

LINEAR DIFFERENTIAL EQUATIONS

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INTRODUCTION

A linear partial differential equation on \mathbf{R}^n (n the number of variables) is an equation of the form

$$P(x, \partial)f = g$$

where $P = P(x, \partial) = \sum a_\alpha(x)\partial^\alpha$ is a linear differential operator, taking a differentiable function f into the linear combination of its derivatives:

$$\sum a_\alpha(x)\partial^\alpha f$$

in this notation α is a multi-index i.e. a sequence of n integers $\alpha = (\alpha_1, \dots, \alpha_n)$ and the corresponding derivative is

$$\partial^\alpha f = \frac{\partial^{|\alpha|} f}{\partial^{\alpha_1} \dots \partial^{\alpha_n}} \quad (|\alpha| = \alpha_1 + \dots + \alpha_n)$$

one also considers systems of such equations, where f (and the right hand side g) are vector valued functions and $P = (P_{ij})$ is a matrix of differential operators.

Linear equations appear systematically in error computations, or perturbation computations; they also appear in evolution processes which are axiomatically linear, i.e. where linearity enters in the definition, e.g. in field theories. So they are a very important aspect of all evolution processes.

For linear partial differential equations, a general theory was developed (although as yet certainly not complete). In many cases general rules of behaviour, or for computing solutions exist, much more than for non linear partial differential equations, for which important mathematical (like functional analysis) were invented and used, but which consists mostly of a smaller number of fundamental systems of equations (like the Navier-Stokes equations describing the flow of a fluid) which model important physical phenomena and whose analysis also uses physical intuition.

1 LINEARITY AND CONTINUITY

1.1 Continuity

In our world, the state of a physical system is usually defined by a finite or infinite collection of real numbers (furnished by measures). These completely describes the state of the system if

1) the result of any other reasonable numerical measure one could perform on the system are determined by these numbers - i.e. in mathematical language, is a function of these, and

2) the future states of the system, i.e. the values of these numbers at future times $t \geq t_0$, are completely determined by these numbers at an initial time $t = t_0$.¹ These numbers may satisfy some relations; mathematically one thinks of the set of all possible states as a manifold (possibly with singularities).²

¹ in quantum physics things are a little different; one must use complex numbers, and the states of a system can no longer be thought of as a set or a manifold, where coordinates (measures) are real numbers; see below.

² for instance if the set of measures contains twice the same identical one, the corresponding numbers must of course be the same. There are many other possible ways of choosing measures that will have correlations,

Above we mentioned only “reasonable” measures. The reason is that our measures are never exact, there is always some error or imprecision (hopefully quite small) - this means that we can only make sense of measures that are continuous functions of the coordinates (the smaller the error on the coordinates, the smaller the error on the measure). Many usual functions are continuous. The elementary function $E(x)$ =integral part of x (largest integral number contained in x , for x a real number) is not continuous, and a computer is not really able to compute it: for instance the computer is not usually able to distinguish between the two numbers $x = 1 - \varepsilon$ and $y = 1 + \varepsilon$ if ε is a very small number (e.g. $\varepsilon = 10^{-43} = 1$ over (1 followed by 43 zeros)), especially if these numbers are the result of an experiment and not given by a sure theoretical argument. Unless it is told to do otherwise, the computer will round up numbers and find $E(x) = E(y) = 1$; this means a huge relative error of 10^{43} (by comparison recall that the size of the known universe is about $10^{20}m$).

1.2 Linearity

Measures provide real numbers. Real numbers can be added, and also multiplied (dilated). Objects or quantities for which addition and dilations are defined are usually called “vectors” and form a vector space; there is also a notion of complex vector spaces where dilations by complex numbers are allowed.³

A real function of real numbers $f(x_1, \dots, x_n)$ is linear if it takes sums into sums:

$$f(x_1 + y_1, \dots, x_n + y_n) = f(x_1, \dots, x_n) + f(y_1, \dots, y_n)$$

(as limiting case f also takes a dilation to the same dilation (if it is continuous):

$$f(\lambda x_1, \dots, \lambda x_n) = \lambda f(x_1, \dots, x_n)$$

if λ is any real number)

Almost equivalently f is linear if it takes straight lines to straight lines or linear movements to linear movements (its “graph is straight”)

Linear functions are simple to compute and to manipulate. The physical quantities we measure are not intrinsically linear; in fact the notion of linearity depends on the choice of basic coordinates (measures) one makes to describe a system, so it is not really well a priori defined for physical systems.

However one of the main points of the differential calculus which was developed during the 17th-18th centuries is the fact that usual functions or measurable physical phenomena are linear when restricted to infinitely small domains, or approximately linear when restricted to small domains (the smaller the domain, the better the relative linear approximation); in other words, error calculus is almost additive, the error produced by the superposition of two fluctuations in an experiment is the sum of the errors produced by each fluctuation separately (up to much smaller errors).

and in fact it will usually be impossible to give a complete description by a set of measures with no correlations at all. A simple example is a system whose states are points on a sphere, as on the surface of our planet Earth: a state can be determined by 3 measures (height, breadth, depth) x, y, z satisfying one quadratic relation $x^2 + y^2 + z^2 = 1$. It is also determined by two real numbers (angles): the latitude and the longitude, but note however that the longitude is not well defined at the north and south poles.

³ this is not a complete mathematical definition; mathematicians also use, in particular in number theory or group theory, additive objects which have not much to do with vectors.

Functions which have this property of being approximately linear on small domains are called differentiable. The linear function which best approximates a differentiable function at a given point is called the tangent linear map, and its slope is the derivative at the given point. Most “usual” function given by simple explicit formulas have this property, e.g. the sine, exp, log function, and linear algebra appears systematically in error calculus concerning these functions.

Note however that Weierstrass showed that there are many continuous functions which are nowhere (or almost nowhere) differentiable - e.g. the function

$$y(x) = \sum 2^{-n} \sin 3^n x,$$

or the function which describes the shape of a coast: this is continuous, but its oscillations in small domains grow sharper and sharper. Many similar functions describing some kind of “fractal chaos” are in the same way continuous but not differentiable. Although they may be quite lovely to look at, they are difficult to handle and compute with quantitative precision - although small, the error on the result becomes comparatively very large when the increment of the variable is very small.

1.3 Perturbation theory and linearity

Anyway for usual “nice” functions the error calculus is always linear. Since the mathematical computation which to a differential equation or partial differential equation (and suitable boundary or initial data) assigns its solution looks nice (and at least in many important cases is nice), one also expects that if one makes a small perturbation of the equation or of the initial data, the resulting error on the solution will depend linearly on the errors on the data and one expects that it is governed by a system of linear differential equations. This is in fact true and not hard to prove in many good cases (although not always - e.g. there are equations without solutions, for which perturbation arguments do not make sense since there is nothing to begin with, see below). In any case, good or not, it is very easy, just using ordinary differential calculus, to write down the linear differential system that the error should satisfy.

For example if an evolution process is described by a differential equation depending on a parameter λ :

$$(1) \quad \frac{dx}{dt} = \Phi(t, x, \lambda)$$

with initial condition at time $t = t_0$

$$(2) \quad x(t_0) = a(\lambda)$$

and $x_0(t)$ is a known solution for $\lambda = \lambda_0$, with initial value $x_0(t_0) = a(\lambda_0)$, then for close values $\lambda = \lambda_0 + \mu$, the solution $x = x(t, \lambda) = x_0 + u$ satisfies

$$\frac{dx}{dt} = \frac{dx_0}{dt} + \frac{du}{dt} = \Phi(t, x, \lambda) = \Phi(t, x_0, \lambda_0) + \nabla_x \Phi \cdot u + \nabla_\lambda \Phi \cdot \mu + \text{small error}$$

so the variation u satisfies, up to an infinitely small error, the linear equation:

$$(3) \quad \frac{du}{dt} = \nabla_x \Phi \cdot u + \nabla_\lambda \Phi \cdot \mu$$

with linear initial condition (with respect to μ):

$$(4) \quad u(t_0) = \nabla_\lambda a \cdot \mu$$

(the argument above, based on physical considerations, is intuitively convincing but it is still not a mathematical proof. In fact for differential equations (one variable), the proof is quite straightforward and taught in undergraduate courses. But although analogues for many good partial differential equations are true, for general partial differential equations the arguments above may be very hard to prove - and sometimes completely false.)

1.4 Axiomatically linear equations

Some theories are linear by their own nature, or axiomatically, so the differential equations that describe the behavior or the evolution of their objects must be linear.

1.4.1 Fields, Maxwell equations

The description of several important physical phenomenons uses fields and field theory. Fields are in the first place vector-functions (defined over time or space-time, or some piece of this, or some suitable space), but the point is that they are vector valued and make up a vector space. So equations for fields should be linear.

A striking example is the system of the Maxwell equations for the electro-magnetic field, in “condensed” form, and suitable units of time, length, electric charge, etc.:

$$\begin{aligned}\nabla \cdot E &= 4\pi\rho \\ \nabla \cdot B &= 0 \\ \nabla \times E + \frac{1}{c} \frac{dB}{dt} &= 0 \\ \nabla \times B - \frac{1}{c} \frac{dE}{dt} &= \frac{4\pi}{c} j\end{aligned}$$

in these equations, $E = (E_1, E_2, E_3)$, resp. $B = (B_1, B_2, B_3)$ is the electric, resp. magnetic field; c is the speed of light, ρ is the electric charge density, and j the electric current density, which is related to the charge by $\nabla j = d\rho/dt$.

For the vector field E with components (E_1, E_2, E_3) , the usual notation $\nabla \cdot E$ (or $\text{div } E$) means the scalar (one component) function

$$\nabla \cdot E = \frac{\partial E_1}{\partial x_1} + \frac{\partial E_2}{\partial x_2} + \frac{\partial E_3}{\partial x_3}$$

and $\nabla \times E = \text{rot } E$ is the vector field with components

$$\nabla \times E = \begin{cases} \partial E_2 / \partial x_3 - \partial E_3 / \partial x_2 \\ \partial E_3 / \partial x_1 - \partial E_1 / \partial x_3 \\ \partial E_1 / \partial x_2 - \partial E_2 / \partial x_1 \end{cases}$$

the same notation is used for $B = (B_1, B_2, B_3)$. The operations div , rot , ∇ are also described elsewhere.

We should note that these equations have a large group of symmetries: the Lorentz group - or the Poincaré group if one includes translations in space time (these leave the equations invariant - not the solutions); these symmetries intertwine non-trivially space and time and were the starting point of relativity theory. The Maxwell equations are essentially the simplest

system of first order linear which remain unchanged under transformations of the Poincaré group.

Mathematical complement: let E be a vector space, equipped with a nondegenerate quadratic form q (in the electromagnetic or relativistic setting $E = \mathbf{R}^4, q = t^2 - x^2 - y^2 - z^2$). Let Ω be the space of all differential forms on E (fields).⁴

The exterior derivation is the first order partial differential operator $d : \Omega \rightarrow \Omega$, which takes k -forms to the $k + 1$ -forms:

$$d \sum a_{i_1 \dots i_k}(x) dx_{i_1} \dots dx_{i_k} = \sum da_{i_1 \dots i_k}(x) dx_{i_1} \dots dx_{i_k}$$

with $df = \sum \frac{\partial f}{\partial x_i} dx_i$. The quadratic form d extends canonically to give a quadratic form or a scalar product on forms, and an infinitesimal volume element dv ; the adjoint of d is the differential operator d^* such that

$$\int \langle d\omega | \phi \rangle = \int \langle \omega | d^* \phi \rangle$$

One then defines the Dirac operator $D = d + d^*$ which takes even forms to odd forms, and is canonically associated to the quadratic form q . The system of Maxwell's equations is

$$DE = A \quad (E \in \Omega^{even}, A \in \Omega^{odd})$$

The electromagnetic field itself is described by 2-forms, and in the actual physical system certain components vanish - e.g. there are no magnetic charges.

1.4.2 Densities on phase space in classical physics

In fact one can associate to any evolution process other ones which are linear and very closely related (although maybe not always completely equivalent). For example in classical mechanics one wants to describe the evolution of a system, i.e. a point in a suitable space X , subjected to an evolution law which can be described by a differential equation. Because the position of a point cannot be determined exactly and there is usually an error, it is often reasonable to use, instead of exact coordinates, probability densities for the mobile to be at a point, i.e. positive functions of integral (total mass) 1; slightly more generally one looks at the evolution of all integrable functions on X ; these form a vector space. It is often more agreeable to use the space $H = L^2(X)$ of half-densities (square integrable functions f , i.e. such that $\int |f|^2 < \infty$) - this set has a very natural mathematical definition and it is also a vector space, where one can add and dilate vectors; furthermore it has a very nice geometry where one can compute distances and angles very much as in our usual 3-dimensional world: it is a "Hilbert space", particularly nice to handle. Movements or transformations of X lift naturally to this vector space H , and the lifting is by definition linear, since H is axiomatically a vector space (note that although the lifting is linear it is infinite dimensional and by no means simpler than the original equation - in fact it is essentially equivalent to it, except that this linearization will ignore any set of motions with set of initial data of probability zero).

The following example is too simple and naive to give a complete idea of all mathematical difficulties that may arise, but will adequately illustrate what precedes: let us consider the very

⁴ a form of degree k can be written $\sum a_{i_1 \dots i_k}(x) dx_{i_1} \dots dx_{i_k}$; the product of forms is defined and is anti-commutative i.e. for odd forms $ba = -ab$.

simple dynamical system in which a mobile in usual 3-dimensional space \mathbf{R}^3 is animated by a constant speed v ($v = (v_j)$ is a vector): the differential equation which describes this is

$$\frac{dx}{dt} = v$$

where $x = x(t)$ is the position of our mobile at time t . The obvious solution is

$$x(t) = x_0 + v(t - t_0)$$

where $x_0 \in \mathbf{R}^3$ is the position at the initial time t_0 .

If $f_{t_0} = f(x)$ is a function on \mathbf{R}^3 (or a probability density, or a half density) at time t_0 , it evolves according to the law

$$f_t(x) = U_t f = f(x - (t - t_0)v)$$

so that the evolution is described by the linear partial differential equation for $F(t, x) = f_t(x)$:

$$\frac{\partial F}{\partial t} = \sum v_j \frac{\partial F}{\partial x_j}$$

with initial condition $F(t_0, x) = f(x)$.⁵

In this context of classical mechanics it is natural to use the “real” vector space H of real valued functions. One could just as well use the complex vector space of square-integrable complex valued functions; then the whole analysis becomes similar to that of quantum mechanics (see below).

1.4.3 Quantum mechanics and Schrödinger equation

In “classical physics”, “observables” form a commutative algebra A , which can be thought of as the algebra of functions on the space X of physical states (continuous, usually real-valued, but we may as well use complex valued functions - which is indispensable for the analogy with quantum physics). The space X can be completely reconstructed from the algebra A : it is its spectrum $Spec A$, a point of X corresponds to a character of A , i.e. an additive and multiplicative map $\chi : A \rightarrow \mathbf{C}$ ($\chi(f + g) = \chi(f) + \chi(g)$, $\chi(fg) = \chi(f)\chi(g)$); thus the only thing which really counts is the algebra A of observables.

In quantum physics, i.e. in the physical world as we know it since the beginning of the 20th century, there is still an algebra A of observables (measurable quantities), but it is no longer a commutative algebra, and the “phase space” can no longer in any way be thought of as a manifold where points are defined by a finite or infinite collection of real numbers (measures), or even as a set : there are not enough characters of the algebra A to determine it. At best the spectrum (phase-space) should be thought of as a “non-commutative space”, in the sense of A. Connes, and this is not a set at all even though many concepts of differential geometry can be adapted. Practically this means again that the only thing that really counts is the algebra A of observables.

However there still remains a quantum phase space which is a complex Hilbert space H . It is more the analogue of the space of half-densities in classical physics than of the classical

⁵ in this example the evolution operator preserves the canonical measure of \mathbf{R}^3 so the transformation formulas for functions, densities or half-densities are the same; in general the transformation formulas for densities or half densities, should include a jacobian determinant or square root of such

“phase space” itself; the algebra A is realized as an algebra of linear operators on H (there is a slight ambiguity on H because, for a given algebra A , there are several non equivalent such Hilbert-space representations, but this does not really matter if one keeps in mind that only the observables are accessible experimentally; one also must bear in mind that the algebra of observables A is in the permanent state of being discovered, and not fully known; anyway this is a very general and abstract set-up, which does not contain the laws of physics - these must be determined and/or verified experimentally).

Still H is axiomatically a vector space, and all constructions about it must be linear. In particular the differential equation describing an evolution process is the Schrödinger equation:

$$(5) \quad \frac{1}{i} \frac{df}{dt} = Af$$

where A is a suitable linear operator (in “real” processes A is self-adjoint, so the evolution operator “ e^{itA} ” preserves lengths). The Schrödinger equation is axiomatically linear, because the “phase-space” H is axiomatically a vector space. In practice only the evolution operator e^{itA} is a continuous linear operator; its generator A (the quantum Hamiltonian) is a more elaborate unbounded operator, not everywhere defined. There are many Schrödinger equations, corresponding to many possible operators A . For all of these there are many possible models, which are often quite complicated; in a typical case (description of the free electron), H is the Hilbert space $L^2(\mathbf{R}^3)$ and $A = -\Delta$ is the Laplace operator (up to sign - see below). The Schrödinger equation has given, and is still giving, a lot of work to many physicists and mathematicians.

2 EXAMPLES

2.1 Ordinary differential equations

Ordinary differential equations (one variable) appear systematically in evolution problems where the evolving system is determined by a finite collection of real numbers (has a finite number of “degrees of freedom”). An outstanding example is the system of Newton’s equations, which describes the movement of the solar system subjected to gravitational forces (where planets and the sun are assimilated to dimensionless massive points). As mentioned above, linear equations occur frequently, in particular to describe small perturbations of solutions of non linear equations; they occur also in several variable problems when there are many symmetries and the problem reduces to a one variable problem; for example for the rotation invariant eigenstates of a vibrating circular drum (eigenstates of the 2 dimensional Dirichlet problem) reduces to a 1 dimensional Bessel equation (cf. below).

A completely elementary but instructive example is the linear equation

$$\frac{dy}{dt} = y$$

whose solutions are the exponential functions

$$y(t) = ce^t,$$

$c = y(0)$ the initial value (a constant). Note that an obvious feature of the solution is that it grows exponentially, so that a small error on the initial data becomes quickly very large for increasing times; for non linear equations even worse accidents may happen: the solution may blow up and cease to exist after a finite time, as in the equation $\frac{dy}{dt} = y^2$ whose non-zero solutions are the functions $y(t) = \frac{y_0}{1-ty_0}$ (the blowing up time is $t_0 = \frac{1}{y_0}$).

This phenomenon is quite general; for equations with a higher number of degrees of freedom, the error (fluctuation) may still grow relatively very large, with rapidly varying direction, even though the solution itself remains bounded. This is one origins of the phenomenon exhibited by H. Poincaré and now often referred to as “chaos”, meaning that even if the equations and models were exact (which they are not), our computations concerning them become quickly meaningless: after some time, shorter or longer depending on the phenomenon and the precision of the initial data, the error is so large that the prediction is false (it takes time to measure precise initial data - essentially the time of the measure is proportional to the number of digits required, and anyway the precision is limited by our present techniques); for example local meteorological predictions do not presently exceed much more than 5-6 days.

However for many differential equations, even when one cannot compute the solutions by means of already known “elementary” functions, one can make quite precise mathematical predictions about their asymptotic behaviour at infinity or near a singular point (this does not contradict the possibility of chaotic behaviour - for instance the fact that a function behaves asymptotically as e^{t^2} or $\sin t^2$ gives meaningful geometric information about it, but certainly does not mean by itself that the function can be computed accurately for large t).

Among ordinary differential equations, many - in particular many of those with physical origin - have “nice” coefficients, e.g. polynomial or rational coefficients, and have been extensively studied and given rise to a huge literature. Typical examples are the hypergeometric equation

$$(6) \quad z(1-z)\frac{d^2y}{dz^2} + [\alpha + \beta z]\frac{dy}{dz} + \gamma u = 0$$

or the Bessel equation

$$(7) \quad \frac{d^2y}{dt^2} + \frac{1}{z}\frac{dy}{dt} + \left(1 - \frac{\nu^2}{t^2}\right)y = 0$$

which is a limiting case of equations similar to the hypergeometric equation, where singular points mesh together. There are many more “classical” equations.

Solutions of such equations extend to the complex plane, except at the singular points where the coefficients have poles. There the solution have “ramified singularities”, like the complex logarithm $\text{Log } z$: they are ramified functions, i.e. they have several branches (determinations) and are not really uniformly defined. But, near a singular point $z = z_0$, general solutions of such equations can be expressed as uniform functions of $\text{Log}(z - z_0)$. A typical example of the singularities one obtains is

$$(8) \quad (z - z_0)^\alpha (\text{Log}(z - z_0))^k \quad \text{with} \quad (z - z_0)^\alpha = e^{\alpha \text{Log}(z - z_0)}$$

(α a complex number, $k \geq 0$ an integer); these occur at “regular” singularities, as for the hypergeometric equation or the Bessel equation at $z = 0$. Other more complicated (irregular) singularities also occur, as for the Bessel equation at $z = \infty$, where the solutions may grow or oscillate faster than products of fractional powers and logarithms (e.g. $e^{\frac{1}{z}}$ at $z = 0$). It is a remarkable fact that one can form “formal” asymptotic expansions in terms of elementary functions such as polynomials of z , $\text{Log } z$, fractional powers of z and exponentials of fractional

powers z (e^{z^d} , d a rational number), and that these still determine essentially completely the solution.

The manner in which the solutions ramify or reproduce themselves, when one turns around the singular points, is called the “monodromy” of the equation, and is an important geometric invariant. In fact, in his admirable work on the hypergeometric equation, Riemann showed that the knowledge of the monodromy essentially determines the equation, so that, not quite obviously, the ramification of the holomorphic extensions of the solutions around complex points is deeply connected with their asymptotic behaviour on the real domain at these points.

There are several other important problems besides the problem of finding solutions with given initial data. Two of the most important are the following:

- the study of equations with periodic coefficients, important in particular because in many problems with geometrical relevance the variable t is an angle, so the coefficients are periodic (of period 2π) one first looks for periodic solutions. This is the purpose of the theory of Floquet.

- eigenfunction and eigenvalue problems (Sturm-Liouville problems). Typically if P is a linear (preferably self-adjoint) operator one looks for a function f satisfying

$$Pf = \lambda f$$

and suitable side conditions (e.g. the boundary conditions $f(a) = f(b) = 0$ if the domain is a finite interval $[a, b]$, or that f be square integrable if the domain is \mathbf{R}). Although the first equation always has solutions (in fact 2 linearly independent solutions), the boundary condition is not usually satisfied by any of them except for exceptional values of λ (eigenvalues). It is sometimes convenient to transform the problem into an equivalent integral equation: this was done by several authors at the beginning of the century (in particular in the Hilbert-Schmidt theory) and this is one of the first example of systematical use of functional analysis in problems concerning differential equations.

This problem is of crucial importance in quantum physics where the equation above is usually referred to as the stationary Schrödinger equation; the eigenvalues λ are energies, or authorized values of the observable, and the corresponding solutions are “eigenstates”. (In our world, which is rather 3-dimensional, partial differential operators arise naturally, and for those the problem is usually much harder. The 1-dimensional problem still arises in problems where there are many symmetries, and gives rise to remarkable symmetries or holomorphic extension properties, which could not possibly hold in higher dimension, so it also gives rise to a huge literature.)

2.2 The Laplace equation

The Laplace equation on \mathbf{R}^n (usually $n = 3$) is the equation

$$\Delta f = g$$

where Δ is the Laplace operator on \mathbf{R}^n :

$$(9) \quad \Delta = \sum \frac{\partial^2}{\partial x_k^2}$$

The Laplace operator is (up to a constant factor) the only linear second order operator which is invariant under translations and rotations. Physical laws are very often described by second

order equations, are linear (for the reasons explained above) and are expected to be translation and rotation invariant (independent of the position of the observer, and of the direction in which he is looking), so it is not surprising that the Laplace operator, or closely related operators, appear very often in many partial differential equations with physical or geometrical origin.

A typical case where the Laplace equation appears is in electrostatic theory, where f is the electric potential and g is the electric charge density (we ignore the dielectric constant ε_0 , which can be taken equal to 1 if units are accorded suitably).

For an elementary charge +1 at a point x_0 in 3-dimensional space, the potential is

$$(10) \quad V = \frac{1}{4\pi|x - x_0|}$$

so in general the solution is given by the linear superposition rule:

$$(11) \quad f(x) = \int d^3y \frac{g(y)}{4\pi|x - y|}$$

This rule was known since the 18th century. In fact it is easy to prove mathematically (to undergraduate students) that the integral above does give a solution if the charge density g vanishes outside of a bounded region of space (or decays suitably fast at infinity so that the integral is still defined); there are other solutions of the Laplace equation, but the formula above gives the only solution that vanishes at infinity.

(if $n = 3$ in general the denominator should be replaced by $|x - y|^{n-2}$ and the constant factor by $1/\text{volume of the unit sphere}$).

Another important problem in electrostatics is the *Dirichlet problem* where one computes a potential f inside a bounded room (domain Ω) when there is no charge, and f is known on the walls (boundary $\partial\Omega$):

$$(12) \quad \Delta f = 0 \quad \text{in } \Omega, \quad f = f_0 \quad \text{on } \partial\Omega \quad (f_0 \text{ given})$$

If Ω is the unit ball of \mathbf{R}^3 ($\partial\Omega$ the unit sphere), the solution f is

$$(13) \quad f = \frac{1}{4\pi} \int_{\text{sphere}} \frac{1 - |x|^2}{|x - y|^3} f(y) d\sigma(y)$$

where $d\sigma(y)$ denotes the usual volume element of the sphere. The function

$$K(x, y) = \frac{1}{4\pi} \frac{1 - |x|^2}{|x - y|^3}$$

is the Poisson kernel (on the unit ball of \mathbf{R}^n it is $\frac{1}{v_{n-1}} \frac{1 - |x|^2}{|x - y|^n}$ where $v_n = \frac{\pi^{n/2}}{\Gamma(n/2)}$ is the $(n - 1)$ -volume of the unit sphere. In general there is always a similar integral formula, although the kernel, i.e. the function replacing $\frac{1 - |x|^2}{|x - y|^3}$, may be harder to compute explicitly).

Thus a boundary data for which the Laplace equation is well-posed is the Dirichlet boundary condition:

$$(14) \quad f = 0 \quad \text{or} \quad f = f_0 \text{ (given)} \quad \text{on the boundary } \partial\Omega.$$

There are other good boundary conditions which, together with the Laplace equation, produce a posed problem; an important one is the Neuman boundary condition: $\frac{\partial f}{\partial n} = g$ where $\frac{\partial f}{\partial n}$ denotes the normal derivative - i.e. the derivative in the normal direction of the boundary (when the boundary is smooth enough to have a tangent plane).

2.3 The wave equation

The wave equation concern functions on $\mathbf{R} \times \mathbf{R}^n$ (for functions on \mathbf{R}^n depending also on the time t - $n + 1$ variables in all) representing a quantity which will vibrate or propagate with constant speed in time.

$$(15) \quad \frac{\partial^2 f}{\partial t^2} - \Delta f = 0$$

one also must study the wave equation with right hand side:

$$\frac{\partial^2 f}{\partial t^2} - \Delta f = g$$

The wave equation is satisfied in particular by the components of the electromagnetic field, as a consequence of the more complete Maxwell equations ($g = 0$ if there are no electric charges or currents). But it also appears in the description of the vibrations of a violin string or of a drum, of the propagation of waves in the sea or on a lake, of sound in the air etc. Note that in these examples the wave equation is only approximate: it is the linearization of a non linear wave equation describing more globally these phenomenons; for example the propagation of waves in the sea depends on the wave length - this is a typically nonlinear phenomenon, and the linear wave equation only gives a good approximation for wavelengths in a relatively narrow band, and it does not give account at all of deep waves like Tsunamis.

The side condition for which the wave equation is well posed is the full Cauchy data:

$$(16) \quad f = f_0, \quad \frac{\partial f}{\partial t} = f_1 \quad \text{for } t = 0$$

one must be careful that this works for the space-like initial surface $\{t = 0\}$ but would not work for arbitrary initial surface (e.g. it would not work for the initial surface $x = y = t = 0$ which is not space-like since the relativity form $x^2 + y^2 + z^2 - t^2$ changes sign on it).

The solution is given in terms of initial data (f_0 given, $f_1 = 0$) by

$$(17) \quad f(t, x) = \frac{1}{4\pi t^2} \int_{|x-y|=t} f_0(y) d\sigma(y)$$

where $d\sigma(y)$ is the natural volume element on the sphere $|y - x| = t$, i.e.; the integral is the mean value of f_0 on the sphere of center x and radius t .

The wave equation is (up a change of scale in time and length) the model equation for propagation of linear (or small) waves. A more sophisticated hyperbolic system is the system of Maxwell equations which describe the oscillations and propagation of the electromagnetic field described above; the Maxwell equations imply the wave equation, in the sense that each component of the solution field must satisfy the wave equation.

2.4 The heat equation and Schrödinger equation

The heat equation (for functions $f(t, x)$ on $\mathbf{R} \times \mathbf{R}^3$) is:

$$(18) \quad \left(\frac{\partial f}{\partial t} - \Delta\right)f = 0 \quad (\text{or some other right-hand side } g)$$

It is a model for the diffusion of heat in a conductive material.

The well posed boundary condition associated to the heat equation is the initial condition

$$(19) \quad f = 0 \quad \text{for } t = 0$$

Note that this is not the Cauchy condition for which there should be two conditions (since the heat operator is second order; one or 2 initial conditions on other initial surfaces do not work).

The solution of the initial problem for the heat equation is given by the formula

$$(20) \quad f(t, x) = (4\pi t)^{-\frac{3}{2}} \int \exp -\frac{(x-y)^2}{4t} f_0(y) dy$$

The heat equation is closely connected to probability theory: the heat kernel

$$(21) \quad \frac{1}{4\pi t} \exp \frac{-x^2}{4t}$$

is the probability of presence at time t (in suitable units of space and time) of a particle governed by brownian motion - as an afterthought this is quite reasonable if one remembers that the heat diffusion is made possible by the agitation of many particles which essentially follow the rules of brownian motion, as was explained by Einstein. Here again this physical intuitive reason can be proved mathematically (and anyway the heat kernel was known long before Brownian motion).

The Schrödinger equation

$$(22) \quad \left(\frac{1}{i} \frac{\partial f}{\partial t} - \Delta\right) f = 0$$

is, at least formally, closely related to the heat equation. Its solution with initial data f_0 for $t = 0$ is given by the formula

$$(23) \quad f(t, x) = (4i\pi t)^{-\frac{3}{2}} \int \exp -\frac{(x-y)^2}{4it} f_0(y) dy$$

However besides the formal analogy, which means that over the field of complex numbers the two equations are the same, there are deep differences for the analysis of real solutions. In particular the formula (23) gives the solution of the Schrödinger equation for all times - in fact it defines a group of isometries in the space L^2 of square integrable functions. On the other hand formula (20), which solves the heat evolution problem, usually only makes sense for $t \geq 0$ for an arbitrary initial data $f_0 \in L^2$ (when one cooks a cake, it is mostly impossible to go backwards and uncook it).

2.5 Equations of complex analysis

2.5.1 The Cauchy-Riemann equation

Many usual functions have one remarkable property: they extend in a natural fashion as functions over the complex plane, or part of it. This is not only true of polynomials (for which this extension property is obvious since their definition only uses additions and multiplications

which make sense for complex numbers), but it is also true for more complicated functions as exponentials, sine and cosine functions, logarithms, and many functions satisfying differential equations, like elliptic functions, hypergeometric functions etc. Functions of a complex variable had been known for some time but at the beginning of the 19th century Cauchy, and shortly afterwards Riemann, noticed and exploited in a very deep manner the fact that they are those functions of the complex variable $z = x + iy \in \mathbf{C}$ which satisfy the partial differential equation

$$(24) \quad \bar{\partial}f = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) = 0$$

\mathbf{C} is the complex plane, which is isomorphic to the real plane \mathbf{R}^2 : a complex number $z = x + iy$ is identified with the pair of real numbers (x, y) given by its real and imaginary parts; i is the basic complex number $i = \sqrt{-1}$; the conjugate of z is $\bar{z} = x - iy$.

Holomorphic functions of n variables $z = (z_1, \dots, z_n)$ satisfy the system of Cauchy-Riemann equations: $\bar{\partial}f = 0$ where $\bar{\partial}$ is the system

$$(25) \quad \bar{\partial} = (\bar{\partial}_1, \dots, \bar{\partial}_n), \quad \bar{\partial}_k = \frac{\partial}{\bar{z}_k} = \frac{1}{2}\left(\frac{\partial f}{\partial x_k} + i\frac{\partial f}{\partial y_k}\right)$$

These equations express that the infinitesimal increment (differential) df of f for infinitesimal increments dx_k, dy_k of the variables is \mathbf{C} -linear, i.e. it is a linear combination of the holomorphic increments $dz_k = dx_k + idy_k$, and does not contain the conjugates $d\bar{z}_k = dx_k - idy_k$

All partial differential equations shown so far have constant coefficients, i.e. their coefficients are constant real or complex numbers. Equations with constant coefficients are those which are invariant under translations - i.e. under a large commutative group. Such linear equations are well understood (see below); an equation with constant coefficients always has solutions, locally and globally, and for many of them one can find good boundary conditions which define a well-posed problem⁶. However if one picks a partial differential equation randomly, there is little chance that it will be a constant coefficient equation, even if one can choose new coordinates suitably. Non constant equations (potentially the majority, especially if no linear structure is specified) are often more complicated; they do not always have solutions, globally nor even locally, even if they arise very naturally in geometric problems, as in the two examples below. It was all the more an agreeable surprise when microlocal-analysis, developed between 1965-1975, showed that they still have in "generic" cases many common features, and explained how and why.

2.5.2 The Hans Lewy equation

It is the equation for functions on $\mathbf{C} \times \mathbf{R}$ or on \mathbf{R}^2 :

$$(26) \quad Hf = \frac{\partial f}{\partial \bar{z}} + \frac{iz}{2} \frac{\partial f}{\partial t} = g$$

As above we have set $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$. The Hans Lewy equation is a boundary Cauchy-Riemann equation in the following sense: let U be the domain of \mathbf{C}^2 (2 complex variables

⁶ This is linked to the fact that Fourier analysis on the group \mathbf{R}^n is particularly agreeable and well understood; some other equations have also a large group of symmetries, such as the Heisenberg group, which is not commutative but as close as possible to being so - e.g. the H. Lewy equation below, and nevertheless present the typical complications of general equations, e.g. they are not solvable.

$z = x + iy, w = s + it$) defined by the inequality $z\bar{z} - s < 0$ with boundary the “paraboloid $s = z\bar{z}$ (this is, after a suitable change of coordinates, equivalent to the unit ball, with boundary the unit sphere; complex domains of \mathbf{C}^2 are not all, by far, equivalent to this, but this is in some sense a rather generic and universal model). The operator $L = \frac{\partial}{\partial \bar{z}} + z \frac{\partial}{\partial w}$ kills all holomorphic functions, because it is a combination of the two basic Cauchy-Riemann derivations $\frac{\partial}{\partial \bar{z}}, \frac{\partial}{\partial w}$, and it is “tangent” to the boundary, so Lf makes sense if f is only defined on the boundary. The variables $z = x + iy, t$ are good parameters for the boundary (the last s being $= z\bar{z}$), and for a function $f = f(z, t)$ we get $L(f) = \frac{\partial f}{\partial \bar{z}} + \frac{i}{2}z \frac{\partial f}{\partial t} = H(f)$. Thus $Hf = 0$ is a necessary (and in fact also sufficient) condition for f to be the boundary value of a holomorphic function defined in the domain U , and $Hf = g$ is a “holomorphic boundary differential form”.

Now it turns out that the Hans-Lewy equation for arbitrary right hand side usually has no solution, i.e. not all functions g are (neither globally nor locally) derivatives of the form Hf . This is surprising, and slightly annoying since, by the Cauchy-Kowalewski theorem, it certainly has a solution, at least locally, if the right hand side g is analytic (cf. §3.1.1 below).

2.5.3 The Misohata equation

The Misohata equation is a weaker and simplified 2-variables version, on \mathbf{R}^2 , where we denote the variables x, t , which cannot usually be solved along the line $t = 0$:

$$(27) \quad Mf = \frac{\partial f}{\partial t} + it \frac{\partial f}{\partial x} = g$$

The equation $Mf = 0$ expresses that f is a holomorphic function of the variable $z = \frac{1}{2}t^2 + ix$ defined in the right half plane $t \geq 0$.

In particular the locally integrable function

$$K(x, t) = \frac{1}{x + it^2/2} = \lim_{\varepsilon \rightarrow +0} \frac{1}{x + i(t^2/2 + \varepsilon)} \quad (\varepsilon \rightarrow +0)$$

is a solution of the Misohata equation, in the distribution sense (see below).

Let us we define an integral boundary operator Bg by

$$Bg = G(x) = \int K(y - x, t) g(y, t) dy dt$$

Because the Misohata operator is self-adjoint (up to sign) we have, for any f vanishing outside of a bounded set:

$$(28) \quad \int K(y - x, t) Mf(y) dy dt = - \int MK(x + y, t) f(y) dy dt = 0$$

In general, if g coincides with a “derivative” Mf near a point $(x_0, t = 0)$, it follows that $G(x) = Bg$ is real analytic near the point x_0 , because the Schwartz Kernel $K(y - x, t)$ is real analytic for $x \neq y, t \neq 0$ (it is a rational function).

This necessary analyticity condition is in fact also sufficient for the Misohata equation $Mf = g$ to have a solution near x_0 ; but anyway the point is that the Misohata equation is not solvable (even near a given point x_0) for arbitrary right hand sides g .

The analysis can be better understood by making a partial Fourier transform (see below) with respect to x ; the Misohata equation becomes:

$$\widehat{Mf}(t, \xi) = \frac{\partial f}{\partial t} + t\xi \widehat{f} = \widehat{g}$$

which reduces the study to that of a family of ordinary differential equations, where one must take on account the fact that \widehat{f} must grow “moderately” (not too fast) for $\xi \rightarrow \infty$ because it is a Fourier transform.

It is convenient to expand everything in series of eigenfunctions of the “harmonic oscillator” (depending on ξ):

$$\frac{\partial^2}{\partial t^2} + \xi^2 t^2$$

for this operator the “ground state” is the quadratic exponential $u_0 = e^{-|\xi| \frac{t^2}{2}}$. The other eigenstates u_k can be expressed elementarily as products of Hermite polynomials and the same exponential. For $\xi < 0$, \widehat{M} behaves as a creation operator: $\widehat{M}u_k = c^{st} u_{k+1}$ (the successive eigenfunctions are, up to constant factors, the $u_k = \widehat{M}^k u_0$). On the other side for $\xi > 0$ it behaves like an annihilation operator $\widehat{M}u_k = c^{st} u_{k-1}$. It follows that for $\xi > 0$, \widehat{M} is onto but not one to one (at least for functions which do not grow too fast and can be expanded in a series of eigenfunctions u_k), and symmetrically for $\xi < 0$ it is one to one but not onto: the range is orthogonal to the ground state u_0 (this is essentially equivalent to equation (28); the fact that the final condition is that G be real analytic rather than identically 0 comes from the fact that the Fourier analysis does not reach functions of too rapid growth at infinity).

The Hans Lewy and Misohata equation are deeply and naturally connected with complex analysis, and with the Cauchy-Riemann equations, and it was at first unexpected that such unsolvable equations should exist. It is even more remarkable that, as microanalysis shows, all such generic equations are microlocally equivalent [the precise result is that if P is a partial differential or pseudodifferential operator of degree m such that at a characteristic point (x_0, ξ_0) (i.e. such that the symbol-leading part vanishes $p_m(x_0, \xi_0) = 0$), the commutator $[P, P^*]$ is elliptic (i.e. its leading part, of order $(2m - 1)$, does not vanish), then near this point P is equivalent, via an elliptic Fourier integral operator (playing the role of a change of coordinates) to the product of an elliptic operator (essentially invertible) and of the Misohata operator (with possibly more hidden variables, playing the role of parameters). The theory of Fourier integral operators, or microlocal analysis, is described in another chapter].

3 METHODS

3.1 Well posed problems

A first important question concerning partial differential equations is to say something about the “general solution” (or the class of all solutions), and this was the main concern till the 19th century.

However differential and partial differential equations usually have many solutions, and from physical examples one expects that one should give additional data, or side conditions such as boundary conditions, to pick out one specific solution. In many problems originating in physics, there are usually natural such side conditions, which together with the given equation or system of equations specify uniquely the solution. For instance if the solution-function $f(t, x)$ depends on the time t (mathematically, time could be any of the variables of the problem), one may want to prescribe initial conditions i.e. the values $f(0, x)$ of f at time $t = 0$ (or at some initial

time $t = t_0$); if the problem is defined on a bounded domain in space, one may assign boundary conditions, i.e. conditions on f and some of its derivatives on the boundary of the domain (those should be linear conditions in a linear problem).

Thus for a given equation one first important question is to determine if the equation is solvable, and what kind of problem can be solved concerning it, i.e. what kind of boundary conditions one ought to add to get what J. Hadamard called a well-posed problem.

For a good problem one usually expects that, in conjunction with the system of partial differential equations, these side conditions give rise to a *well-posed problem*, i.e. the solution is uniquely defined, and depends reasonably (continuously) on the data, so that a small error on the data will produce a small error on the result.

As mentioned above, for many problems originating in physics, the type of side conditions which should be added to get a well-posed problem is well known from physical experience (e.g. the Dirichlet boundary condition for the Laplace equation, or the Cauchy initial data for the wave equation). In general theory however, i.e. for abstract equations with no special physical meaning, the fact that there exist boundary conditions for which the problem is well posed, is generally true in dimension 1 (for differential equations), but in higher dimensions it may be false and anyway it may be quite hard to find what kind they should be ⁷

3.1.1 Initial value problem, Cauchy-Kowalewsky theorem

For differential equations (functions of one variable - \mathbf{R} -valued or vector valued) it was shown by A. Cauchy that the initial problem has a unique solution, i.e. there exists a unique solution of the differential equation

$$\frac{dy}{dt} = \Phi(t, y)$$

taking a given value $y(t_0) = 0$ at a given initial time t_0 , providing that the defining function Φ is regular enough.

For a linear equation $\frac{dy}{dt} = a(t)y + b(t)$ the solutions exist (is well defined) as far as the equation itself, and it depends continuously on the initial data, although the error may grow exponentially (very fast). [As mentioned above things are more complicated for non linear equations, for which solutions may not be defined for all times, as the functions $y(t) = \frac{y_0}{1-ty_0}$ which are solutions of the equation $\frac{dy}{dt} = y^2$ and explode at the blow-up time $t_0 = \frac{1}{y_0}$].

For partial differential equations (functions of two variables or more), the situation is much more complicated. A straightforward extension of Cauchy's theorem for the initial value problem was proved by S. Kowalewski (announced by Cauchy): for a partial differential equation of order n :

$$\frac{\partial^n f}{\partial t^n} - P(x, t, \partial^\alpha f) = 0$$

where the differential operator P (linear or not) contains only derivatives $\partial^\alpha f$ of order $\leq n$ and $< n$ with respect to t , the Cauchy data, or initial data of order m , is the collection of t -derivatives of order $< m$, restricted to the initial plane ($t = 0$):

$$\frac{\partial^j f}{\partial t^j} \Big|_{t=t_0}, j = 0, 1, \dots, n-1$$

⁷ the boundary data of solutions of a system of partial differential equations will usually itself satisfy an induced system of differential equations on the boundary. If one is only interested in solutions on one side of the boundary, there may be further integral (non local) relations, making things often complicated.

Then the Cauchy problem (initial value problem):

$$\frac{\partial^n f}{\partial t^n} - P(x, t, \partial^\alpha f) = 0 \quad \frac{\partial^j f}{\partial t^j} = f_j(x) \quad \text{for } j = 0, 1, \dots, n-1, \quad t = t_0$$

has a unique solution, given by a convergent power series (in small enough domain).

However this theorem requires a rather heavy restriction on the equation and on the data: they must be real analytic (i.e. locally expressible by convergent series of powers of the fluctuations of x and t). This is not so bad for the operator P , because most natural or useful equations are analytic in the first place; but it is a very heavy restriction on the initial data, reflecting the fact that, even for linear equations the solution usually exists only in a very small domain, and that it does not in any reasonable sense depend continuously on the initial data, and there may be no solution at all in general if the initial data is not regular enough.⁸

3.1.2 Other boundary conditions

For some equations, like the wave equation, the solution of the Cauchy problem is everywhere defined, and depends reasonably continuously on the initial Cauchy data. Such equations are called hyperbolic equations, and following J. Hadamard we say that for these the initial problem (or Cauchy problem) is well posed.

However there are other kinds of useful equations, e.g. the Laplace equation and heat equation, which are not hyperbolic. For such equations the solution of the Cauchy problem, and even its domain of definition, do not depend continuously or reasonably on the initial data; the initial Cauchy problem is not well-posed, and does not usually correspond to a problem with physical meaning.

For the heat equation

$$\frac{\partial f}{\partial t} - \Delta f = 0 \quad \text{or some other right-hand side } g$$

a reasonable initial condition is $f(0, x) = f_0$, i.e. one specifies f at time $t = 0$.

This is well-posed, provided the initial data f_0 does not grow too fast at infinity. It is not the Cauchy problem since the heat equation is second order.

For the Laplace equation in a domain Ω with boundary $\partial\Omega$, a well-posed problem is the Dirichlet problem

$$\begin{aligned} \Delta f &= 0 \quad \text{or} \quad \Delta f = g \\ f &= f_0 \quad \text{on} \quad \partial\Omega \end{aligned}$$

where the boundary data f_0 is a given function on the boundary. This again is not the Cauchy problem which for a second order operator would require one more condition on the derivatives of f .

Operators such as the Laplace operator Δ are called elliptic; they have the following remarkable property: a solution of the equation $\Delta f = 0$ is always a very smooth (regular) function - in fact analytic. The heat operator has a similar property, but not the wave operator.

At least for linear operators one can distinguish important interesting properties such as hyperbolicity or ellipticity. For nonlinear equations things are more complicated - in particular

⁸ Some genuinely physical phenomena, such as those which appear in the Kelvin-Helmholtz perturbation problem, are very unstable, and this reflects in the fact that they are only posed in an analytic context.

the fact that an equation behaves like an elliptic or hyperbolic equation may depend on the solution itself, so that it may be awkward to predict what kind of boundary conditions may be well-posed (e.g. the equation $f_{tt} - ff_{xx}$ is “hyperbolic” for $f > 0$ and elliptic for $f < 0$; it is very complicated to handle if f can change sign). Such difficulties occur in the study of models for stationary transonic flows.

3.2 Distributions

3.2.1 Distributions

Distributions or generalized functions are a useful and important tool in the theory of partial differential equations (more for linear equations). They appear in several manners.

In physical theories, they appear quite naturally in the following manner: points and functions are usually at best only defined up to some errors, and what is accessible to measure is not the values of a function f at a point, but rather its means on neighborhoods of points; almost equivalently f is known through the integrals

$$\int f\varphi$$

where the φ are suitable “test functions”. In measure theory the test functions are continuous functions with bounded support (i.e. vanishing outside of some bounded set), or characteristic functions of bounded sets. In distribution theory, the test functions are smooth functions φ with compact support (i.e. φ vanishes outside of a bounded set, and has continuous derivatives of arbitrarily high order). These form a vector space C_0^∞ , and distributions appear as linear forms on C_0^∞ (satisfying obvious continuity conditions).

There are many test functions (an example is the function $\varphi(x) = \exp \frac{1}{1-|x|^2}$ if $|x| < 1$, 0 if $|x| \geq 1$). In fact there are so many, that if f is a continuous function, it is completely determined by the distribution it defines ($f = g$ if $\int f\varphi = \int g\varphi$ for all test functions).

Furthermore derivatives of distributions are well defined: if f has continuous derivatives, integration by parts shows that we have

$$\left\langle \frac{\partial f}{\partial x_j}, \varphi \right\rangle = \int \frac{\partial f}{\partial x_j} \varphi = -\left\langle f, \frac{\partial \varphi}{\partial x_j} \right\rangle$$

and this gives the definition of the derivative of a distribution, extending the usual definition for differentiable functions.

Thus distributions appear as objects which generalize functions, and for which derivatives are always defined. Note that the derivatives of a continuous function is a well defined distribution, but is usually not a function at all. It can be shown that in this respect distributions are somehow a minimal extension of functions: any distribution is, in neighborhoods of a point, a finite sum of derivatives of continuous functions.

3.2.2 Weak Solutions

Distributions first appear in the description of generalized or weak solutions, which are used for all partial differential equations (linear or not), and were first introduced by B. Riemann in his memoir on atmospheric waves (1860). It is often convenient to replace a system of differential equations by an essentially equivalent system of integral or integro-differential equations (i.e. which is equivalent, at least for functions which are regular enough), or simply to require that the equations hold in the distribution sense, when this makes sense. In fact in physical problems,

e.g. in the electromagnetic theory, it is often the integral formulations which appeared first (the differential formulation is still more agreeable, because it is written in a very condensed algebraic form, and also because it is obvious from the way it is written that it deals with phenomena for which causes and effects are local).

For example the initial problem for an ordinary differential equation

$$\frac{dy}{dt} = \Phi(t, y), \quad y(t_0) = y_0$$

is equivalent to the integral equation

$$y(t) = y_0 + \int_{t_0}^t \Phi(s, y(s)) ds$$

In this case the two problems are equivalent, i.e. their solutions are the same, although the integral makes sense for less regular functions y (e.g. continuous or measurable functions, not necessarily differentiable). For partial differential systems (more than one variable) it often happens that the integral problem has solutions which are not differentiable so the differential problem does not make sense for them. Such solutions are called “weak” solutions of the problem. They are important and lead to a more complete description of the set of solutions; also the integral formulation is often better adapted to the powerful methods of functional analysis. For non linear equations (e.g. the Navier-Stokes equations) the consideration of weak solutions has led to important progress.

Weak solutions are sometimes still a little awkward to use, because in nonlinear problems there may be several possible formulations of a related integral problem and the notion of weak solution may depend on this choice⁹. In linear problems, “weak” solutions are always special cases of distributions, whose definition is quite universal.

3.2.3 Elementary solutions

Another instance where distributions appear is in formulas such as those which give the solution of a partial differential equation (the fundamental solutions above). These formulas often involve “singular integrals”, i.e. limits of integrals which “involve” higher derivatives, over \mathbf{R}^n or suitable submanifolds, which behave like integrals but are not integrals in the sense of measure theory.

For example the Laplace operator is $\Delta = \sum \partial_{x_j}^2$, and for $n \geq 3$ the Laplace equation $\Delta f = g$ is solved by the integral formula

$$f(x) = \int g(x - y)K(y)dy$$

where K is the elementary solution:

$$K(y) = C \left(\sum y_j^2 \right)^{-\frac{n-2}{2}}$$

⁹ A simple example is the following: the equations $\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x}$ or $\frac{\partial}{\partial t} \frac{u^p}{p} + \frac{\partial}{\partial x} \frac{u^{p+1}}{p+1}$ (p real > 0) are equivalent for a differentiable function u . But for $u \in L^\infty$ (bounded but possibly discontinuous), in the distribution sense, they are not - e.g. if u has a simple jump along a line $x = kt$, the equation in the distribution sense implies a relation between the values on the left and on the right of the line which explicitly depends on p , although this dependence vanishes for infinitely small jumps.

(the constant C is the inverse of the $(n-1)$ -volume of the unit sphere; if g is a bounded function with compact support, f is the only continuous solution which vanishes at ∞).

The D'Alembertian (wave operator) is the second order operator:

$$\square = \partial_{x_1}^2 - \sum_2^n \partial_{x_j}^2$$

(all signs changed but one). If $n \geq 3$ is odd, the wave equation $\square f = g$ is solved by a similar formula

$$f(x) = \langle K_1(y), f(x-y) \rangle$$

where again $K_1(y) = C (y_1^2 - \sum_2^n y_j^2)^{-\frac{n-2}{2}}$ if $y_1 > (\sum_2^n y_j^2)^{\frac{1}{2}}$, = 0 otherwise. However if $n \geq 5$, K_1 is not integrable and must be reinterpreted as a distribution. (If n is even, e.g. in the physical case $n = 4$, there is a similar integral formula, where the integral is over the forward light-cone $y_1 = (\sum_2^n y_j^2)^{\frac{1}{2}}$, and is again a singular integral if $n \geq 6$, so the elementary solution should be viewed as a distribution).

The theory of distributions leads to a rather universal description of such "singular integrals". Special cases of distributions appeared quite early in formulas containing singular integrals - they are also quite visible in the work of P. Dirac; general closely related constructions were described by S.L. Sobolev; the general definition, based on a systematic use of integration by parts as sketched above, was given by L. Schwartz in 1952.

3.3 Fourier analysis

3.3.1 Fourier transform

One of the oldest and most powerful and important methods is the Fourier transformation, which corresponds to the physical idea of decomposing a movement as a superposition of vibrations of various frequencies (waves).

A periodic function of period 2π can be written as a Fourier series:

$$f(x) = \sum_{-\infty}^{+\infty} f_n e^{nix}$$

where the coefficients f_n are the numbers:

$$f_n = \frac{1}{2\pi} \int_0^{2\pi} f(y) e^{-nit} dy.$$

For a general function f on \mathbf{R}^n the Fourier transform is defined as the integral

$$\hat{f}(\xi) = \int_{\mathbf{R}^n} e^{-ix\xi} f(x) dx$$

and f can be reconstructed from its Fourier transform by a similar formula (reciprocity formula):

$$f(x) = (2\pi)^{-n} \int_{\mathbf{R}^n} e^{ix\xi} \hat{f}(\xi) d\xi$$

Fourier first used such series to study the heat equation: the periodic solution of the heat equation $\partial_t f = \Delta f$ with initial data $f(t) = \sum f_n e^{nit}$ is

$$f(t, x) = \sum f_n e^{nix - n^2 t}$$

The Fourier transformation is at first defined for integrable functions, but it can be extended as a one to one linear transformation on a large class of distributions: the space of tempered distributions (these are distributions which in the mean do not grow faster than a polynomial at infinity). In particular it is well defined for square-integrable functions, and in fact produces a linear isometry on the space $L^2(\mathbf{R}^n)$ (except for a factor $(2\pi)^{-n}$).

The Fourier transformation is an extremely useful and powerful tool, in particular for the study of differential equations with constant coefficients. It is closely connected to the additive group law of vectors in \mathbf{R}^n , but microanalysis has shown that ideas related to the Fourier transformation (decomposition of functions in vibrations of various frequencies) are still quite useful in the study of equations with non-constant coefficients, and yield quite exact results in asymptotic problems.

3.3.2 Equations with constant coefficients

For non linear equations, deep and difficult methods have been elaborated in many important cases, but no really general method exists. In contrast the theory of linear partial differential equations has given rise to general methods and constructions.

A partial differential equation with constant coefficients is an equation on \mathbf{R}^n of the form

$$(29) \quad \sum a_\alpha \partial^\alpha f = g$$

where the coefficients are real or complex numbers (constants). One also considers systems of such equations (i.e. where f and g are vector-valued functions, and the a_α are constant matrices).

Except the two last examples, the examples of §2 are partial differential equations with constant coefficients.

For such equations Fourier analysis and rather general considerations on limits give an essentially complete treatment. Fourier transformation turns the problem into a multiplication (or rather division problem)

$$(30) \quad P(\xi) \hat{f} = \hat{g}$$

where $P(\xi) = \sum a_\alpha (i\xi)^\alpha$ is a polynomial (with constant complex or matrix coefficients), and the unknown f or the data g are in a suitable space of Fourier transforms of distributions (depending on the problem one wants to solve). It turns out that this rather algebraic problem of division by such polynomials, although difficult, is now quite well understood, and gives reasonably complete answers to many questions concerning the corresponding partial differential equations, such as the existence of local or global solutions.

However even in the one variable case there remain many useful cases which are not translation invariant and cannot be completely treated by frequency analysis alone.

3.3.3 Asymptotic analysis, microanalysis

The Fourier transformation depends on the additive group structure of \mathbf{R}^n , and one could expect that it is less useful for problems unrelated to this group law (e.g. equations with

nonconstant coefficient, i.e. which are not translation invariant). Microlocal analysis, which was developed in the late sixties, shows that high frequency asymptotics, so as vanishing Planck constant asymptotics, can be studied using the ideas of the Fourier transformation, and that the results are completely intrinsic and do not depend on a choice of coordinates.

Some motivations and aspects of microanalysis are described in another chapter, and here we will just briefly sketch what it achieves.

A first ingredient is the definition of the microsupport, or wave-front set, of a distribution: a distribution T is smooth (C^∞) in a neighborhood of a point x_0 if there exists a truncation function $\varphi \in C_0^\infty$ such that $\varphi(x_0) \neq 0$ and $\varphi T \in C_0^\infty$. On the other hand if T has compact support, it is smooth if its Fourier transform \widehat{T} is rapidly decreasing at ∞ , and it is natural to say that T is smooth in the direction of a non zero covector ξ_0 if $\widehat{T}(\xi)$ is of rapid decrease in a small conic sector around ξ (of the form $\{\frac{\xi}{|\xi|} - \frac{\xi_0}{|\xi_0|} < \varepsilon\}$).

Putting the two together, one says that T is smooth at a non-zero covector (x_0, ξ_0) ($\xi_0 \neq 0$) if there exists a smooth cutoff function φ with $\varphi(0) \neq 0$ such that $\mathcal{F}(\varphi T)$ is of rapid decrease in a small conic sector around ξ_0 as above.

The microsupport $SS T$ is the set of all covectors at which T is not smooth. It turns out that this is intrinsic, i.e. does not depend on a choice of local coordinates used to define it.

Closely related to the consideration of wave front sets is the study of rapidly oscillating functions: let φ be a real smooth function with non vanishing derivative $d\varphi$. If $P = P(x, D)$ is a differential operator of order m , t a large real parameter, then

$$e^{-it\varphi} P e^{it\varphi} = t^m p_m(x, d\varphi) + t^{m-1} P_{1,\varphi} + \dots + t^{m-j} P_{j,\varphi} + \dots$$

is a polynomial in t ; the coefficient P_j of t^{m-j} is a differential operator of order j ; the leading term can be read out of the leading term of P : it is $\sum_{|\alpha|=m} p_\alpha(x) (d\varphi)^\alpha$, the ‘‘symbol’’ of P evaluated at $d\varphi$. (this is an easy remark, starting with the fact that we have $e^{-it\varphi} D_j e^{it\varphi} = t \frac{\partial \varphi}{\partial x_j} + D_j$.)

It is useful to study asymptotic equations of the form

$$(31) \quad P(e^{it\varphi} a) = e^{it\varphi} b$$

when a, b are asymptotic expansions w.r. to negative powers of t of the form

$$(32) \quad a \sim \sum_{k \leq k_0} t^k a_k(x)$$

The leading term of $P(e^{it\varphi} a)$ is $p_m(x, d\varphi) a$ so if $P(x, d\varphi) \neq 0$ (31) has a unique asymptotic solution as in (32).

It is manifest that this asymptotic calculus is local, not only in the basic x variables, but also in $\xi = \frac{\partial}{\partial x}$, i.e. microlocal calculus lives locally on the set of cotangent vectors T^*X . It cannot be completely local because it only works if something - the parameter t above, or the length of a frequency - goes to infinity. In fact there is a sharp limit to localization in both the x and ξ variables, closely related to the Heisenberg uncertainty principle, but most of the time microlocalisation remains safely below this limit.

It has been noted that the cotangent bundle T^*X is equipped with a symplectic structure: to each smooth function $f(x, \xi)$ is associated a hamiltonian vector field

$$H_f = \sum \frac{\partial f}{\partial \xi_j} \frac{\partial}{\partial x_j} - \frac{\partial f}{\partial x_j} \frac{\partial}{\partial \xi_j}$$

Also the symbol (leading term) of the commutator of two differential operators P, Q with symbols $p = \sigma(P), q = \sigma(Q)$ (homogeneous polynomials of ξ) is the Poisson bracket

$$\sigma([P, Q]) = \frac{1}{i}\{p, q\} = \frac{1}{i}H_p(q)$$

So it is natural that symplectic geometry is an essential ingredient in microlocal analysis.

Microanalysis has constructed powerful tools. The first is the theory of pseudo-differential operators. These are a microlocal generalisation of differential operators: the symbol is a homogeneous function of ξ , not necessarily a polynomial; the calculus is very similar i.e. although there is no longer a simple “algebraic-differential” formula for compositions $P \circ Q$, the same Leibniz rule for differential operators remains valid as an asymptotic formula, and just as useful. In particular a pseudo-differential operator P has a “symbol” (dominant term) $p = \sigma(P)$ which is a function $p(w, \xi)$ on the cotangent bundle, homogeneous with respect to ξ (of degree d if P is of degree d) and we have, as for differential operators (if $d = \deg(P), d' = \deg(Q)$):

$$\sigma_{d+d'}(PQ) = \sigma_d(P)\sigma_{d'}(Q) \quad \sigma_{d+d'-1}([P, Q]) = \{\sigma_d(P), \sigma_{d'}(Q)\}$$

Another important construction is that of “Fourier integral distributions” and “Fourier integral operators”. When studying a partial differential equation, it is of course useful to choose local coordinates in which it takes a simplified form (e.g. for an equation $\sum a_j(x) \frac{\partial f}{\partial x_j} = g$ near a point where the a_j do not all vanish, there is always a set of local coordinates (y_1, \dots, y_n) (as smooth as the data) in which the equation reads $\frac{\partial f}{\partial y_1} = g$). Fourier integral operators make it possible to perform “canonical” local changes of coordinates (i.e. homogeneous, preserving the symplectic structure), keeping track of microsupports.

Microanalysis has provided many striking results about the location of singularities of solutions of differential equations.

We will just describe one of the first and easiest: let $P = P(x, D)$ be a differential operator. We will say that P is real with simple characteristics if the symbol $p = \sigma(P)$ is a real valued polynomial of ξ and the hamiltonian H_p never vanishes for $\xi \neq 0$:

Theorem 1 ((Propagation of singularities)) *Let P be a real operator with simple characteristics. If f is a distribution solution of P , or just $Pf \in C^\infty$, the microsupport $SS f$ is contained in the characteristic set $\text{char} P = \{p = 0\}$ and is invariant by H_p , i.e. is a union of “bicharacteristics” curves (integral curves of the hamiltonian field H_p , contained in the characteristic set $\{p = 0\}$).*

This is proved by showing that if the Hamiltonian field H_p is not parallel to the radial vector $\sum \xi_j \frac{\partial}{\partial \xi_j}$ (generator of homotheties) then there exists a Fourier integral operator F (microlocal change of coordinates) which transforms P in $FPF^{-1} \sim A \circ D_1$ with A elliptic (invertible). For D_1 or AD_1 the result is easy to prove.

For example if $P = \frac{\partial^2}{\partial t^2} - \Delta_x$ is the wave operator, the bicharacteristics are the straight lines $x = x_0 + (t - t_0) \frac{\xi}{\tau}, \xi, \tau = \text{constants}, |\tau| = \|\xi\| \neq 0$ corresponding to light rays, and the propagation theorem says that the singular support of any distribution solution of the wave equation is a union of light rays. This is one way of understanding the link between geometric and wave optics as explained in the chapter on microlocal analysis.

Fourier integral distributions are also quite important because they systematically appear in many “explicit” formulas. For analytic such objects, the theory of microfunction and microdifferential of M. Sato, T. Kawai, M. Kashiwara [17] gives a very profound insight, showing how “holonomic” distributions extend to the complex domain as ramified holomorphic functions, singular along suitable complex hypersurfaces. This theory uses deeply the modern theory of analytic functions of several complex variables; we will not describe it further here.

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