says that one cannot have at the same time access to the position and the velocity of small objects as elementary particle is "obvious": φ and $\hat{\varphi}$ cannot both have a 1-point support - in fact if one vanishes outside of a bounded set, the other is analytic and cannot vanish outside of a bounded set. A more precise yet elementary formulation, for functions f of one variable, is given by the inequality:

$$\int_{\mathbb{R}} |xf(x)|^2 dx \int_{\mathbb{R}} |\xi \hat{f}(\xi)|^2 d\xi \geq 2\pi \left(\int_{\mathbb{R}} |f(x)|^2 dx \right)^2 \tag{1.3}$$

which is true for any square integrable function $\varphi \in L^2(\mathbb{R})$ (equality is reached with the gaussian function: $\varphi(x) = e^{-\frac{1}{2}x^2}$). Observe that the first integral in the left hand side of (1.3) measures by its smallness the fact that f "is localized" near zero, and the second does the same for \hat{f} .³

Mathematical tools dealing with these difficulties and with asymptotic calculus generally were developed systematically; they can be considered as the starting point of *microlocal analysis*. Two basic examples have motivated this theory : the Schrödinger equation and the theory of the wave equation. Asymptotic analysis with respect to the parameter \hbar is quite natural for the Schrödinger equation; it leads to the notion of pseudo-differential calculus with a parameter, or semi-classical calculus. Expansions in terms of high frequencies are even older and also quite natural in the description of light waves ; they lead to the calculus of pseudo-differential operators, developed by L. Nirenberg, J.J. Kohn, L. Hörmander etc. in the late sixties.

Microlocal analysis develops a very geometrical manner of dealing with this asymptotic calculus, reconciling asymptotically in a remarkable manner the techniques of wave analysis (Fresnel's description of light) and of particle analysis (geometrical optics). The name was pinpointed in 1970 when were announced in the Nice international congress the definition of the wavefront set (L. Hörmander, M. Sato), the possibility of using canonical transformations (J.V. Egorov), and the description of remarkably simple microlocal models for complicated systems of differential equations (M. Sato)

This program has several outgrowths, such as:

³this is an elementary quadratic inequality: integrating by parts one gets, for any real α : $\int ((xf)^2 - \alpha f^2 + \alpha^2 f'^2) = \int (xf + \alpha f')^2 \geq 0$ hence $\int (xf)^2 \int f'^2 \geq \int f^2$, which is equivalent to (1.3).

- The construction of high frequency approximations of the solutions of the Maxwell and wave equations, including significant qualitative and quantitative information. This is used e.g. in the design of antennas and in the evaluation of the radar stealthiness of a plane or a missile. Much work was devoted to this analysis, both in the U.S.A. and in the Soviet Union, e.g. in the work of J. Keller, V. Babich and their collaborators. An important point was evaluating what part of the wave is diffracted by the obstacle. Later work in this direction used in a significant manner propagation of “Gevrey regularity” in fact Gevrey₃) and “Gevrey microanalysis”, see § 3.9.4.
- The precise analysis of the distribution of eigenvalues of the Laplace operator in a bounded domain (see § 3.8). This begins with H. Weyl’s remarkable asymptotic estimate of the eigenvalues. The error term in this evaluation turned out to be very difficult to control, and the best result was given by L. Hörmander in his paper on the spectral function [49], where Fourier integral operators are used the first time, to give more precise estimates than was possible before. Many other beautiful results relating the spectrum of the Laplace operator on a domain and the geometry of the domain, such as suggested by the paper of Mark Kac “can you hear the shape of a drum” [15], or relations between the spectrum and the configuration of closed geodesics suggested by R. Balian and C. Bloch [63], are made possible and precise using microlocal analysis, see § 3.8.
- Analysis of the scattering of quantum particles by a localized potential, or of a wave by an obstacle, with in particular the description of the scattering frequencies (see [66] [101])

The tools introduced are general enough to produce results for many problems related to those above: Dirac equation, Maxwell equations, equations of elasticity with no stress on the boundary etc.

Microlocal analysis is also used for more theoretical results, e.g. in the proof of sophisticated variants of Holmgren’s uniqueness theorem.

In this contribution we have concentrated on the Schrödinger and the wave equations, which illustrate particularly well the motivations and methods of microlocal analysis, and for which results are quite striking. We have mostly followed the presentation of L. Hörmander, which remains close to our usual insight in analysis or geometry. A more algebraic approach, based on the fact that distributions are superpositions of boundary (edge) values of holomorphic functions defined in angular complex sectors, and using deeply and systematically the theory of holomorphic functions of several variables, was developed by the Japanese mathematicians (M. Sato, T. Kawai, M. Kashiwara).

2 THE SCHRÖDINGER EQUATION AND SEMICLASSICAL ANALYSIS

2.1 Schrödinger equation

This is the equation

$$\frac{\hbar}{i} \partial_t \phi - \frac{\hbar^2}{2} \Delta \phi + V \phi = 0 \quad (2.1)$$

where $V(x)$, the potential, is a real valued function. We will not go into the explanation of the role of the Schrödinger equation in modern physics; it is however important to recall the following facts.

1. The solution of the equation with prescribed initial data $\phi(x, 0) = \phi_0$, is given by a unitary group of operators in the Hilbert space $L^2(\mathbb{R}^n)$:

$$\phi(x, t) = \phi_t = e^{\frac{i}{\hbar} H} \phi_0$$

The generator H is a selfadjoint operator extending the differential operator $\frac{1}{2} \hbar^2 \Delta_x - V$ (differential operators are not bounded operators, and there may be several “natural” selfadjoint extensions, in

particular on bounded domains, and it is not always easy to prove that there is one if V is unbounded and not positive; but here we will usually write $H = \frac{1}{2}\hbar^2\Delta_x - V$ and ignore this difficulty which was extensively studied by many authors, see e.g. K. Yosida [32], T. Kato [16], or M. Reed B. Simon [58]. Since e^{itH} is unitary, the L^2 -norm $\|\phi_t\|$ is invariant:

$$\|\phi_t\|^2 = \int_{\mathbb{R}^n} |\phi(x, t)|^2 dx = \|\phi_0\|^2. \quad (2.2)$$

In quantum physics this is normalized to 1 and $|\phi(x, t)|^2$ is then interpreted as the probability density, at time t , of the presence of a particle at a point x .

2. The Planck constant \hbar is a physical constant determined by experiment. What is not constant is the scale at which one makes measures or observations. At the atomic scale (very small lengths and short times) \hbar is not at all negligible and one must deal with the complete Schrödinger equation above (this accounts very accurately for what is observed experimentally). On our macroscopic scale however, \hbar is comparatively very small and is treated as a vanishing quantity ($\hbar \rightarrow 0$). This does not mean that one can replace \hbar by 0 in the equation, because we are typically dealing with a “singular perturbation”, i.e. the terms which vanish involve higher order derivatives than those which remain. What one is really interested in is the asymptotic behavior of solutions for $\hbar \rightarrow 0$, hoping that this will turn out to be simpler than the global behavior of complete solutions, realize a connection between quantum and classical mechanics and eventually produce new tools to analyze the original equation.

More generally semi-classical analysis studies asymptotic solutions of differential equations $Pf \simeq 0$, or $Pf \simeq g$, for $\hbar \rightarrow 0$, where P is an asymptotic differential operator of the form:

$$P(x, \hbar D_x, \hbar) \sim \sum \hbar^k P_k(x, \hbar D), \quad (2.3)$$

associated to a function (symbol) $p(x, \xi, \hbar)$, and $f \sim \sum \hbar^k f_k$ is an asymptotic function or distribution, so as g . Here and in all the sequel, we have set:

$$D = \frac{1}{i}\partial = \frac{1}{i}(\partial_{x_1}, \dots, \partial_{x_n})$$

A remarkable feature of semi-classical analysis is that it relates the study of asymptotic solutions to classical mechanics, using what is now called “micro-local analysis”. The Schrödinger equation is its first and most important motivation and application.

2.2 WKB Asymptotics

It is customary to look for oscillating asymptotic solutions of the Schrödinger equation (2.1) in the form:

$$\phi_{\hbar}(x, t) \simeq e^{\frac{i}{\hbar}S(x, t)}(a(x, t) + O(\hbar)) \quad (2.4)$$

The exponent $S(x, t)$ is called a *phase* and $a(x, t)$ an *amplitude*. The W.K.B. method is the method for producing such asymptotic solutions (W.K.B. stands for Wentzel-Kramer-Brillouin): note that we have

the operator relation

$$\begin{aligned}
 e^{-\frac{i}{\hbar}S} \left(\frac{\hbar}{i} \partial_t - \frac{\hbar^2}{2} \Delta + V \right) e^{\frac{i}{\hbar}S} &= \\
 (S_t + \frac{1}{2} |S_x|^2 + V) a &+ \\
 + \frac{\hbar}{i} \left(a_t + S_x \cdot a_x + \frac{1}{2} \Delta_x S a \right) &- \\
 - \frac{1}{2} \hbar^2 \Delta_x a &
 \end{aligned} \tag{2.5}$$

which follows from the elementary relations $e^{-\frac{i}{\hbar}S} \left(\frac{\hbar}{i} \partial_k \right) e^{\frac{i}{\hbar}S} = \frac{\hbar}{i} \partial_k + \partial_k S$. Inserting this in (2.4), we get for 0-order terms the “*eiconal equation*”:

$$\frac{\partial S}{\partial t} + \frac{1}{2} (\nabla_x S)^2 + V(x) = 0 \tag{2.6}$$

and for 1st order terms the “*transport equation*”:

$$\partial_t a + \frac{1}{2} \nabla_x S \cdot \nabla_x a + \frac{1}{2} (\Delta_x S) a = 0. \tag{2.7}$$

The eiconal equation (2.6) is familiar in fluid mechanics and control theory, where it is called the *Hamilton-Jacobi equation*, and there are several ways of analyzing it. For our purpose the natural route is the connection with hamiltonian systems. Let

$$E = E(x, \xi) = \frac{1}{2} |\xi|^2 + V(x)$$

be the hamiltonian function associated with the operator $-\frac{1}{2} \hbar^2 \Delta_x + V(x)$; the hamiltonian vector field (see section 2.10) L_E is defined as

$$L_E = \sum \frac{\partial E}{\partial \xi_j} \partial_{x_j} - \frac{\partial E}{\partial x_j} \partial_{\xi_j} = \sum \xi_j \partial_{x_j} - \frac{\partial V}{\partial x_j} \partial_{\xi_j}.$$

Assuming that S is smooth (at least twice differentiable), we introduce the Lagrangian manifold $\Lambda_S \subset \mathbb{R}^{2n} \times \mathbb{R}$, set of all points (x, ξ, t) with $\xi = \nabla_x S(x, t)$. (Lagrangian means that the differential 2-form $\sum d\xi_j dx_j$ induces 0 on Λ_S , which follows from the Schwarz identities $\frac{\partial^2 S}{\partial x_i \partial x_j} = \frac{\partial^2 S}{\partial x_j \partial x_i}$. Lagrangian manifolds play a crucial role in microanalysis, as we will see further in chapter 3).

Because Λ_S is Lagrangian, the Eiconal equation implies that the Lagrangian manifold is tangent to $\partial_t + L_a$, i.e. Λ_S is the union of integral curves of the system

$$\begin{aligned}
 \frac{dx}{dt} &= \nabla_\xi H = \xi, \\
 \frac{d\xi}{dt} &= -\nabla_x H.
 \end{aligned} \tag{2.8}$$

with initial point (for $t = 0$) $x(0) = x_0 \in \mathbb{R}^n$, $\xi(0) = \nabla_x S(x_0)$.

For any $x_0 \in \mathbb{R}^n$ there exists a unique integral curve $x(t), \xi(t)$ with initial data $x(0) = x_0, \xi(0) = \xi_0 = \nabla_x S(x_0)$; the eiconal equation implies that along this curve we have $\frac{dS}{dt} = -E(x, \xi)$ i.e.

$$S(x(T)) - S(x(0)) = - \int_0^T \left(\frac{1}{2} \left(\frac{dx}{dt} \right)^2 + V(x) \right) dt. \tag{2.9}$$

This determines the phase S completely, at least for small t , because the map $(x, t) \mapsto (x(t), t)$ is one to one, at least for small t (this follows from the implicit function theorem).

Once S is known, the amplitude a is determined by integrating the transport equation along the integral curve above. This gives a solution mod. \hbar^2 . One can improve and get an asymptotic solution mod. $\hbar^{-\infty}$, replacing a by an asymptotic sum $\sum \hbar^k a_k$: a_k is computed recursively as a solution of a transport equation $Ta_k = *$, where the transport operator T is the first order operator appearing in (2.7), and the right hand side is given in terms of the preceding a_j .

2.3 Newtonian and Hamiltonian Mechanics

In Newtonian mechanics, the movement of a particle of mass m in a potential $V(x)$ is determined by the second order differential equation

$$m\ddot{x} = \frac{d^2x}{dt^2} = -\nabla_x V(x). \quad (2.10)$$

In the Lagrange or Hamilton formulation of mechanics, the basic data is the Lagrangian function

$$L(v, x) = \frac{1}{2}m\|v\|^2 - V(x)$$

where the potential function V is a function of the position x alone (v stands for the speed).

The movements of a point of mass m in a potential V are the extremals of the action integral:

$$\int_0^t L(\dot{x}(s), x(s)) ds. \quad (2.11)$$

The movements $x(t)$ are the paths for which this integral is stationary (e.g. maximal or minimal), the end-points $x(0) = x_0, x(t) = x_t$ being fixed. They are the integral curves of Euler's differential equation (one out of so many "Euler equations"):

$$m \frac{d^2x}{dt^2} = \nabla_x L(x, v) = -\nabla_x V(x)$$

which here is equivalent to Newton's equation, and which we also write

$$\frac{dx}{dt} = v, \quad \frac{d}{dt}mv = \nabla_x L(x, v).$$

In this variational formulation, the variables are the position x and the speed $\dot{x} = v$. One switches to the position-impulsion variables (q, p) by taking the new variables $q = x, p = \nabla_v L$ (above we have rather written $p = \xi$; in a more intrinsic setting (x, v) appears as an element of the tangent space, while $(q, p) = (x, \xi)$ is an element of the cotangent space).

The Hamiltonian $H = H(p, q)$ and the Lagrangian $L = L(v, x)$ are related by the Legendre transform:

$$H + L = \langle p, v \rangle$$

(note that this is symmetric since we also have $v = \nabla_p H$, so L is also the Legendre transform of H).

With $L = \frac{1}{2}m|v|^2 - V(x)$, we get $p(\xi) = mv, H = \frac{1}{2m}|\xi|^2 + V(x)$.

In the position-impulsion variables (q, p) , the equations of the movement take the hamiltonian form:

$$\frac{dq}{dt} = \nabla_p H, \quad \frac{dp}{dt} = -\nabla_q H. \quad (2.12)$$

These are the same as the equations (2.8) above (with mass $m = 1$). In this manner the classical Newtonian or Hamiltonian mechanics describing the evolution of a particle in a potential V appear as a limiting case for $\hbar \rightarrow 0$ of the Schrödinger equation. Observe that the classical principle of conservation of energy, or conservation of the hamiltonian along the trajectories (which is obvious in the hamiltonian formulation), states that:

$$\frac{d}{dt}H = 0.$$

It is the classical counterpart of the quantum conservation of energy (cf. equation (2.2)).

The impulsion variable p is dual to the speed v variable (or to the position q), so $\xi = p$ appears as a Fourier variable. In the Schrödinger formulation it corresponds to the operator $\frac{\hbar}{i}\nabla_x$. According to Heisenberg's principle p and q cannot be apprehended both at the same time, whereas there classical limits can: classical mechanics is local with respect to both space and impulsion variables.

2.4 Caustic

The phase function S is reconstructed from the Lagrangian Λ_S , which in turn was constructed using the hamiltonian curves $x(t), \xi(t)$, solution of (2.8).

At this point several accidents may occur : first the integral curves might “blow up” and not be defined for all times; this is often the case for abstract equations, but usually not in situations which arise from physics where the “movements” of particles are expected to go on forever, e.g. if V is differentiable and $V' = O(\|x\|)$. In this case Λ is globally well defined, so as the function s which varies according to formula (2.9) along hamiltonian curves.

The next accident, however, will usually always happen, even if the Lagrangian manifold Λ is well defined: the projection map $\Pi : (x, \xi) \mapsto x$ may cease being one to one. It ceases to be so on the set of critical points of the projection Π , which is the set of zeros of the Jacobian determinant of the map $\pi|_{\Lambda}$; this is a hypersurface of Λ (defined by one numerical equation), whose projection is the caustic, a hypersurface of $\mathbb{R}^n \times \mathbb{R}$ (usually with singularities, of a rather remarkable type, even if the critical set in Λ has none. This difficulty is essentially removed in the next section on microlocal analysis, which shows that the extension of the solution is really well defined as an asymptotic \hbar -distribution, cf. § 2.7, 3.6.

For example if $V = 0$, $S(x, t) = \frac{1}{2} \frac{\|x\|^2}{t-t_0}$ is the solution of the Eiconal equation with initial data $S(x, 0) = -\frac{\|x\|^2}{2t_0}$; the hamiltonian curves are $X(t) = (1 - \frac{t}{t_0})x$, they all converge to the origin at time t_0 .

It is elementary that the caustic at time t is the set of end-points $X(x, t)$ of solutions of the hamiltonian evolution equation (2.6) with initial point x :

$$\frac{d^2 X}{dt^2} = -\nabla_x V(X), \quad X(x, 0) = x,$$

for which the Jacobian determinant $J = \det \frac{DX}{Dx}$ vanishes (i.e. for which the map $x \mapsto X(x, t)$ ceases to be infinitesimally one to one).

By definition of the Lagrangian manifold we have $\frac{dX}{dt} = \nabla_x S$, so for the Jacobian matrix $\frac{DX}{Dx}$ we get

$$\frac{d}{dt} \left(\frac{DX}{Dx} \right) = \nabla^2 S \cdot \left(\frac{DX}{Dx} \right), \quad \left(\frac{DX}{Dx} \right)(0, x) = I,$$

where $\nabla^2 S = \frac{\partial^2 S}{\partial x^2}$ is the hessian matrix of S . So along the integral curves $J = \det \frac{DX}{Dx}$ satisfies:

$$\frac{dJ}{dt} = \text{Tr} \nabla^2 S \cdot J = \Delta S \cdot J$$

The transport equation for the amplitude a means that along integral curves we have

$$\frac{da}{dt} = -\frac{1}{2}\Delta_x S.a$$

so $J^{\frac{1}{2}}a$ is constant along these curve and a behaves as a *half density* on Λ (this idea was emphasized by L. Hörmander [50] who used it to define the symbol of Fourier integral distributions).

In physical situations the caustic corresponds to points where $\Delta_x S \rightarrow -\infty$. The amplitude, solution of (2.7) usually also blows up at those points.

Note that the caustic indicates at most a singularity of the solution of the Schrödinger equation $\phi_t = e^{\frac{t}{i\hbar}H}\phi_0$ (with initial value $\phi_0 = e^{\frac{i}{\hbar}S_0}a_0$); the solution itself should be defined, at least as a distribution, for all times.

2.5 Feynman Integral

The considerations above and the connection with Newtonian dynamics can also be obtained formally in the manner described below, leading quite naturally to a description by *Feynman Integrals*. What follows is heuristic; rigorous proofs are difficult in this setting, even more so since Feynman integrals are not mathematically completely understood.

Let us first recall Trotter's formula: if A, B are linear operators, the equality $e^{t(A+B)} = e^{tA}e^{tB}$ only holds when A and B commute. However one always has the following limit identity:

$$e^{t(A+B)} = \lim_{m \rightarrow \infty} \left(e^{\frac{tA}{m}} e^{\frac{tB}{m}} \right)^m \tag{2.13}$$

This also works for suitable unbounded operators generating groups e^{tA}, e^{tB} .

Let us rewrite the Schrödinger equation (2.1) $\partial_t \phi = (A+B)\phi$ i.e. $\phi_t = e^{t(A+B)}\phi_0$ if the initial data is $\phi = \phi_0$ for $t = 0$ (we write as above $\phi = \phi(x, t) = \phi_t(x)$), with

$$A = \frac{i\hbar}{2}\Delta, \quad B = -\frac{i}{\hbar}V$$

The solutions of the two corresponding equations

$$\begin{aligned} \partial_t \phi &= A\phi, & \phi &= \phi_0 & \text{for } t &= 0, \\ \partial_t \psi &= B\psi, & \psi &= \psi_0 & \text{for } t &= 0, \end{aligned}$$

are given by the explicit formulas:

$$\begin{aligned} \phi(x, t) &= e^{-\frac{it}{\hbar}V(x)}\phi_0(x), \\ \widehat{\psi}(\xi, t) &= e^{-\frac{1}{2}iht\xi^2}.\widehat{\psi}_0(\xi). \end{aligned}$$

The second relation is the Fourier transform of $\psi_t = e^{t\frac{i\hbar\Delta}{2}}\psi_0$; it implies

$$\psi(x, t) = \left(\frac{i\hbar}{2\pi}\right)^{-\frac{n}{2}} \int e^{\frac{i|x-y|^2}{2i\hbar}} \psi_0(y) dy.$$

Cutting the interval $[0, t]$ in m equal parts, and introducing m intermediate end-points x_0, \dots, x_{m-1} , we get

$$\left(e^{\frac{tA}{m}} e^{\frac{tB}{m}} \right)^m \phi_0 = \phi_m(x, t) = C_m \int \dots \int e^{\frac{i}{\hbar}S_m} \phi(x_0) dx_0 dx_1 \dots dx_{m-1}, \tag{2.14}$$

with

$$S_m = S_m(x, x_{m-1}, x_{m-2} \dots x_0) = \frac{t}{m} \left[\sum_1^m \frac{1}{2} \left| \frac{x_k - x_{k-1}}{\frac{t}{m}} \right|^2 - \sum_1^m V(x_{k-1}) \right] \quad (\text{we set } x_m = x) \quad (2.15)$$

and

$$C_m = \left(\frac{i\hbar t}{2\pi m} \right)^{-\frac{mn}{2}} \quad \text{so that} \quad C_m \int \dots \int e^{\frac{i}{\hbar} S_m} dx_0 \dots dx_m = 1.$$

The phase S_m is the discrete version of the action integral S :

$$S = S(\underline{x}) = \int_0^t \left(\frac{1}{2} |\dot{\underline{x}}(s)|^2 - V(x(s)) \right) ds \quad (2.16)$$

along a path $\underline{x}(t)$ with $\underline{x}(\frac{kt}{m}) = x_k$, i.e. we have $S_m \rightarrow S(\underline{x})$ for $m \rightarrow \infty$ if \underline{x} is sufficiently regular.

Trotter's formula means that the solution ϕ of the Schrödinger equation is the limit of ϕ_m for $m \rightarrow \infty$, so it is tempting to write it as a Feynman integral:

$$\phi(x, t) = C \int e^{\frac{i}{\hbar} S(\underline{x})} \phi(y) \mathcal{D}\underline{x} \quad (2.17)$$

over paths \underline{x} with $\underline{x}(0) = x$. This is quite heuristic, since the sign $\mathcal{D}\underline{x}$, the integration sign \int and "the constant" C such that $C \int e^{\frac{i}{\hbar} S(\underline{x})} \mathcal{D}\underline{x} = 1$ are certainly not defined individually. Together $C \int e^{\frac{i}{\hbar} S(\underline{x})} \Phi(\underline{x}) \mathcal{D}\underline{x}$ should at least remotely behave as an integral, but the class of functionals $\Phi(\underline{x})$ for which it is defined is not mathematically well understood. Still formulas such as (2.17) are very suggestive and of constant use by physicists.

A similar construction was done for the heat equation $(\hbar \partial_t - \frac{\hbar^2}{2} \Delta_x + V)u = 0$ for $t \geq 0$, $u(x, 0) = u_0(x)$. In this case formula (2.17) should be replaced by:

$$u(x, t) = C \int e^{-\frac{1}{\hbar} S(\underline{x})} u_0(y) \mathcal{D}\underline{x}.$$

Here the exponential is small because now the phase exponent is real and essentially negative for $t > 0$. The heuristic derivation above can be made rigorous, and the limit is a *Wiener integral*, as the notation suggests (although individual terms $C, \int, \mathcal{D}\underline{x}$ still do not make much sense separately).

2.6 Stationary Phase.

If A is a complex $n \times n$ invertible symmetric matrix with positive real part, we have

$$\int_{\mathbb{R}^n} e^{-\frac{1}{2} \langle Ax, x \rangle} dx = \left(\det \frac{A}{2\pi} \right)^{-\frac{1}{2}}$$

(this reduces to the computation of a gaussian integral if A is real positive, and follows by analytic continuation; $(\det A)^{\frac{1}{2}}$ denotes the holomorphic branch which is > 0 if A is real > 0).

In particular if $A = iB$ with B a real invertible $n \times n$ symmetric matrix we get

$$\int_{\mathbb{R}^n} e^{\frac{i}{2} \langle Bx, x \rangle} dx = (2\pi)^{\frac{n}{2}} e^{i\sigma \frac{\pi}{4}} |\det B|^{-\frac{1}{2}}$$

with $\sigma = \sigma(B)$ the signature: $\sigma = n_+ - n_-$ where n_{\pm} is the number of positive resp. negative eigenvalues of B .

The formula of the stationary phase concerns the asymptotic behaviour for $\lambda \rightarrow \infty$ of a phase integral

$$\int e^{i\lambda\phi} a \, dx$$

with a a smooth compactly supported function on \mathbb{R}^n . If the phase ϕ has no critical point, the phase integral is of rapid decrease ($O(\lambda^{-\infty})$) with respect to λ (this is easily proved by integrating by parts). Thus the asymptotic behaviour of the phase integral is governed by the behaviour of the amplitude a near the critical points of the phase ϕ .

If ϕ has only non-degenerate critical points, integration by part and a convenient nonlinear change of variables using the Morse lemma, gives the following generalization: the phase integral has a simple leading part

$$\int e^{i\lambda\phi} a \, dx = \sum e^{i\lambda\phi(x_k)} e^{i\sigma_k \frac{\pi}{4}} \left| \det \left(\frac{\lambda}{2\pi} \phi''(x_k) \right) \right|^{-\frac{1}{2}} \left(a(x_k) + O\left(\frac{1}{\lambda}\right) \right) \quad (2.18)$$

where the sum is over all critical points x_k . Non-degenerate critical point means $\phi'(x_k) = 0$ and $\det \phi''(x_k) \neq 0$ so the formula makes sense: the sum is finite because nondegenerate critical points are isolated so there are only a finite number of them in $\text{supp } a$, and for each $\left| \det \frac{\lambda\phi''}{2\pi} \right|^{-\frac{1}{2}} = \left(\frac{2\pi}{\lambda} \right)^{\frac{n}{2}} \left| \det \phi'' \right|^{-\frac{1}{2}}$ is well defined.

This can be refined, and one gets a full asymptotic expansion

$$\int e^{i\lambda\phi} a \, dx \sim \sum_k e^{i\lambda\phi(x_k)} \left(\frac{2\pi}{\lambda} \right)^{\frac{n}{2}} e^{i\sigma_k \frac{\pi}{4}} \left| \det \phi''(x_k) \right|^{-\frac{1}{2}} \left(\sum_j a_{k,j} \lambda^{-j} \right)$$

where the coefficients $a_{k,j}$ are “local” i.e. only depend on a finite number of derivatives of a at x_k (they also depend polynomially on the derivatives of ϕ). One may also allow ϕ and a to depend smoothly on a parameter, and replace a by an asymptotic expansion $a \sim \sum a_j \lambda^j$. In this slightly more general version, the formula of the stationary phase is a universally useful tool for microanalysis (or for the \hbar Fourier integral operators below).

We emphasize that the main contributions to the integral come from the stationary points and are local, i.e. only depends on a finite number of derivatives at x_0 of a and ϕ .

The method of the stationary phase is also applied, still very heuristically but convincingly, to the Feynman path integral

$$\psi(x, t) = C \int e^{\frac{i}{\hbar} S(\underline{x})} \psi_0(x_0) \mathcal{D}\underline{x}$$

where S is the action integral (2.16), depending on a path \underline{x} with fixed end $\underline{x}(t) = x$. It is a natural idea to do this integral in two steps, as one would do for a finite dimensional integral: first integrate over all paths with given ends $\underline{x}(t) = x, \underline{x}(0) = y$, then integrate the result with respect to y . The formula of the stationary phase leads to expect that the first integral is essentially a finite sum of the form (2.18), produced by the critical paths. As we mentioned before, the critical paths, which extremize the Lagrangian (when extremities are fixed), are the trajectories of classical mechanics; the critical value is the action for these paths.

For instance if we are dealing with a free field, i.e. $V = 0$, the classical paths are straight lines running at constant speed, the unique extremal path joining y to x in time t has the speed $\frac{x-y}{t}$. The phase for

⁴For some questions it is also useful to allow a complex phase with positive (≥ 0) imaginary part ($|e^{i\lambda\phi}|$ must remain bounded; for integrals depending on a parameter, the computation of critical points and critical values can be adequately reinterpreted.

the reduced integral is the critical action $\frac{|x-y|^2}{2t}$ so the dominant term in the reduced integral should look like

$$\phi(x, t) = C' \int e^{\frac{i}{2\hbar t}|x-y|^2} \phi_0(y) dy$$

(with $C' \int e^{\frac{i}{2\hbar t}|x-y|^2} dy = 1$): this is indeed the exact formula.

In general things may be more complicated because the critical paths cannot be followed continuously for all y, t ; in fact there may be branching points forming a caustic, which again can be guessed from the heuristic Feynman integral.

2.7 \hbar Fourier Integral Operator

Superpositions of asymptotic solutions of the Schrödinger equation such as in (2.4) define special cases of \hbar Fourier integral operators. These are (asymptotic) integral operators of the form:

$$\phi \mapsto F_t \phi \sim \hbar^{-N/2} \int_{y, \theta \in \mathbb{R}^{n+N}} e^{\frac{i}{\hbar} S(x, y, t, \theta)} a(x, y, t, \theta) \phi(y) dy d\theta.$$

a is a smooth function or more generally an asymptotic expansion $a(x, y, t, \theta, \hbar) \sim \sum a_k \hbar^k$ (the factor $\hbar^{-N/2}$ is chosen so that $\phi_t = F_t \phi$ remains bounded for $\hbar \rightarrow 0$ if $a_0 \neq 0$).

In such an integral the main contributions always come from stationary points, i.e. points where $d_\theta S = 0$; other points have negligible contributions (i.e. $O(\hbar^{-\infty})$, see below section 2.6, and chapter 3).

Applying an \hbar -Fourier integral operator to an oscillating asymptotic expansion such as those which appear in §§ 2.2, 2.6 will give another one in good cases, when the resulting critical points are non-degenerate, or otherwise a superposition of such “asymptotic distribution”. For the same reason composition of two \hbar -Fourier operators is another one. We do not describe this analysis further here since it is clearer (and somewhat more general) for the Fourier integral operators described in chapter 3, and we will just try to give an idea with the simplest examples.

The solution ϕ of the free Schrödinger equation ($V = 0$) is given by its Fourier transform $\hat{\phi}(\xi, t) = e^{-\frac{i}{2}|\xi|^2 t} \hat{\phi}_0$. This can be rewritten

$$\phi(x, t) = \iint e^{\frac{i}{\hbar} \langle x-y, \xi \rangle - \frac{1}{2}|\xi|^2 t} \phi_0 dy d\frac{\xi}{2\pi\hbar} \tag{2.19}$$

which is obviously an \hbar -Fourier integral operator.

In the general case ($V \neq 0$) the solution is still defined by an \hbar -Fourier integral operator, provided the extremal curves are defined for all t , but there is usually no obvious formula. (one may construct such a formula

$$\phi = \iint e^{\frac{i}{\hbar} S(x, y, \xi, t)} a(x, y, \xi, t, \hbar) \phi_0(y) dy d\frac{\xi}{2\pi\hbar}$$

by the WKB method of § 2.2, with initial data $S = \langle x - y, \xi \rangle$, $a = 1$ for $t = 0$. There may be a caustic problem (depending on the geometry of extremal curves in particular the existence of focal points), so that the formula is only constructed for small t ; but it can then be iterated to give a similar formula (with possibly more integration variables) for arbitrary t ; the derivation of the canonical Feynman integral uses a similar iteration, but here the dimension remains bounded.

More generally for a semi-classical operator (2.3) $P(x, \hbar D_x, \hbar) f \simeq 0$, associated to a function

$$P(x, \xi, \hbar) \simeq \sum P_k(x, \xi) \hbar^k$$

(ξ stands for $D_x = \frac{\hbar}{i}\partial_x$), the eiconal and transport equation read:

$$P_0(x, d_x S) = 0, \quad La = 0$$

where the transport operator L is the coefficient of \hbar (leading term) in the asymptotic operator $e^{-\frac{i}{\hbar}S} P e^{\frac{i}{\hbar}S}$:

$$L = \sum \partial_{\xi_k} P_0(x, dS) D_{x_k} + [P_1(x, dS) - \frac{i}{2} \sum \partial_{\xi_k \xi_l} P_0(x, dS) D_{x_k x_l} S].$$

At “bad points” the effect of an \hbar -Fourier integral operator on an oscillating asymptotic expansion is no longer of this type; it is still a continuous superposition of such expansions, and a special kind of “asymptotic distribution” (where the singularities concentrate along a caustic). Here again this is better understood in the framework of Fourier integral distributions of Chapter 3, and we just illustrate it by an example.

Let us take up again the example of § 2.4 (with center $x_0 = 0$) and investigate the solution of the Schrödinger equation with initial data $\phi_0 = e^{-\frac{i}{2\hbar}|x|^2}$. The formula for gaussian Fourier transforms shows that the Fourier transform of the solution Φ is $\hat{\phi} = \left(\frac{2\pi\hbar}{i}\right)^{\frac{n}{2}} e^{-\frac{1}{2}i\hbar(t-1)|\xi|^2}$. In the inverse Fourier transform the powers of $2\pi, \hbar, i$ cancel out, and only the ambiguity on $(\det(1-t))^{-\frac{1}{2}}$ remains: we get

$$\phi = \begin{cases} (1-t)^{\frac{n}{2}} & e^{-\frac{i}{2\hbar}(1-t)^{-1}|x|^2} & \text{if } t < 1 \\ i^{-n}|1-t|^{\frac{n}{2}} & e^{-\frac{i}{2\hbar}(1-t)^{-1}|x|^2} & \text{if } t > 1. \end{cases}$$

Notice that there is a change in the phase ($i^{-\frac{n}{2}}$) when crossing $t = 1$; this is often the case when crossing a caustic; it is a topological phenomenon linked with the so-called Maslov index of the corresponding Lagrangian. For $t = 1$ we get $\phi = \left(\frac{2\pi\hbar}{i}\right)^{\frac{n}{2}} \delta$; this is no longer an oscillating asymptotic expansion (but certainly a superposition of such).

3 HIGH FREQUENCY ASYMPTOTICS AND MICROLOCAL ANALYSIS

In the previous section, there is a “small parameter” with respect to which asymptotic expansions are done (possibly the Planck constant \hbar). In many problems in P.D.E.s the small parameter is a small wavelength - or equivalently the large parameter is the size of a large frequency. High frequency asymptotics give rise to the same asymptotic calculus, and microanalysis is the resulting theory.

3.1 Differential Operators

We first recall notations for partial differential operators⁵. Let

$$a(x, \xi) = \sum_{|\alpha| \leq m} a_\alpha(x) \xi^\alpha$$

be a polynomial of ξ with coefficients smooth functions of x . Then we introduce the differential operator $A = a(x, D)$:

$$a(x, D) = \sum a_\alpha(x) D^\alpha.$$

By the Fourier inversion formula we have

$$a(x, D)f = (2\pi)^{-n} \int e^{ix \cdot \xi} a(x, \xi) \hat{f}(\xi) d\xi. \tag{3.1}$$

⁵we use standard notations for differential operators: if $\alpha = (\alpha_1, \alpha_2 \dots \alpha_n) \in \mathbf{N}^n$ is a multi-index we write $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n$, $\alpha! = \alpha_1! \alpha_2! \dots \alpha_n!$, $\partial_x^\alpha = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \dots \partial_{x_n}^{\alpha_n}$, $D = \frac{1}{i} \partial$ ($D^\alpha = i^{-|\alpha|} \partial^\alpha$).

The function $a(x, \xi)$ is called the total symbol of the operator $a(x, D)$ and its top-order part is its symbol (or principal symbol): $\sigma_m(A) = \sum_{|\alpha|=m} a_\alpha(x, \xi)$ (or simply σ_A if there is no confusion about the degree). Let us single out the following properties of differential operators:

1. Differential operators are local: if u is a distribution vanishing near a point x , then so does Au if $A = a(x, D)$ is a differential operator. Equivalently A diminishes supports, and likewise singular supports (singsupp u is the smallest closed set outside of which u is a smooth function): $\text{supp } u, \text{ :}$

$$\text{supp } Au \subset \text{supp } u, \quad \text{singsupp } Au \subset \text{singsupp } u.$$

2. If $A = a(x, D)$ and $B = b(x, D)$ the total symbol of $A \circ B = C = c(x, D)$ is given by Leibniz' rule:

$$c(x, \xi) = \sum \frac{1}{\alpha!} \partial_\xi^\alpha a D_x^\alpha b \quad (\text{usually denoted } = a \circ b). \quad (3.2)$$

The second assertion follows from Leibniz' rule for derivations.

For principal symbols one gets

$$\sigma_{p+q}(A \circ B) = \sigma_p(A)\sigma_q(B), \quad \sigma_{p+q-1}[A, B] = -i\{\sigma_p(A), \sigma_q(B)\}$$

where $[]$ denotes the commutator $[A, B] = AB - BA$ and $\{\}$ is the Poisson bracket:

$$\{f, g\} = \sum \partial_{\xi_j} f \partial_{x_j} g - \partial_{x_j} f \partial_{\xi_j} g.$$

In particular if ϕ is a smooth function we have $e^{-it\phi} A(e^{it\phi}) = \sigma_A(td\phi) + O(t^{m-1})$ which shows that the principal symbol is invariant if viewed as a function of covectors (on $T^*\mathbb{R}^n$).

3.2 Microsupport

Microlocal analysis takes advantage of the fact that the composition formula (3.2) is local both in the x and ξ variables, not x alone, and pushes as far as possible the idea of localizing in the set of cotangent vectors (x, ξ) (the phase space of mechanics) rather than in the x -space alone. The first important ingredient of microanalysis is the notion of microsupport, or wave front set, which gives a precise formulation of the fact that singularities of distributions can be localized asymptotically in phase space (i.e. simultaneously in position space and asymptotically in moment space).

Starting from the fact that a distribution u with compact support is smooth if and only if its Fourier transform \hat{u} is of rapid decrease at infinity (i.e. $\|\xi\|^N \hat{u}$ is bounded for all N), the micro support is defined (by negation) as follows:

Definition 3.1 *A distribution u is said to be microlocally smooth (or just smooth) near a covector (x_0, ξ_0) ($\xi_0 \neq 0$) if there exists a smooth cutoff function ϕ with compact support, with $\phi(x_0) \neq 0$, such that the Fourier transform $\widehat{\phi u}$ is of rapid decrease at ∞ in a small conic neighborhood of ξ_0 ⁶. The microsupport SSu is the set of all nonzero covectors at which u is not smooth.*

The micro support $SS(u)$ is a closed set; it is conic, i.e. a union of rays $(x, \lambda\xi), \lambda > 0$; it is some times useful to reinterpret it as a closed subset of the cotangent sphere bundle. Any closed conical set can actually occur as the microsupport of some distribution; e.g. an elementary Fourier transform computation shows that the microsupport of the (locally integrable) distribution $u(x, t) = \frac{1}{x^2 - it}$ is reduced to one single ray: $\{x = t = \xi = 0, \tau > 0\}$.

The microsupport has two important properties:

⁶this means that for all N , $\|\xi\|^N \widehat{\phi u}(\xi)$ is bounded in some small conic neighborhood $\left\| \frac{\xi}{\|\xi\|} - \frac{\xi_0}{\|\xi_0\|} \right\| < \epsilon$ of ξ

- it does not depend on the set of local coordinates used to define it, when viewed as a set of covectors,
- its projection on the x -space is the singular support,

The first statement is now familiar, although not completely obvious since the definition above uses the Fourier transformation, and seems to privilege linear coordinates or operators with constant coefficients. The second statement means that the microsupport is a refinement in phase space, of the singular support. The proofs are not hard but rather inseparable of those concerning pseudo-differential operators described below.

The appropriateness and advantage of the microsupport in phase space rather than in x -coordinate space appears clearly in the following example: recall that the wave operator is $\square = \partial_t^2 - \sum \partial_{x_j}^2$, if units are chosen so that the speed of light is 1. Its bicharacteristic curves are the hamiltonian curves of its symbol $(\xi^2 - \tau^2)^7$ which lie in the characteristic set $\{\xi^2 - \tau^2 = 0\}$; their projection on the (t, x) -space are the light rays. The theorem of propagation of singularities below (theorem 3.1) states that if f is a distribution solution of the wave equation $\square f = 0$, its microsupport $\text{SS } f$ is a union of bicharacteristic curves. This statement cannot be expressed neatly in terms of the singular support alone. It also gives a posteriori an obvious justification of geometric optics as a consequence of wave optics.

Microanalysis leads to study microsingularities and to ignore smooth objects. If u is a distribution, one writes $u \sim 0$ near a covector (x, ξ) if u is smooth at this point in the sense of definition 3.1; if U is an open set of covectors, one writes $u \sim 0$ in U if it is smooth at all points of U .

3.3 Pseudo-differential Operators

Formula (3.1)

$$Af = A_a f = (2\pi)^{-n} \int e^{ix \cdot \xi} a(x, \xi) \hat{f}(\xi) d\xi. \tag{3.3}$$

still makes sense and defines an operator $a(x, D)$, for almost any function $a(x, \xi)$ (for instance it suffices that a be of polynomial growth with respect to ξ so that the integral converges when \hat{f} is of rapid decrease).

Pseudo-differential operators are special operators $a(x, D)$, where the function a is required to satisfy suitable regularity conditions for $\xi \rightarrow \infty$, ensuring “good microlocal properties”. The main requirement is that pseudo-differential operators should be “microlocal”, i.e. that they diminish microsupports (see below), and as much as possible that they give rise to an asymptotic symbolic calculus similar to that of differential operators. Many classes of total symbols (functions $a(x, \xi)$) have been proposed, for which the corresponding operators deserve the name “pseudo-differential”. One of the first is the class of “classical” symbols, whose order of magnitude for $\xi \rightarrow \infty$ behaves as that of homogeneous functions:

Definition 3.2 *A classical symbol of degree m is a smooth function $a(x, \xi)$ such that for any α, β :*

$$\partial_x^\alpha \partial_\xi^\beta a(x, \xi) \lesssim (1 + \|\xi\|)^{m-|\beta|}$$

where \lesssim means that the quotient of l.h.s. by r.h.s. remains bounded for $x \in$ any compact set.

Among classical symbols, the simplest, which already generalize differential operators, are the “**regular symbols**”, i.e. classical symbols which admit an asymptotic expansion of the form

$$a \sim \sum a_k(x, \xi) \quad \text{for } \xi \rightarrow \infty$$

⁷i.e. the integral curves of the hamiltonian $2(\xi\partial_x - \tau\partial_t)$

with a_k smooth, homogeneous of degree $m - k$ (a of degree m). The \sim sign means that for any α, β we have

$$\partial_x^\alpha \partial_\xi^\beta (a - \sum_{k < N} a_k) \lesssim (1 + \|\xi\|)^{m-N|\beta|} \quad (3.4)$$

A classical pseudo-differential is an operator $A = a(x, D)$ with a a classical symbol.

The fact that an operator A is a pseudo-differential operator does not depend on a choice of local coordinates (same as the definition of the microsupport; short proofs of this were given by Friedrichs and Hörmander); thus pseudo-differential operators are also well defined on manifolds, such as a sphere or a torus.

Pseudo-differential operators are not local (i.e. if f vanishes near a point x it does not follow that Af does, unless A is a differential operator); however the following result states that they are microlocal (better than local) for singularities:

Proposition 3.1 *Pseudo-differential operators are microlocal, i.e. if A is such an operator, we have, for any distribution f :*

$$SSAf \subset SSf$$

In fact this is one more important property of the microsupport, and its proof goes with the proof of the invariance of the microsupport under changes of coordinates.

3.4 Symbolic calculus, principal Symbol

Pseudo-differential are not local operators, so they cannot obey to the exact (finite) Leibniz formula which holds for differential operators; however if, in the spirit of microlocal analysis, we ignore operators of degree $-\infty$ and more generally smooth objects: then we get an asymptotic Leibniz formula: if $A = a(x, D), B = b(x, D)$ then $A \circ B = c(x, D)$ ⁸ where $c(x, \xi)$ is a symbol which satisfies asymptotically the Leibniz formula (3.2):

$$c(x, \xi) \sim \sum \frac{1}{\alpha!} \partial_\xi^\alpha a D_x^\alpha b.$$

This is no longer a finite sum if a is not polynomial w.r. to ξ ; however the degrees in the l.h.s. go down to $-\infty$, so the asymptotic expansion makes sense in the sense of (3.4).

If A is a pseudo-differential operator, its principal symbol (or just symbol if there is no ambiguity) is

$$\sigma_m(A) = a \pmod{O(|\xi|^{m-1})}$$

$\sigma_m(A)$ = the homogeneous function a_0 if A is regular. This is a function on T^*X ; it is characterized by the fact that, if ϕ is a real function with $d\phi \neq 0$, we have:

$$e^{-i\lambda\phi} A(e^{i\lambda\phi}) = a(x, \lambda d\phi) + O(|\lambda|^{m-1}). \quad (3.5)$$

For the principal symbol the Leibniz rule above reduces to

$$\sigma_{p+q}(A \circ B) = \sigma_p(A) \sigma_q(B) \quad (3.6)$$

$$\sigma_{p+q-1}([A, B]) = \frac{1}{i} \{ \sigma_p(A), \sigma_q(B) \} \quad (3.7)$$

where $\{ \}$ is the canonical Poisson bracket of T^*X : in local coordinates:

$$\{f, g\} = \sum \partial_{\xi_j} f(x, \xi) \partial_{x_j} g(x, \xi) - \partial_{x_j} f(x, \xi) \partial_{\xi_j} g(x, \xi). \quad (3.8)$$

⁸ A or B should also satisfy a mild compacity condition ensuring that the composition $A \circ B$ is well defined

Note that on a manifold only the principal symbol of an operator is intrinsically defined as a function on the cotangent bundle. The total symbol is not defined - at least not canonically; there is no canonical rule which to a function a on T^*X assigns an operator such as $a(x, D)$ on \mathbb{R}^n . This is one of the difficulties of quantification.

Thus pseudo-differential operators form an algebra, i.e. the composition of two such operators is another one. An important new feature is that there are many invertible operators, in contrast with what happens for differential operators, where the only invertible operators are of order 0.

Proposition 3.2 *A pseudo-differential operator of order m said to be elliptic if its principal symbol is invertible ($|\sigma_A| \gtrsim \|\xi\|^m$). Then A is quasi-invertible, i.e. there exists a pseudo-differential operator B of degree $-m$ such that $AB \sim Id$ and $BA \sim Id$ where \sim means that both sides differ by an operator of degree $-\infty$ (such an operator B is called a parametrix of A).*

The construction of inverses can be microlocalized, and one gets

Proposition 3.3 *Let A be a pseudo-differential operator, and f a distribution. Then*

$$SSf \subset \text{char}A \cup SS Af$$

(the characteristic set $\text{char}A \subset T^*X$ is the set of cotangent vectors near which σ_A (or A itself) is not invertible).

3.5 Symplectic geometry

As we have mentioned, the definition of the microsupport is local and does not depend on the choice of local coordinates, so as the fact that an operator is pseudo-differential; they make sense on any manifold.

But microlocal calculus will exploit much further the fact that the pseudo-differential calculus described above is local on phase space (microlocal), and allow changes of coordinates there. There are obvious constraints on the change of coordinates χ : it should be a homogeneous since the symbols used for pseudo-differential are essentially homogeneous functions; more importantly it should preserve the Poisson brackets, which appears as leading term in the commutator law.

At this point symplectic geometry comes back as an essential and unavoidable ingredient in the whole picture - with a vengeance since it was first invented for problems in mechanics and dynamical systems closely related to those of partial differential equations, but its role in PDE tended to fade away with Hadamard's definition of well-posed problems, and the emergence of distributions theory.

A complete review on symplectic geometry is quite beyond the scope of this review. We will just recall the following facts:

- A symplectic manifold is a manifold X equipped with a closed nondegenerate 2-form, or equivalently with a Poisson bracket $\{f, g\}$ - in local coordinates

$$\{f, g\} = \sum c^{pq} \partial_p f \partial_q g$$

where the matrix (c_{pq}) is skew symmetric and invertible. The corresponding 2-form is $\sum c_{pq} dx^p dx^q$ with c_{pq} the inverse matrix; the fact that it is closed is equivalent to the Jacobi identity for Lie brackets:

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0.$$

X must be even-dimensional because the rank of a non-degenerate skew-symmetric form is even.

- In the pseudo-differential context, we are dealing with homogeneous (or conic) symplectic manifolds, i.e. the group \mathbb{R}_+ of homotheties acts and the symplectic form is homogeneous of degree 1 (or equivalently the Poisson bracket is of degree -1 , as one sees on the formulas above). In this case the symplectic form σ is exact, there exists a unique horizontal primitive λ , the “*Liouville form*” (i.e. λ is a 1-form orthogonal to rays, $d\lambda = \sigma$).

The basis of a symplectic cone (set of rays) is $Y = X/\mathbb{R}_+$; it is a contact manifold, i.e. it is equipped with the pull down λ_Y , which is well defined up to a smooth positive factor, and such that $\lambda_Y d\lambda_Y^{2n-1}$ is a volume element on Y ($\dim X = 2n$). The data consisting of a symplectic cone is equivalent to that of a contact manifold, and it is sometimes practical to switch from one language to the other.

- two symplectic manifolds of the same dimension are locally isomorphic; the same is true for contact structures (or homogeneous symplectic structures).
- a submanifold $Y \subset X$ is said to be involutive if $\{f, g\} = 0$ on Y implies $\{f, g\} = 0$ on X (equivalently the tangent space TY contains its orthogonal for the symplectic form); then $\dim Y \geq \frac{1}{2}\dim X$.
- a submanifold Y is called Lagrangian if it is involutive of minimal dimension ($\frac{1}{2}\dim X$).

For instance if X, X' are two symplectic manifolds of the same dimension and $u : X \rightarrow X'$ is a differential map, then u is a symplectic map iff its graph $\Gamma_u \subset X \times X'^0$ is Lagrangian (X'^0 denotes X' with the opposite symplectic form; this just means that the 2-form $\sigma_X - \sigma_{X'}$ should vanish on the graph). More generally a Lagrangian submanifold $\Lambda \subset X \times X'^0$ should be thought of as a relation between the objects X and X' .

A typical example of a lagrangian submanifold $\Lambda \subset T^*X$, X a manifold, is the section defined by df , f a smooth function on X , that is the set of all $(x, \xi = df(x))$: the fact that this is lagrangian i.e. $\sum d\xi_j dx_j$ vanishes on Λ is just a reformulation of the Schwarz lemma: $\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}$.

One very systematic manner of producing lagrangian submanifolds of T^*X (X a manifold) is by using generating functions: a generating function is a smooth function $S(x, \theta)$ on $X \times \mathbb{R}^N$ ($\theta \in \mathbb{R}^N$ is a parameter); the critical set C_S is the set where all vertical derivatives $\frac{\partial S}{\partial \theta_j}$ vanish; we require that the differentials $d\frac{\partial S}{\partial \theta_j}$ be linearly independent so that C_S is a manifold. The associated lagrangian manifold (shade) is the image Λ_S of C_S by the differential map $U : (x, \theta) \mapsto (x, d_x S(x, \theta))$; the independence of the $d\frac{\partial S}{\partial \theta_j}$ ensures that this is always a smooth submanifold of T^*X .

$U|C_S$ is an immersion i.e. its derivative is injective); it is lagrangian because it is of dimension $n = \dim X$ as C_S , and the inverse image of the Liouville form $\lambda = \sum \xi_j dx_j$ on C_S is $U^*\lambda = dS$ by definition, so $U^*\sigma = ddS = 0$.

In fact, at least locally⁹ and away from the zero section ($\xi \neq 0$) any lagrangian submanifold can be defined in that manner. A homogeneous lagrangian submanifold is defined by a homogeneous generating function (or phase function, see below).

Homogeneous involutive and lagrangian submanifolds play an important role in microanalysis. The changes of coordinates we wish to reach are homogeneous canonical changes, in the sense of the following definition.

Definition 3.3 A homogeneous canonical map is a map $\chi : (x, \xi) \mapsto (y, \eta) = \chi(x, \xi)$ such that

- χ respects the symplectic structures ($\Leftrightarrow \{f \circ \chi, g \circ \chi\} = \{f, g\} \circ \chi$).
- χ is homogeneous: $\chi(x, \lambda\xi) = (y, \lambda\eta)$.

Such a map (or rather its graph) can always be defined, at least locally, by a generating function.

⁹in fact globally if one allows further a positive imaginary part in S .

3.6 Fourier Integral Distributions, Fourier Integral Operators

In section 2.7, we mentioned \hbar -Fourier integral operators. In the present framework one also defines Fourier integral operators. They are a special case of “integral operators” $f \mapsto \int K(x, y)f(y)dy$ where the distribution K satisfies a maximal set of differential equations (“holonomic” in the terminology of [86, 87, 88]).

One important virtue of Fourier integral operators is that they can be microlocalized, and used to perform canonical changes of coordinates on the cotangent bundle - they are also called “quantized canonical transformations. A Fourier integral operator is defined as a locally finite sum of generalized oscillatory integrals:

$$Af = \int e^{iS(x,y,\theta)} a(x, y, \theta) f(y) d\theta dy, \quad (x, y, \theta) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \times \mathbb{R}^N).$$

More generally considers Fourier integral distributions, also defined as locally finite sums of generalized oscillatory integrals:

$$T(x) = \int e^{iS(x,\theta)} a(x, \theta) d\theta \quad (\text{i.e. } \langle T, \phi \rangle = \int e^{iS(x,\theta)} a(x, \theta) \phi(x) d\theta dx).$$

There are technical conditions: a is a symbol as above ($\theta \in \mathbb{R}^N - \{0\}$), and the phase function S is homogeneous of degree 1 with respect to θ and has no stationary point (the total differential $dS \neq 0$) (this condition is necessary to give sense to the generalized oscillatory integral); the “kernel” $K(x, y) = \int e^{iS(x,y,\theta)} a(x, y, \theta) d\theta$ is then well defined as a distribution, and only the critical point of the phase S ($\{\partial_\theta S\} = 0$) contribute to the singularities of K .

A typical example of Fourier integral distributions are those associated to a smooth oriented hypersurface Σ of equation $u = 0$, u smooth with $du \neq 0$ on Σ . the associated phase function is $S = \theta_1 u(x)$ and the corresponding distributions (a a regular symbol of degree s) are of the form

$$T(x) = f(x)(u + i0)^{-s-1} + g(x) \quad \text{resp.} \quad f(x)(u + i0)^{-s-1} + g(x) \log(u + i0)$$

with f, g smooth functions, if $s \in \mathbf{C}$ is not an integer, resp. $s \in \mathbf{Z}$ (s may be any complex number, but in most useful cases it is an integer or a half integer). $(u + i0)^s$ and $\log(u + i0)$ are defined as the limits (which always exist as distributions) $\lim_{\epsilon \rightarrow +0} (u + i\epsilon)$ resp. $\lim_{\epsilon \rightarrow +0} \log(u + i\epsilon)$.

Here we are at first interested in those Fourier integral operators which enable to perform canonical changes of coordinates: the phase function S the critical set C_θ of points where the θ -derivatives vanish is a manifold, the derivatives $d\partial_{\theta_j} S$ are independent, and its image C under the map (x, y, θ) is the graph of a canonical isomorphism $\chi : T^*Y \rightarrow T^*X$ (canonical means compatible with the Poisson brackets). A special case is the case of pseudo-differential operators: $S = \langle x - y, \xi \rangle$ where χ is the identity map (we wrote ξ instead of θ).

Important facts of the theory are the following:

- The set of operators thus constructed only depends on the canonical map χ (i.e. two phase functions with the same χ will yield the same operators for suitable choices of the symbols), and any canonical map χ can actually occur. There is a notion of degree - we denote F_χ^m the set of operators of degree m corresponding to a given χ . Operators of degree 0 are locally L^2 -continuous.
- Fourier integral operators can be composed: if $A \in F_\chi^m$ and $B \in F_{\chi'}^{m'}$ then $A \circ B \in F_{\chi \circ \chi'}^{m+m'}$.
- There is a notion of elliptic Fourier integral operator (the symbol a is elliptic). If $A \in F_\chi^m$ is elliptic, it has a parametrix $B \in F_{\chi^{-1}}^{-m}$.

- (Egorov) If $A \in F_{\chi}^m$ we have $SS Af \subset \chi(SS f)$ (equality if A is elliptic). Moreover if $A \in F_{\chi}^m$ is elliptic and P is a pseudo-differential operator on Y , then APA^{-1} is a pseudo-differential operator on X with principal symbol $\sigma_P \circ \chi^{-1}$.

Remark Although this does not lead to notable simplifications, WKB or semiclassical analysis on \mathbb{R}^n can be essentially reduced to microlocal analysis, adding one variable x_{n+1} and interpreting \hbar as ξ_{n+1}^{-1} : semiclassical analysis is essentially the same as the study of the x_{n+1} -Fourier transforms of solutions of pseudo-differential $P(x, D)$ independent of ξ_{n+1} in the region $\xi_{n+1} \neq 0$ (P does not depend on x_{n+1} , i.e. it is translation invariant, but it does depend on $\xi_{n+1} = \hbar^{-1}$).

3.7 Models, propagation of singularities

All these constructions can be “microlocalized”: operators, distributions, etc. modulo smoothing objects, make sense microlocally (on small open conic subsets of T^*X). Local Fourier integral operators can then be used to transform locally a given system of differential equations into a much simpler one, in the same way as a suitable choice of local coordinates can greatly simplify the study of a geometrical object. They are also often called “quantized contact (or canonical) transformations”.

Among the simplest operators that occur (models) are simple vector fields such as

$$\partial_{x_j}, \tag{3.9}$$

$$\partial_{x_j} + i\partial_{x_{j+1}}, \tag{3.10}$$

$$\partial_{x_j} + ix_j\partial_{x_{j+1}}. \tag{3.11}$$

It was proved by M. Sato, T. Kawai and M. Kashiwara that “most” systems of partial differential equations¹⁰ are microlocally equivalent to a system composed of some of these models. In particular one proves the following result:

Theorem 3.1 (Propagation of singularities) *Let P be a pseudo-differential operator, real with simple characteristics (more precisely: the principal symbol σ_P is real, and the hamiltonian vector field H_{σ_P} is not parallel to the radial vector field $\sum \xi_j \partial_{\xi_j}$). Then*

- 1) P is microlocally equivalent to AD_1 , with A elliptic.
- 2) If Pf is smooth, $SS f$ is a union of bicharacteristic curves of P .

1) means that near any (x, ξ) ($\xi \neq 0$) there exists an elliptic Fourier integral operator F such that $FPF^{-1} = AD_1$ (mod. smoothing operators). As often in this theory the proof contains two parts: first the geometric construction of the canonical map attached to F (in physics this usually corresponds to solving the eiconal equation); then (one is reduced to an operator of the form $D_1 +$ perturbation) construct the total symbol of F , solving inductively the analog of transport equations. In the present example this is natural and easy; for complicated models it can be more involved.

2) is easy for the model operator ∂_{x_1} , which is the generator of the group of translations parallel to the x_1 -axis: solutions are translation invariant, so their microsupport are also translation invariant; but in this case the orbits of the translation group are precisely the bicharacteristic curves. The general case follows because the statement is invariant under transport by Fourier integral operators.

The statement about the microsupport of solutions of the wave equation in § 3.4, is a special case of this theorem.

¹⁰ i.e. generic systems at generic points

3.8 Eigenvalues of elliptic operators

Here is another remarkable consequence of the geometric microanalysis described above, which was guessed by physicists (R. Balian, C. Bloch [63]), and proved precisely by Colin de Verdière [76], J. Chazarain [74, 75], J. Duistermaat and V. Guillemin [80]:

Let A be a positive selfadjoint elliptic pseudo-differential operator, of degree $m > 0$, on a compact manifold X . Then it is known (at least since the 19th century for the Laplace operator), that there exists an orthonormal basis of smooth eigenfunctions $u_k \in L^2(X)$, where the corresponding eigenvalues form an increasing sequence $\lambda_0 \leq \lambda_1 \leq \dots \rightarrow \infty$ (this follows from the Riesz theory of integral equations, and the fact that A has a parametrix which is a pseudo-differential operator of degree $-m < 0$ hence a compact integral operator).

H. Weyl gave a striking formula giving the order of magnitude of λ_k . This is better described using the inverse function $N_A(s)$ = number of eigenvalues $\leq s$:

Theorem 3.2 *If A is as above, then*

$$N_A(s) \sim (2\pi)^{-n} \text{vol}\{\sigma_A \leq s\}$$

where vol denotes the symplectic volume of sets in T^*X .

Thus $N_A(s)$ is of order $s^{\frac{n}{m}}$, $m = \text{deg } A$. This is easy to understand when $A = a(D)$ is a differential operator with constant coefficients on the torus $\mathbb{R}^n/2\pi\mathbb{Z}^n$: then the eigenfunctions are the exponentials $e^{ix \cdot \xi}$, ξ with integral coefficients; the corresponding eigenvalue is $a(\xi)$, so $N_A(s)$ is the number of integral points of the region $\{a \leq s\} \subset \mathbb{R}^n$, approximately equal to its volume $\int_{a \leq s} d^n \xi$, hence the formula (the symplectic measure of T^*X is $d^n x d^n \xi$ and the universal constant $(2\pi)^n$ is the volume of the basis X).

In the torus example above the order of magnitude of the error term is at most $s^{(n-1)/m}$, as the $(n-1)$ -volume of the boundary $a(\xi) = s$. This is also true in general but is much harder to prove; the first complete proof was given by L. Hörmander [49] using there for the first time a Fourier integral operator.

A similar estimate also holds for a manifold with boundary. The error term (of order $s^{\frac{n-1}{2}}$) can be erratic, but for the Dirichlet problem R. Balian, C. Bloch announced a sharper result with a second term $c s^{\frac{n-1}{2}}$ in the asymptotic expansion whose coefficient is given by an integral on the boundary, valid if the geometry of geodesics is nice (the set of initial points of closed geodesics should be of measure 0).

The most precise statement can be found in the book of L. Hörmander [51] IV chapter 23.

Hörmander's theory was refined and globalized a little later by J. Chazarain (loc. cit.), J.J. Duistermaat and V. Guillemin (loc. cit.) who proved the following theorem:

Theorem 3.3 *Let A be a real first order pseudo-differential operator on a compact manifold X . Then*

1) (Hörmander) $\text{exp } itA$ is well defined; it is a one parameter group of Fourier integral operators, whose underlying group of symplectic isomorphisms is the dynamical flow of the Hamiltonian vector field H_A .

2) Suppose A positive elliptic. Then $\text{Tr}(e^{itA})$ is well defined as a distribution of T , its microsupport is contained in the set $\{(t, \tau), \tau > 0, t \text{ a period of a closed characteristic curve of } A\}$.

The characteristic curves are the integral curves of the Hamiltonian H_A . If $A = \sqrt{-\Delta}$ for a Riemannian metric on X , the characteristic curves are exactly the geodesics (more precisely their lift to T^*X) parametrized by arclength, so the periods are the multiples of the lengths of closed geodesics (including 0). With a mild supplementary condition¹¹ $\text{Tr}(e^{itA})$ is a Fourier integral distribution of t ; its singularities are of the type described in § 3.6, and were precisely described in [80].

¹¹namely for any T the set of equations $\Phi_T(x, \xi) = (x, \xi)$ defining the initial points (x, ξ) of closed geodesics of length T (Φ_T the geodesic flow) is of constant rank, so that these initial points form a smooth manifold. This is always true is $T = 0$ and was used in [49]

In the case of the Laplace operator on a Riemannian manifold, a link between the asymptotics of eigenvalues and lengths of closed geodesics was first announced by the physicists R. Balian and C. Bloch, [63] (including the case where X has a boundary, for which Fourier integral operators are not enough), and a somewhat different (but essentially equivalent) statement was proved by Y. Colin de Verdière [76].

3.9 Miscellaneous

3.9.1 Microlocal regularity

There are many manners of evaluating or measuring the regularity of a distribution (other than C^∞ regularity), and many of these can be microlocalized. If E is a space of distributions, we will say that a distribution u is microlocally in E if $Au \in E$ for any compactly supported pseudo-differential operator of order 0; this definition is only reasonable if E itself is microlocalizable, i.e. $AE \subset E$ for any compactly supported pseudo-differential operator of order 0. One can then define $SS_E u$: $(x, \xi) \notin SS_E u$ if there exists a pseudo-differential operator A of degree 0, elliptic at (x, ξ) (i.e. σ_A is invertible near (x, ξ)) such that $Au \in E$.

One very useful example is L^2 regularity: a fundamental result is that L^2 is microlocalizable, i.e. compactly supported pseudo-differential operators of degree 0 are L^2 continuous. The Sobolev spaces $W^{s,p}$ (the space of distributions whose derivatives of order s belong locally to $L^p, 1 < p < \infty$) are microlocalizable, so are the Hölder spaces C^s, s not an integer (if $s = k + \sigma$ with $k \geq 0$ an integer, $0 < \sigma < 1, f \in C^s$ if its derivatives of order k are Hölder continuous of order σ).

It is all the more striking that the most elementary function spaces are not microlocalizable: the space C^k of k -times continuously differentiable functions (k an integer): continuity is not a microlocal property, and in fact it is rather subtle to detect it on the Fourier transform; neither are the spaces L^1, L^∞ ¹². This fact is the first important difficulty in the theory of partial differential equations in several variables (i.e. more than one), and the reason why so many complicated functions or distributions spaces had to be introduced in this theory.

3.9.2 Weyl Calculus

In the symbolic calculus of pseudo-differential operators the principal symbol is intrinsically defined as a function on the cotangent bundle, but the rest of the total symbol is not; it depends for instance on a choice a phase function. For instant the “normalized” assignment $a(x, \xi) \rightarrow a(x, D)$ used above is only defined on \mathbb{R}^n and is linked with the standard phase function.

There are other manners of constructing a correspondence “total symbol” \rightarrow operator giving rise to different although equivalent formulas. One of the most useful was used by H. Weyl and E.P. Wigner. To a function $a(x, \xi)$ on \mathbb{R}^{2n} they associate the operator Op_a defined by

$$Op_a f(x) = \frac{1}{(2\pi)^n} \int e^{i\langle x-y, \xi \rangle} a\left(\frac{x+y}{2}, \xi\right) f(y) dy d\xi.$$

If a is a symbol, Op_a is a pseudo-differential operator $b(x, D)$ with b a symbol,

$$b \sim \exp\left(-\frac{ix \cdot \xi}{2}\right) a. \tag{3.12}$$

So one gets essentially the same operators and closely related formulas for the symbolic calculus. One advantage of this presentation is that it is obvious that Op_a is self-adjoint iff a is real.

¹²The microlocalizable substitutes are the Hardy space H_1 and its dual BMO , the space of functions of bounded mean oscillation - and the detailed study of these is one of the hard recent achievements of real analysis.

Another more subtle point of interest is the following: in the study of more complicated operators it may be useful to use symbols $a(x, \xi)$ with less regularity than the classical “symbols” above. In limiting cases, it may still be true, but not obvious, that the corresponding operators are microlocal and L^2 continuous, but the asymptotic expansions used above no longer work. In such limiting cases normal assignment $a \mapsto a(x, D)$ and Weyl assignment $a \mapsto \text{Op}_a$ do not always give the same classes of operators. In many cases the Weyl rule, because it has more symmetries, seems to be a better choice, e.g. it is easier to relate not very good regularity properties of a to L^2 continuity of $\text{op } a$ rather than of $a(x, D)$.

3.9.3 Analytic Pseudo-differential Operators and Analytic Wave Front set

For some questions it is useful to microlocalize notions which are beyond C^∞ smoothness, the most important case being analyticity. The construction of the wave front set indicated above, following Hörmander’s, is not suitable for the study of the propagation of analyticity because, among other reasons, it uses multiplication by C^∞ functions (with compact support) which destroys analyticity.

The theory of analytic pseudo-differential equations, microfunctions and analytic microsupport was developed by the Japanese mathematicians M. Sato, T. Kawai and M. Kashiwara [54]¹³. The manner (suited to the study of analyticity) in which these authors define distributions, and their generalization as “hyperfunctions” or “microfunctions” is by viewing them as sum of boundary values of holomorphic functions in small complex sectors ($\mathbb{R}^n + i\Gamma$, Γ a small complex cone), using systematically the cohomological theory of holomorphic sheaves. This theory is very beautiful and powerful, but exceeds what can be described here.

There are other equivalent definitions of the analytic microsupport. One which is still linked with the point of view of boundary values is the following: any temperate distribution f on \mathbb{R}^n is the boundary value for $t \rightarrow +0$ of a harmonic function $F(x, t)$ defined on the half-space $t \geq 0$ in \mathbb{R}^{n+1} ; in fact, as any harmonic function on this half-space, F extends holomorphically to the complex region $\{z = x - i\xi, t\} \in \mathbb{C}^{n+1}, \|\xi\| < \text{Re } t\}$. One can then read off analytic microsupport of f in terms of analytic continuation of F at points of the boundary of the above domain :

Proposition 3.4 *A point (x, ξ) does not belong to SS_{anf} iff the harmonic continuation F extends analytically near the point $(x - i\xi, t = \|\xi\|)$.*

An equivalent manner of seeing the microsupport is via the Bros-Iagolnitzer transform:

$$Bu(x, \xi) = \int e^{-i(y-x) \cdot \xi - |y-x|^2} u(y) dy \tag{3.13}$$

or its semi-classical variant

$$B_s u(x, \xi, \lambda) = \int e^{-\lambda(i(y-x) \cdot \xi + |y-x|^2)} u(y) dy$$

This combines the Fourier transform (the imaginary part in the exponent) and a cutoff away from x_0 , since $e^{-|\xi_0||x-x_0|^2}$ is of exponential decrease (the function $F(x - i\xi, \|\xi\|) = 1$ above is related to the integral $\int_0^\infty Bf(x, r\xi) dr$ which is clearly analytic when Bf is of exponential decrease):

Theorem 3.4 *A distribution (or hyperfunction) u is smooth (resp. analytic) near the point (x_0, ξ_0) iff Bu is of rapid decrease (resp. of exponential decrease) with respect to ξ (resp. λ) in a small conic neighborhood of (x_0, ξ_0) .*

(as above “conic neighborhood” means that for some small ϵ it contains the set of all (x, ξ) such that $\|x - x_0\| < \epsilon, \|\frac{\xi}{\|\xi\|} - \frac{\xi_0}{\|\xi_0\|}\| < \epsilon$.)

¹³a first definition of analytic pseudo-differential operators was given by L. Boutet de Monvel and P. Krée [37].

3.9.4 Gevrey Classes

A smooth function f belongs to the Gevrey class G_s ($s > 0$) if for any x there exist a neighborhood V_x and constants c, a such that

$$\sup_{x \in U} |f^\alpha(x)| \leq C |\alpha|!^s a^{|\alpha|}.$$

For $s = 1$ this just means that f is analytic in U .¹⁴

For $s > 1$, an example of Gevrey function is the function of one variable $f(t) = e^{-t^\sigma}$ ($f = 0$ if $t \leq 0$), which is G_s with $s = 1 + \frac{1}{\sigma}$.

Gevrey_s regularity is linked to exponential decay of order $\frac{1}{s}$ of the Fourier transform. In fact a function $f \in G_s$ ($s > 1$) can be compactly supported (there are G_s partitions of unity), and for compactly supported functions, $f \in G_s$ is equivalent to

$$|\hat{f}| \leq c e^{a|\xi|^{\frac{1}{s}}} \quad \text{for some constants } c, a > 0.$$

(in the limit case $s = 1$, analytic regularity is also linked with exponential decay of the Fourier transform, but in the slightly more subtle manner described above (§ 3.9.3) since one does not dispose of partitions of unity).

Using the analytic theory mentioned above, one shows that G_s is microlocalizable; the Gevrey_s microsupport can be seen using the harmonic extension F : f is G_s at a point (x, ξ) if F is G_s up to the boundary in a small neighborhood of the boundary point $(x - i\xi, t = \|\xi\|)$; equivalently using the Bros-Iagolnitzer transform (3.13):

Proposition 3.5 *f is G_s near a point $(x, \xi) \in T^*(\mathbb{R}^n)$ ($\xi \neq 0$ iff Bf is of exponential type $\frac{1}{s}$ in a small conic neighborhood of (x, ξ) i.e. in a suitably small conic neighborhood one as:*

$$|Bf(y, \eta)| \leq C e^{-c|\eta|^{\frac{1}{s}}}.$$

There is a similar definition for semiclassical analysis.

Gevrey classes and Gevrey analysis appear technically, but naturally, in many places. In particular they appear quite naturally in the study of irregular singular points of ordinary differential equations.

A little more unexpectedly, propagation of Gevrey₃ singularities (linked to the Airy function and $e^{-k^{\frac{1}{3}}}$ decay of the wave) turns out to be essential in the study of the diffraction of a wave by an obstacle. For instance consider a convex obstacle illuminated by a source in front of it. In the back of the obstacle, in the shaded region there is still some light due to the diffraction. Propagation of C^∞ singularities is not a sharp enough tool to describe this phenomenon because such singularities do not reach this region. At variance analytic singularities propagate outside the obstacle like the C^∞ singularities but when they reach the obstacle they creep on the boundary and in the mean time produce Gevrey singularities which in turn propagate in the medium. These singularities (the function is C^∞ but not analytic) turn out to be exactly Gevrey₃. The factor 3 is linked to the appearance of the Airy function:

$$A(x) = \frac{1}{2\pi} \int e^{i(x\xi + \frac{\xi^3}{3})} d\xi.$$

This effect was known in elementary configuration since ([61]). It was computed long ago on some natural examples (e.g. by J.B. Keller or V.M. Babich) to contribute to the design of antennas and to the evaluation of the radar stealthiness of a plane (cf. [70]). A complete understanding of the phenomena in general configuration with rigorous proofs was eventually obtained G. Lebeau [91], using microlocal analysis in an essential manner.

¹⁴For $0 < s < 1$, $f \in G_s$ means that f extends as an entire holomorphic function on \mathbf{C}^n , such that $|f(z)| \leq c e^{a|z|^{\frac{1}{1-s}}}$ for some constants $c, a > 0$; this is obviously not a localizable or microlocalizable property and $G_s, s < 1$ is not much used in this context.

3.9.5 Uncertainty principle

One underlying idea of microanalysis is that it is possible to perform “cutoffs” in both the x (space) and ξ (moment) variables at the same time, of the form $f \mapsto a(x, D)f$, for which things remain in good control, provided that the symbol a is sufficiently well-behaved, and to dissect phase-space in boxes (regions) which are approximately the size of $\text{supp } a$. For the microlocal analysis above a is a classical pseudo-differential symbol, i.e. essentially a homogeneous function w.r. to ξ , and its support is essentially a conical set, and this gives significant information, e.g. an exact analysis of singularities, in simple cases. However there are many questions (e.g. the study of operators with multiple characteristics) for which the microlocal analysis above is no longer sufficient; a more refined dissection is required, and one must use non classical pseudo-differential operators (symbols a which are still reasonably well behaved but not classical) so as to reach smaller boxes than conical ones (it is also indicated to use Weyl’s normalization (cf. § 3.9.2) rather than the usual one). Heisenberg’s uncertainty principle (cf. (1.1)) can be grossly reinterpreted by saying it is not possible to analyze a distribution punctually (point by point) at the same time in position space and its Fourier transform in momentum space, and boxes one can reach cannot be very small: their symplectic volume must be $\geq (2\pi)^n$. Still it is tempting to try to cut down to boxes of that size. This turns out to be very difficult; volume considerations are far from sufficient, and the symplectic geometry of bounded regions is quite complicated and not completely understood; the symplectic boxes, so as the symbols $a(x, \xi)$ used, must be sufficiently “regular” so as to remain within the possibilities of pseudo-differential calculus.

Such a program was achieved by C. Fefferman and DH. Phong [44, 82] who used it to prove, among other things, very subtle results on positivity of partial differential operators, refining the so-called “sharp Gårding inequalities”.

3.9.6 Carleman estimates

A typical unique continuation property for a differential operator on a manifold X is the following: let $\Sigma \subset X$ be a hypersurface defined by an equation $\Phi = 0$ where ϕ is a smooth function; then if f (function, or distribution), a solution of $Pf = 0$ near Σ vanishes near Σ if it vanishes for $\Phi > 0$ (on one side of Σ).

Many unique continuation results in the analytic setting (P and Σ analytic), such as those proved by Holmgren, are consequences (or linked to) propagation of analyticity.

To avoid analyticity conditions T. Carleman introduced the type of estimates that bear his name in 1939. They were also used to prove the first type of general results concerning propagation of singularities (cf. Hörmander [12], chap. 8).

A typical Carleman estimate looks like

$$\|u\|_1 \leq C(\tau) \|P_\phi u\|_2 \quad \text{with } P_\phi = e^{-\tau\phi} P e^{\tau\phi}$$

where $\|\cdot\|_1; \|\cdot\|_2$ are Sobolev norms. exploiting some “convexity” properties of P and ϕ with respect to the data. To a large extent the method can be microlocalized, replacing the function ϕ by a zero-order pseudo-differential operator when suitable convexity conditions can be analyzed. But still Carleman estimates retain their interest beyond microlocal analysis because

- They are used in applied problems which involve “more” than the high frequency asymptotic (wave front set) and therefore this type of calculus is closely related to pseudo-differential calculus with parameter as it the case for the Schrödinger equation.
- They require a minimum of regularity and can be adapted to the analysis of some non linear evolution equations. Some results having been obtained for the Navier Stokes equations [84]

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