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Complex Star Algebras

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Abstract. We describe a classification of star algebras on the cotangent bundle of a complex manifold, locally isomorphic to the algebra of pseudo-differential operators ; this requires a slight extension of the usual definition of star algebras. We show that in dimension ≥ 3 these are essentially trivial and come from algebras of differential operators on X ; in dimension 1 and 2 there are many more, which we describe.

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Contents

1 Introduction

Let us first recall what a star-product is (detailed definitions are given in section 2) : let X be a manifold and let $\widehat{\mathcal{O}}$ denote the algebra of formal series

$$f = \sum_{k \geq k_0} f_k h^k$$

where the f_k are smooth functions on X and h is a "small" formal parameter. A star product on X is a unitary algebra law on $\widehat{\mathcal{O}}$ for which the unit is 1 and the product is local, i.e. given by a formula :

$$f, g \rightarrow B(f,g) = fg + \sum_{k \geq k_0} h^k B_k(f,g)$$

where the B_k are bidifferential operators on X : in local coordinates $B_k(f,g) = \sum a_{\alpha\beta}\partial^\alpha f \partial^\beta g$ with smooth coefficients $a_{\alpha\beta}$ (it is further required that the unit is 1, i.e. $B_0(f,g) = fg$ and $B_k(1,f) = B_k(f,1) = 0$ for any $k > 0$ and any f ; the addition law is the usual addition of $\widehat{\mathcal{O}}$. A star product can be thought of as a non-commutative deformation of the usual product. The leading term of commutators $\{f,g\} = hB_1(f,g) - hB_1(g,f)$ defines a Poisson bracket on X (star products are also called "deformation quantization of Poisson manifolds").

In this paper I will use a slightly extended definition, where star products live on cones. A cone Σ with basis $B\Sigma = X$ is the complement of the zero section in a line bundle $L \rightarrow X$ (a complex line bundle if X is a complex manifold, and preferably a half-line bundle if X is real) ; in the semi-classical case above $\Sigma = X \times \mathbf{R}_+$ and $h = \frac{1}{r}$ if r denotes the fiber variable (the small "Planck constant" plays the role of the inverse of a large frequency). In this context $\widehat{\mathcal{O}}$ is the set of formal series $f = \sum_{k \leq k_0} f_k$ where for each k, f_k is a function homogeneous of degree k on Σ and, locally, a star product is defined as above as a bidifferential product law on $\widehat{\mathcal{O}}$

$$f, g \rightarrow B(f,g) = fg + \sum_{k \leq k_0} B_k(f,g)$$

where B_k is now a bidifferential operator on Σ, homogeneous of degree $k \rightarrow -\infty$ with respect to fiber homotheties. The B_k may involve derivations in any direction, so there is no longer a distinguished "Planck constant" commuting with the rest [1]. The associated Poisson bracket now lives on Σ and is homogeneous of degree -1. This definition includes the algebras of pseudo-differential operators or Toeplitz operators, which are after all among the most important and belong to the same formalism.

Complex star algebras arrive naturally and unavoidably in many problems concerning differential operators, whose symbols are polynomials and always

[1] There is absolutely no reason that the Planck constant should commute with the rest, especially when it is a parameter without physical significance

live on a complex manifold. So it is important to study them, and to study their relations with "polynomial" objects associated to differential operators.

In his paper [22] M. Kontsevitch has shown that any homogeneous Poisson bracket on a real manifold comes from a star product. His proofs extend without changing a word to star -products on a cone. Kontsevitch's formula giving a star product from a Poisson bracket on an affine space also works without any modification in the complex case (i.e. $\Sigma = \mathbf{C}^n \times \mathbf{C}^\times$). But the argument used to go from local to global does not work for complex manifolds, because it uses in an unavoidable manner partitions of unity and tubular neighborhoods. In general I do not know if a global star product exists for a given Poisson bracket, even in the symplectic case, nor do I know what the classification of such algebras looks like (see however [20], where it is shown that even if such an algebra $\mathcal{E}$ may not exist, the category of sheaves of $\mathcal{E}$-modules can be defined up to equivalence).

In this paper I investigate those star algebras which live on a complex cotangent cone $T^*X - \{0\}$ deprived of its zero section, equipped with its canonical symplectic Poisson bracket. All star algebras associated to this Poisson bracket are locally isomorphic, and there exists a global such algebra, viz. the algebra of pseudo-differential operators ; so there is at least a starting point for the classification. This will turn out to be essentially trivial in dimension $n \geq 3$ (Theorem1), but instructively not in dimension 2. More precisely algebras over a manifold X of dimension 2 or ≥ 2 are described in section 4, and compared to $\mathcal{D}$-algebras, i.e. sheaves of algebras over X locally isomorphic to $\mathcal{E}$, the algebra of pseudo-differential operators coming from differential operators on X. It turns out that if $\dim X \geq 3$ we get nothing new : the functor which takes a $\mathcal{D}$-algebra to the associated star-algebra is an equivalence. If $\dim X = 2$ the same functor is faithful, i.e. two $\mathcal{D}$-algebras are isomorphic if and only if the associated star-algebras are isomorphic, and an isomorphism between such star-algebras comes from a unique isomorphism between the original $\mathcal{D}$-algebras ; however there are in general many more "exotic" star-algebras which do not come from a $\mathcal{D}$-algebra.

If X is of dimension 1 the classification depends on whether X is open, of genus ≥ 2, of genus 1 or of genus 0.

An inner automorphism of the algebra $\mathcal{E}$ of pseudo-differential operators on X ($U : P \to APA^{-1}$) has a symbol $\sigma(U) = d\,\mathrm{Log}\,\sigma(A)$, which is a section of the sheaf ω (on the "basis" $B\Sigma = \Sigma/\mathbf{C}^\times$ of closed forms homogeneous of degree 0 on Σ, and an exponent which is the degree of A ; we will see in section 2 that any automorphism U of $\mathcal{E}$ has likewise a symbol and an exponent $\in \mathbf{C}$. Similarly a star algebra has a symbol $\sigma(\mathcal{A}) \in H^1(B\Sigma, \omega)$ and an exponent $\in H^1(B\Sigma, \mathbf{C})$. We will see in section 3 that if X is an open curve or a curve of genus ≥ 1, star algebras on Σ are completely determined by their exponent. The classification is more subtle when X is closed of genus 1 or 0.

The techniques used in this paper are a mixture of non-commutative cohomology, holomorphic cohomology, and the relation between the cohomology of a sheaf with a filtration and the cohomology of the associated graded sheaf. This contains nothing really new or difficult, but the mixture can be somewhat muddling.

As far as I know the questions studied here have not been investigated before and the results are new.

In sections 2 and 3 we recall the definition of star algebras, and some classification principles.

In section 4 we describe the classification when $\dim X \geq 2$.

In section 5 we describe the case where X is a curve ($\dim X = 1$) : results are substantially different if X is open, $X = P_1$, X is of genus 1, or X is of genus $g \geq 2$.

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2 Star Algebras

2.1 Cones

Definition 1 *A real (resp. complex) cone is a C^∞ (resp. holomorphic) principal bundle Σ with group $\mathbf{R}_+^\times$ (resp. $\mathbf{C}^\times$). The basis is $B\Sigma = \Sigma/\mathbf{R}_+^\times$ (resp. $\Sigma/\mathbf{C}^\times$).*

A real cone is isomorphic to a product cone $B\Sigma \times \mathbf{R}_+^\times$.[2] A complex cone is isomorphic to $L - \{0\}$ (L deprived of its zero section) where L is a complex line bundle over $B\Sigma$. L will usually not be a trivial bundle.

Definition 2 *(i) We denote $\mathcal{O}(m)$ the sheaf on $B\Sigma$ of homogeneous functions of degree m of Σ (holomorphic in the complex case).*

(ii) We denote $\widehat{\mathcal{O}}$ the sheaf on $B\Sigma$ of formal symbols ("asymptotic expansions" for $\xi \to \infty$ in Σ) :

$$(1) \qquad f \in \widehat{\mathcal{O}} \quad \text{if} \quad f = \sum_{m \leq m_0} f_m \quad \text{with} \quad f_m \in \mathcal{O}(m)$$

(m an integer, $m \to -\infty$).

Definition 3 *For an integer $k \geq 1$ we denote $\widehat{\mathcal{D}}_k$ the sheaf (on $B\Sigma$) of formal k-differential operators : $P(f_1, \ldots, f_k) = \sum_{m \leq m_0} P_m(f_1, \ldots, f_k)$ with P_m a k-linear differential operator homogeneous of degree m with respect to homotheties (m an integer, $m \to -\infty$).*

If $k = 1$ we will just write $\widehat{\mathcal{D}}$.

[2] at least if we are dealing with paracompact manifolds, which will always be the case in this article.

Locally Σ is a product cone and we may choose homogeneous coordinates (real or complex) x_j of degree 0 on the basis, and r of degree 1 on the fiber. Then $P_m(f_1, \ldots, f_k)$ is a sum of monomials

$$\varphi(x)\ r^m\, \partial_x^{\alpha_1}(r\partial_r)^{m_1}(f_1)\ldots\partial_x^{\alpha_k}(r\partial_r)^{m_k}(f_k).$$

There is no restriction on the order of P_m.

The presence of two "degrees" is confusing so in what follows **degree** will always refer to the degree with respect to homotheties, and **order** refers to the degree as a differential operator; thus if $P \in \widehat{\mathcal{D}}_k$ each term P_m of degree m is of finite order, although the resulting infinite sum P may be of infinite order.

We will denote $\widehat{\mathcal{D}}^\times \subset \widehat{\mathcal{D}}$ the sheaf of invertible formal differential operators : $P = \sum P_k \in \widehat{\mathcal{D}}^\times$ is invertible iff its leading term $\sigma(P) = P_{m_0}$ is invertible, i.e. P_{m_0} is of order 0, the multiplication by a nonvanishing function homogeneous of degree m_0. We denote by $\widehat{\mathcal{D}}^\times_-$ the subsheaf of those invertible P such that $P(1) = 1$, i.e. P is of degree 0, its leading term is $P_0 = 1$ and terms of lower degree have no constant term : $P_m(1) = 0$ if $m < 0$.

Remark 1 Sheaves are of course useless in the real case but must be used in the complex case where global sections do not necessarily exist.

Remark 2 For analytic cones there is also a notion of convergent symbol (introduced by the author in [6] to define analytic pseudodifferential operators). These are in fact the more important and for many questions it is essential to use convergent rather than formal symbols.[3] However for the classification results below, there is no significant qualitative difference between formal and convergent symbols, so we will stick to formal symbols and avoid convergence technicalities.

2.2 Star Products on a Real or Complex Cone.

Definition 4 *A star product on Σ is a sheaf $\mathcal{A}$ on the basis $B\Sigma$, locally isomorphic to $\widehat{\mathcal{O}}$ as a sheaf of vector spaces (the structural sheaf of groups is described below), equipped with an associative unitary algebra law whose product (star product) $f * g = B(f, g)$ is locally a formal bidifferential operator.*

Locally $f * g = \sum B_m(f, g)$ with B_m a bidifferential operator homogeneous of degree $m \to -\infty$, $B_0 = 1$. The first idea is that the structural sheaf of groups used to patch together local frames of $\mathcal{A}$ is the sheaf $\widehat{\mathcal{D}}^\times$ (on $B\Sigma$) of invertible formal differential operators, but there is a unit that we can choose equal to 1 in all local frames so this obviously reduces to $\widehat{\mathcal{D}}^\times_-$.

[3] e.g. convergent rather than formal symbols are essential in the finiteness theorems of T. Kawai and M. Kashiwara [22], or for going from $\mathcal{E}$-modules to $\mathcal{D}$-modules in the thesis of D. Meyer [23], and probably in most problems involving a comparison between $\mathcal{E}$ and $\mathcal{D}$-modules.

Note that homotheties (hence degrees) are not respected by $\widehat{\mathcal{D}}_-^\times$. However if $P \in \widehat{\mathcal{D}}_-^\times$, f and Pf have the same leading term ; so P respects the filtration defined by degrees ($f \in \widehat{\mathcal{O}}_m$ if $f = \sum_{j \le m} f_j$) and $\mathrm{gr}\, P$ is the identity on $\mathrm{gr}\, \widehat{\mathcal{O}} = \bigoplus \mathcal{O}(m)$.

In the semi-classical definition, Σ is a product cone $\Sigma = B\Sigma \times L$ ($L = \mathbf{R}_+$ or $\mathbf{C}^\times$), the star product law is defined on $\widehat{\mathcal{O}}$ and does not involve vertical derivatives, so the "Planck constant" $h = r^{-1}$ plays the role of a constant. The definition above includes the "semi-classical" case and also the algebras of pseudodifferential or Toeplitz operators. This conic framework for star products was described in [4].

In the real case, using partitions of unity, it is immediate to see that $\mathcal{A}$ is always isomorphic to $\widehat{\mathcal{O}}$ as a sheaf ("there exists a global total symbolic calculus"). This is no longer true in the complex case, and in particular it is not true in the most simple and natural examples as we will see below, so the sheaf theoretic presentation cannot be avoided.

2.3 Associated Poisson bracket

If $\mathcal{A}$ is a star algebra on Σ it has a canonical filtration coming from the filtration of $\widehat{\mathcal{O}}$ by homogeneity degrees, and there is a canonical isomorphism :

$$\mathrm{gr}\, \mathcal{A} \simeq \widehat{\mathcal{O}}$$

because the structural sheaf of groups $\widehat{\mathcal{D}}_-^\times$ induces the identity on $\mathrm{gr}\, \widehat{\mathcal{O}}$). The commutator law then defines a Poisson structure on $\mathrm{gr}\, \mathcal{A} = \widehat{\mathcal{O}}$ i.e. the leading term of the commutator law

$$\{f, g\} = B_1(f, g) - B_1(g, f)$$

is a Poisson bracket on Σ, homogeneous of degree -1. This means that it is a skew-symmetric bivector field

$$\{f, g\} = -\{g, f\}, \quad \{f, gh\} = \{f, g\}h + g\{f, h\}$$

satisfying the Jacobi identity (i.e. it is a Lie bracket) :

$$\{f\{g, h\}\} = \{\{f, g\}h\} + \{g\{f, h\}\}$$

and it is homogeneous of degree -1 with respect to homotheties

$$\deg\{f, g\} = \deg f + \deg g - 1 \quad \text{if } f, g \text{ are homogeneous.}$$

Existence of a global star-algebra on a real symplectic cone Σ was proved by V. Guillemin and myself in [5] (see also [3]), and by M. De Wilde and P. Lecomte ([9],[10]) in the semiclassical symplectic case (cf. also the nice deformation proof of B.V. Fedosov [11]).

In [22] M. Kontsevitch proved that any Poisson bracket comes from a star-product in the real semiclassical case. More precisely he proves that there is a one to one correspondence between isomorphic classes of star-products and isomorphic classes of formal families of Poisson brackets depending on the "small parameter" h. His result extends, without changing a word, to star-products on a real cone with the definition above ; families of Poisson brackets should be replaced by formal Poisson brackets on Σ :

$$c = \sum_{k \leq -1} c_m. \tag{2}$$

Kontsevitch's formula giving a star product from a Poisson bracket on an affine space also works without any modification in the complex case (i.e. $\Sigma = \mathbf{C}^n \times \mathbf{C}^\times$). But as mentioned above the argument used to go from local to global does not work for complex manifolds, and in general I do not know if a global star product exists for a given Poisson bracket, even in the symplectic case, nor do I know what the classification of such algebras looks like (see however [20], where it is shown that even if $\mathcal{E}$ may not exist, the category of sheaves of $\mathcal{E}$-modules is defined up to equivalence).

In the rest of the paper we investigate a special class of star algebras, i.e. those which live on a cotangent bundle $\Sigma = T^*X - \{0\}$, X a complex manifold, equipped with its canonical Poisson bracket. In this case there is a canonical global star-algebra, viz. the algebra $\mathcal{E}$ of pseudo-differential operators, which is the "microlocalization" of the sheaf $\mathcal{D}_X$ of differential operators on X. It is known and easy (cf. below) that any two star algebras with the same symplectic Poisson bracket are locally isomorphic, so our algebras are classified by $H^1(B\Sigma, \operatorname{Aut}\mathcal{E})$. It is also interesting to compare these with algebras of differential operators, locally isomorphic to $\mathcal{D}_X$ on X hence classified by $H^1(X, \operatorname{Aut}\mathcal{D})$: this is done in the next three sections.

3 Pseudo-differential Algebras

3.1 $\mathcal{E}$-algebras

Let $\Sigma = T^*X - \{0\}$ be the cotangent bundle (minus the zero section) of a complex manifold X, equipped with its canonical symplectic structure. The basis is $B\Sigma = \Sigma/\mathbf{C}^\times = PX$, the projective cotangent bundle. There is a canonical star algebra on Σ, viz. the algebra of pseudo-differential operators, microlocalization of the algebra of differential operators on X, whose Poisson bracket is the standard Poisson bracket of T^*X. If we choose local coordinates $x = (x_1, \ldots, x_n)$ on X and the dual cotangent coordinates $\xi = (\xi_1, \ldots, \xi_n)$ on the fibers, the pseudodifferential product is given by the Leibniz rule for symbols $f, g \in \widehat{\mathcal{O}}$:

$$f * g = \sum \frac{1}{\alpha!} \, \partial_\xi^\alpha f \; \partial_x^\alpha g. \tag{3}$$

The patching cocycle is the cocycle defined by changes of coordinates : this is a cocycle because it does patch together total symbols of differential operators (locally : polynomials in ξ), to give the global sheaf $\mathcal{D}_X$ of differential operators.

We are interested in star algebras on Σ associated to the canonical Poisson bracket : we will call $\mathcal{E}$-algebra such an algebra.

Proposition 1 *Any $\mathcal{E}$-algebras is locally isomorphic to $\mathcal{E}$ through an operator $P \in \widehat{\mathcal{D}}_{-}^{\times}$.*

This result is well known and we just give an indication of the proof : locally the pseudo-differential algebra $\mathcal{E}$ has (topological) generators x_i, ξ_i satisfying the canonical relations

$$[x_i, x_j] = [\xi_i, \xi_j] = [\xi_i, x_j] - \delta_{ij} = 0.$$

If $\mathcal{A}$ is a star algebra with the same Poisson bracket, one can construct by successive approximations symbols X_i, Ξ_i with the same principal part as x_i, ξ_i and satisfying the same canonical relations

$$[X_i, X_j]_{\mathcal{A}} = [\Xi_i, \Xi_j]_{\mathcal{A}} = [\Xi_i, X_j]_{\mathcal{A}} - \delta_{ij} = 0.$$

Now there is a unique isomorphism $U : \mathcal{E} \to \mathcal{A}$ which takes x_i to X_i and ξ_i to Ξ_i and this is always a differential operator $U \in \widehat{\mathcal{D}}_{-}^{\times}$.

Remark 3 The construction also works globally over any open subcone $U \subset T^*\mathbf{C}^n$ which is Stein and contractible (e.g. the set $\{\xi_i \neq 0\} \subset T^*B$, B a ball in $\mathbf{C}^n$, or a Stein contractible set). Over such a set, any $\mathcal{E}$-algebra $\mathcal{A}$ is isomorphic to $\mathcal{E}$, and any section α of $\mathcal{O}(m)$ is the symbol of a section of $\mathcal{A}_m$.

Thus one obtains all $\mathcal{E}$-algebras by gluing together models of $\mathcal{E}$ over a covering of Σ by open conic subsets Σ_i, using automorphisms of $\mathcal{E}$ on the intersections. The following proposition sums up what was said above :

Proposition 2 *Star algebras on $\Sigma = T^*X - \{0\}$ are locally isomorphic to the pseudo-differential algebra $\mathcal{E}$. The set $\mathrm{Alg}_{\mathcal{E}}$ of isomorphy classes is canonically isomorphic to $H^1(PX, \mathrm{Aut}\,\mathcal{E})$.*

$\mathrm{Aut}\,\mathcal{E}$ denotes the sheaf of automorphisms of $\mathcal{E}$; the noncommutative cohomology $H^1(PX, \mathrm{Aut}\,\mathcal{E})$ is described below in section 3.4.

3.2 Differential Operators and $\mathcal{D}$-algebras

If X is a complex manifold, the sheaf $\mathcal{D}_X$ of differential operators on X is well defined. If U is an automorphism of $\mathcal{D}_X$ preserving symbols, it fixes the subalgebra $\mathcal{O}_X \subset \mathcal{D}_X$ of operators of order 0, (because it fixes symbols and preserves invertible operators, which are necessarily of order 0). It follows that

U is locally an inner automorphism of the form Int e^f (f holomorphic). We have Int $e^f = \mathrm{Id}$ iff f is (locally) constant, so the automorphism sheaf is

$$\text{Aut } \mathcal{D}_X \simeq \mathcal{O}_X^\times/\mathbf{C}^\times \simeq \mathcal{O}_X/\mathbf{C}. \tag{4}$$

We will call $\mathcal{D}$-algebra a sheaf of algebras on X locally isomorphic to $\mathcal{D}_X$ (such algebras appear in [2] where they are called "twisted algebras of differential operators"). The set $\mathrm{Alg}_\mathcal{D}$ of isomorphic classes of these algebras is canonically isomorphic to $H^1(X, \mathcal{O}_X/\mathbf{C})$.

A $\mathcal{D}$-algebra obviously also defines a star-algebra on PX, and it is natural to compare the two sets $\mathrm{Alg}_\mathcal{D}$ and $\mathrm{Alg}_\mathcal{E}$.

3.3 Automorphisms and Symbols of Automorphisms

To understand how local $\mathcal{E}$-algebras can be patched together to make global objects, we have to know what automorphisms of $\mathcal{E}$ look like.

Let $U \in \widehat{\mathcal{D}}_-^\times$ be an automorphism of $\mathcal{E}$: U preserves symbols and the unit 1, so $U-1$ is of degree ≤ -1 and the logarithm $D = \mathrm{Log}\, U = -\sum_{n\geq 1} -\frac{1}{n}(1-U)^n$ is well defined ; it is a derivation of degree ≤ -1 of $\mathcal{E}$.

Now if D is a derivation of degree $\leq k$ its symbol $\delta = \sigma_k(D)$ is a homogeneous derivation of degree k of the Poisson algebra $\widehat{\mathcal{O}}$, i.e. a symplectic vector field on Σ, homogeneous of degree k. This corresponds, via the symplectic structure of Σ, to a closed differential form α, homogeneous of degree $k+1$.

Let ρ denote the radial vector field, infinitesimal generator of the action of $\mathbf{C}^\times$ ($\rho = \sum \xi_j \partial_{\xi_j}$ in local coordinates on X, T^*X as above) : the associated Lie derivation is $L_\rho = i_\rho d + d i_\rho$ (i_ρ denotes the interior product) so

$$d i_\rho \, \alpha = (k+1)\, \alpha.$$

Hence α is exact (the differential of a homogeneous function) if $k+1 \neq 0$. If $k+1 = 0$, $s = i_\rho \, \alpha$ is locally constant, and α is locally the differential of a homogeneous function of degree 0 iff $s = 0$.

By successive approximations, it follows that locally any derivation D of $\mathcal{E}$ is of the form $s\, \mathrm{ad}(\mathrm{Log}\, P_1) + \mathrm{ad} Q$ with P_1 elliptic of degree 1, $Q \in \mathcal{E}$, and any automorphism of $\mathcal{E}$ is locally of the form

$$U = (\mathrm{Int}\, P_1)^s \mathrm{Int}\, Q_0 \tag{5}$$

with P_1 elliptic of degree 1, Q_0 elliptic of degree 0. [4] Int P denotes the inner automorphism $a \to P\, a\, P^{-1}$.

If U is an automorphism of $\mathcal{E}$, we define its symbol $\sigma(U)$ as the closed 1-form on Σ homogeneous of degree 0 corresponding to the leading term of $\mathrm{Log}\, U$.

We have $\sigma(U) = d\mathrm{Log}\, \sigma(P)$ if $U = \mathrm{Int}\, P$.global section of ω (this is a closed 1-form on Σ). If $\sigma(U) = 0$ ($\mathrm{Log}\, U$ of degree ≤ 2) there exists a unique $P \in \mathcal{E}^\times$ of degree 0 and symbol 1 such that $U = \mathrm{Int}\, P$. Summing up we have proved :

[4] as usual in the context of pseudodifferential operators, elliptic = invertible.

Proposition 3 *There is an exact sequence of sheaves of groups on PX:*

$$0 \to \mathcal{E}_-^\times \to \operatorname{Aut}\mathcal{E} \to \omega \to 0 \tag{6}$$

where $\mathcal{E}_-^\times$ denotes the multiplicative sheaf of groups on $B\Sigma$ of sections of $\mathcal{E}$ of symbol 1, and ω is the sheaf on PX of closed 1-forms homogeneous of degree 0 on Σ.

If $\mathcal{A} \in \mathrm{Alg}_\mathcal{E} \simeq H^1(PX, \operatorname{Aut}\mathcal{E})$ its symbol $\sigma(\mathcal{A}) \in H^1(PX, \omega)$ is defined as the image cocycle.

Remark 4 If U is an automorphism of $\mathcal{A}$, it defines a one parameter group $U^s = \exp s\mathrm{Log}\, U$, $s \in \mathbf{C}$. This is polynomial in s mod.$\mathcal{A}_n$ for any $n < 0$.

3.4 Non Commutative Cohomology Classes

In this section we recall the elementary resuls of noncommutative cohomology that we will use (for more information see [16]). Let Y be a space and $\mathcal{G}$ a sheaf of groups on Y. We denote $H^0(Y, \mathcal{G}) = \Gamma(Y, \mathcal{G})$ the set of global sections of $\mathcal{G}$ over Y : this is a group.

We denote $H^1(Y, \mathcal{G})$ the set of equivalence classes of cocycles

$$u_{ij} \in \Gamma(Y_{ij} = Y_i \cap Y_j, \mathcal{G}) \quad \text{such that} \quad u_{ij}u_{jk} = u_{ik}$$

associated to open coverings $Y = \bigcup Y_i$; two cocycles are equivalent if, after a suitable refinement of the covering, we have $u_{ij} = u_i u'_{ij} u_j^{-1}$ for some family $u_i \in \Gamma(Y_i, \mathcal{G})$.

$H^1(Y, \mathcal{G})$ classifies the set of isomorphy classes of $\mathcal{G}$ principal homogeneous right $\mathcal{G}$ sheaves, i.e. sheaves α on Y, equipped with a right action of $\mathcal{G}$, locally isomorphic to $\mathcal{G}$ considered as a right $\mathcal{G}$-sheaf.

Proposition 4 *Let*

$$0 \to A \to B \to C \to 0 \tag{7}$$

be an exact sequence of sheaves of groups on Y, with A normal in B. Then there is a long cohomology sequence ;

$$\begin{aligned} 0 \to H^0(Y, A) \to H^0(Y, B) \to H^0(Y, C) \to \\ \to H^1(Y, A) \to H^1(Y, B) \to H^1(Y, C). \end{aligned} \tag{8}$$

This is "exact" in the sense that

i) it is exact at the first three places (the H^0 are groups, the H^1 are pointed sets).

ii) The group $H^0(Y, C)$ acts on the set $H^1(Y, A)$, and its orbits are the fibers of the map $H^1(Y, A) \to H^1(Y, B)$ (the action is given by $c \cdot (a_{ij}) = (b_i a_{ij} b_j^{-1})$

if c is a global section of B, and $b_i \in \Gamma(Y_i, B)$ a lifting of c to B over a fine enough covering Y_i).

iii) If $\beta \in H^1(Y,B)$ it defines twisted sheaves of groups $\mathcal{A}_\beta \subset \mathcal{B}_\beta$ (where $\mathcal{B}_\beta$ is the sheaf of B-automorphisms of the principal B-sheaf β), and the fiber of the map $H^1(Y,B) \to H^1(Y,C)$ is the image of $H^1(Y,A_\beta)$ in $H^1(Y,C)$.

More explicitly if β, β' are two principal B-sheaves, then $\gamma = \mathrm{Hom}_B(\beta, \beta')$ is a principal B_β-sheaf. If β, β' have the same image in $H^1(Y,C)$ then $\gamma/\mathcal{A}_\beta = \mathrm{Hom}_C(\beta/\mathcal{A}, \beta'/A)$ has a global section, i.e. is trivial, so γ is the image of a sheaf $\alpha \in H^1(Y, A_\beta)$. Finally $\beta' \sim \alpha \times_{A_\beta} \beta$ is in the image of $H^1(Y, A_\beta)$.

In this paper the noncommutative cohomology sequence stops there, and we will not use higher cohomology $H^j, j \geq 2$ whose definition is more elaborate (the substitutes are more complicated objects sometimes described by means of "stacks"). Exact sequences concerning torsors as above were introduced by J. Frenkel [12, 13]. Of course if A, B, C are commutative, the higher cohomology groups $H^j, j \geq 0$ are well defined commutative groups, and we will occasionally use the long cohomology exact sequence in that case up to order $j = 2$.

3.5 Symbols

If $\mathcal{A} \in \mathrm{Alg}_{\mathcal{E}} \simeq H^1(PX, \mathrm{Aut}\,\mathcal{E})$ we have defined its symbol as the image of its defining cocycle in $H^1(PX, \omega)$. To compute H^0 and H^1 for automorphisms, it will be useful to compute them first for symbols.

The following exact sequences of sheaves are also useful to handle ω :

$$0 \to \mathcal{O}_{PX}/\mathbf{C} \to \omega \to \mathbf{C} \to 0 \tag{9}$$

$$0 \to \mathbf{C} \to \mathcal{O}_{PX} \to \mathcal{O}_{PX}/\mathbf{C} \to 0 \tag{10}$$

These give rise to long exact cohomology sequences. We will call "exponent map" the cohomology maps coming from the map $\omega \to \mathbf{C}$ in (9).

With slight abuse we will call "Chern maps" [5] the maps :

$$\mathrm{ch}: \; H^j(Y, \mathcal{O}/\mathbf{C}) \to H^{j+1}(Y, \mathbf{C}). \tag{11}$$

in the long exact cohomology sequence derived from (10).

The sheaf $\mathcal{O}/\mathbf{C}$ ($Y = X$ or PX) identifies with the sheaf of closed holomorphic 1-forms on Y. If Y is a Stein manifold we have $H^j(Y, \mathcal{O}) = 0$ for $j \geq 1$ so the Chern map $H^j(Y, \mathcal{O}/\mathbf{C}) \to H^{j+1}(Y, \mathbf{C})$ is an isomorphism for $j \geq 1$.

If Y is a compact Kähler manifold, the long exact cohomology sequence from (10) splits into a sequence of short split exact sequences :

$$0 \to H^{j-1}(Y, \mathcal{O}/\mathbf{C}) \to H^j(Y, \mathbf{C}) \to H^j(Y, \mathcal{O}) \to 0 \quad (j \geq 0)$$

[5] The standard Chern map : $H^1(Y, \mathcal{O}^\times) \to H^2(Y, \mathbf{C})$ factors through $H^1(Y, \mathcal{O}/\mathbf{C})$.

and for $j \geq 0$ we have an isomorphism

$$H^j(Y, \mathcal{O}/\mathbf{C}) = \sum_{p+q=j+1, p>0} H^{pq} \tag{12}$$

where (here, and whenever possible) H^{pq} denotes the space of harmonic forms of type p, q on Y.

Proposition 5 *(i) If $n = \dim X \geq 2$, or if X is a closed curve of genus $\neq 1$, the map $H^0(X, \mathcal{O}/\mathbf{C}) \to H^0(PX, \omega)$ is an isomorphism.*

(ii) If X is an open curve or a closed curve of genus 1, then ω is split and

$$H^0(PX, \omega) \simeq H^0(X, \mathcal{O}/\mathbf{C}) \oplus H^0(X, \mathbf{C}).$$

Proof : A global section of ω is a closed 1-form on $T^*X - \{0\}$, homogeneous of degree 0. Locally on X such a form α reads

$$\alpha = \sum \alpha_k dx_k + \beta_k d\xi_k \tag{13}$$

where the coefficients α_k resp. β_k are of degree 0 resp. -1. If $n \geq 2$ this implies $\beta_k = 0$ so the α_k only depend on x. Hence (i) for $n \geq 2$.

If X is a closed curve of genus $\neq 1$ ($n = 1$ so $PX = X$), then the Chern map $H^0(X, \mathbf{C}) \simeq \mathbf{C} \to H^1(X, \mathcal{O}/\mathbf{C}) = \mathbf{C}$ is injective : it maps $s \in \mathbf{C}$ to $s \operatorname{ch} \mathcal{O}(1)$ (where as above $\mathcal{O}(1)$ denotes the sheaf of homogeneous functions of degree 1 on T^*X) and $\operatorname{ch} \mathcal{O}(1) \neq 0$ if $g \neq 1$.[6] So the exponent map $H^0(X, \omega) \to H^0(X, \mathbf{C})$ vanishes, and the map $H^0(X, \mathcal{O}/\mathbf{C}) \to H^0(X, \omega)$ is an isomorphism, hence (i) in this case.

If $n = 1$ and X is open or of genus 1, there exists a global nonvanishing vector field, so ω is split : $\omega = \mathcal{O}/\mathbf{C} \oplus \mathbf{C}$ hence (ii).

Proposition 6 *(i) If $n = \dim X \geq 2$ the map $H^1(X, \mathcal{O}/\mathbf{C}) \to H^1(PX, \omega)$ is an isomorphism.*

(ii) If $n = \dim X = 1$ ($PX = X$) and X is open or closed of genus 1 (ω split), then $H^1(X, \omega) = H^1(X, \mathcal{O}/\mathbf{C}) \oplus H^1(X, \mathbf{C})$.

(iii) If X is a closed curved of genus $g \neq 1$ the exponent map $H^1(X, \omega) \to H^1(X, \mathbf{C}) \simeq \mathbf{C}^{2g}$ is an isomorphism.

This should be complemented as follows in case (ii) : if X is an open curve, $H^1(X, \mathcal{O}/\mathbf{C}) = 0$ so $H^1(X, \omega) \simeq H^1(X, \mathbf{C})$.

If X is closed of genus 1, then $H^{20} = 0$ so $H^1(X, \mathcal{O}/\mathbf{C}) \simeq H^{20} + H^{11} \simeq H^{11} \simeq \mathbf{C}$, and $H^1(X, \omega) \simeq H^{11} + H^1(X, \mathbf{C}) \simeq \mathbf{C}^3$.

[6] The corresponding cocycle is $d\mathrm{Log}\,(\frac{\xi_i}{\xi_j})$ if ξ_i is the symbol of a nonvanishing vector field on a covering X_i of X , whose image in $H^1(X, \omega)$ is $\frac{d\xi_i}{\xi_i} - \frac{d\xi_j}{\xi_j}$, obviously a coboundary.

Lemma 1 *If X is a ball (or more generally Stein contractible space), we have $H^1(PX, \omega) = 0$.*

Proof : We have $PX \simeq X \times P_{n-1}$, so $H^1(PX, \mathbf{C}) = 0$ (PX is simply connected) and the map $H^1(PX, \mathcal{O}/\mathbf{C}) \to H^1(PX, \omega)$ is onto (if $n = 1$ we are finished).

Next we wave $H^j(PX, \mathcal{O}) = 0$ for any $j > 0$ ($\mathcal{O}$ has no cohomology on P_{n-1}) so the Chern map $H^1(PX, \mathcal{O}/\mathbf{C}) \to H^2(PX, \mathbf{C}) \simeq \mathbf{C}$ is one to one. Now, as above for curves of genus $\neq 1$, $H^2(PX, \mathbf{C}) \simeq \mathbf{C}$ is generated by the Chern class of $\mathcal{O}(1)$, corresponding to the cocycle $d\,\mathrm{Log}\,(\frac{\xi_i}{\xi_j})$ for ξ_i an elliptic symbol of degree 1 over a covering U_i of PX. This is also precisely the image of $1 \in H^0(PX, \mathbf{C}) \simeq \mathbf{C}$ by the exponent map $H^0(PX, \mathbf{C}) \to H^1(PX, \mathcal{O}/\mathbf{C})$, so the exponent map is onto and the map $H^1(PX, \mathcal{O}) \to H^1(PX, \omega)$ vanishes.This proves the lemma.

Proof of Proposition 6. (i) Let α be a principal ω-sheaf on PX corresponding to a cocycle in $H^1(PX, \omega)$, and let $X = \bigcup X_i$ be a covering of X by complex balls (or Stein contractible open sets). Then $\alpha_i = \alpha_{|X_i}$ is trivial. The patching isomorphism $u_{ij} : \alpha_j \to \alpha_i$ is the translation by a section $u_{ij} \in H^0(PX_i \cap PX_j, \omega) = H^0(X_i \cap X_j, \mathcal{O}/\mathbf{C})$; thus α is defined by a cocycle $(u_{ij}) \in H^1(X, \mathcal{O}/\mathbf{C})$. If $n \geq 2$ and if $(u_{ij}) = (\alpha_i - \alpha_j) \sim 0$ in $H^1(PX, \omega)$ then again $\alpha_i \in H^0(X_i, \mathcal{O}/\mathbf{C})$ by Proposition 5, so $(u_{ij}) \sim 0$ in $H^1(X, \mathcal{O}/\mathbf{C})$. This proves (i).

If $n = 1$ ($PX = X$) and X is open or of genus 1, ω is split so $H^1(X, \omega) = H^1(X, \mathcal{O}/\mathbf{C}) \oplus H^1(X, \mathbf{C})$.

If X is open then $H^1(X, \mathcal{O}/\mathbf{C}) = 0$ because in the long exact cohomology sequence from (10) we have $H^1(X, \mathcal{O}) = H^2(X, \mathbf{C}) = 0$, so $H^1(X, \omega) \simeq H^1(X, \mathbf{C})$.

If X is of genus $g = 1$, we have $H^1(X, \mathcal{O}/\mathbf{C}) = H^{20} + H^{11} = \mathbf{C}$ and $H^1(X, \omega) \simeq H^{11} + H^1(X, \mathbf{C}) \simeq \mathbf{C}^3$.

If X is a closed curve of genus $g \neq 1$ we have seen that the map $H^0(X, \mathbf{C}) \to H^1(X, \mathcal{O}/\mathbf{C})$ is one to one, and $H^2(X, \mathcal{O}/\mathbf{C}) = H^{30} + H^{21} + H^{12} = 0$ so from the long exact cohomology sequence from (9)

$$\cdots \to H^0(X, \mathbf{C}) \to H^1(X, \mathcal{O}/\mathbf{C}) \to H^1(X, \omega) \to H^2(X, \mathcal{O}/\mathbf{C}) \to \ldots$$

we see that the map $H^1(X, \omega) \to H^1(X, \mathbf{C})$ is one to one.

This proves Proposition 6 and its complement. Note that if X is a curve, the only case where $H^1(X, \omega) = 0$ is when X is simply connected.

3.6 Filtrations

As mentioned above $\mathrm{Aut}\,\mathcal{E}$ has a natural filtration (as well as $\mathcal{E}_-^\times \subset \mathrm{Aut}\,\mathcal{E}$) : any $a \in \mathrm{Aut}\,\mathcal{E}$ is of degree ≤ 0 and $a \in \mathcal{E}_-^\times$ is of degree $n < 0$ if $a = \mathrm{Int}\,(1+b), b \in \mathcal{E}_n$. The corresponding graded sheaf is

$$\text{(14)} \qquad \mathrm{gr}\,\mathrm{Aut}\,\mathcal{E} = \bigoplus_{k \leq 0} (\mathrm{Aut}\,\mathcal{E})_k/(\mathrm{Aut}\,\mathcal{E})_{k-1} \simeq \omega + \bigoplus_{k<0} \mathcal{O}(k).$$

It is commutative, and this will help to extract more information. This also works for any $\mathcal{E}$-algebra $\mathcal{A}$ because the filtration above, and the leading terms, are by definition invariant by automorphisms so $\mathrm{gr}\,\mathrm{Aut}\,\mathcal{A} \simeq \mathrm{gr}\,\mathrm{Aut}\,\mathcal{E}$.

Proposition 7 *Let $\mathcal{A}$ be an $\mathcal{E}$-algebra.*

(i) The natural map $\mathrm{gr}\,H^0(PX, \mathrm{Aut}\,\mathcal{A}) \to H^0(PX, \mathrm{gr}\,\mathrm{Aut}\,\mathcal{E})$ is injective. If $H^1(PX, \mathrm{gr}\,\mathrm{Aut}\,\mathcal{E}) = 0$ it is one to one.

(ii) If $H^1(PX, \mathrm{gr}\,\mathrm{Aut}\,\mathcal{E}) = 0$ then $H^1(PX, \mathcal{A}_-^\times) = 0$.

(iii) If $H^2(PX, \mathcal{E}_-^\times) = 0$ the symbol map induces a surjective map

$$H^1(PX, \mathrm{gr}\,\mathrm{Aut}\,\mathcal{E}) \to \mathrm{gr}\,H^1(PX, \mathrm{Aut}\,\mathcal{A}). \tag{15}$$

Proof : (i) The map $\mathrm{gr}\,H^0(PX, \mathrm{Aut}\,\mathcal{A}) \to H^0(PX, \mathrm{gr}\,\mathrm{Aut}\,\mathcal{E})$ takes any U of degree $m \leq 0$ to its symbol $\sigma_m(U)$ which is a section of ω if $m = 0$ or of $\mathcal{O}(m)$ if $m < 0$. $\sigma_m(U) = 0$ means that U is really of degree $\leq m-1$ so the resulting graded map is injective.

Conversely let X_i be a covering of PX, and $a_i \in H^0(X_i, \mathrm{Aut}\,\mathcal{A})$ be such that $a_i a_j^{-1}$ is of degree $m < 0$ (this is true if $\sigma(a_i) = a$, a global section of $\mathrm{gr}_{m+1}(PX, \mathrm{Aut}\,\mathcal{E})$). Then $\sigma_m(a_i a_j^{-1})$ is a 1-cocycle with coefficients in $\mathcal{O}(m)$. If $H^1(PX, \mathcal{E}_-^\times) = 0$ this is a coboundary i.e. of the form $\sigma_m(b_i) - \sigma_m(b_j)$, $b_i \in A_m$ so the $\mathrm{Int}\,(1+b_i)^{-1} a_i\, \mathrm{Int}\,(1+b_j)$ are equal to a_i mod. $\mathrm{Aut}\,\mathcal{A}_m$ and patch together mod. $(\mathrm{Aut}\,\mathcal{A})_{m-1}$. Note that if the X_i and their intersections are Stein it is not necessary to shrink the covering, so by successive approximations we get a cocycle $a \in H^0(PX; \mathrm{Aut}\,\mathcal{A})$ equal to (a_i) mod. $(\mathrm{Aut}\,\mathcal{A})_m$.

(ii) If $H^1(PX, \mathrm{gr}\,\mathcal{E}_-^\times) = 0$ and if (a_{ij}) is a cocycle of degree $n < 0$ with coefficients in $\mathcal{A}_-^\times$, then $\sigma_n(a_{ij})$ is a 1-cocycle with coefficients in $\mathcal{O}(n)$, hence a coboundary $\sigma_n(b_i) - \sigma_n(b_j)$, $b_i \in \mathcal{A}_n$. So a is equivalent to the cocycle

$$(1+b_i)^{-1} a_{ij} (1+b_j)$$

which is of degree $n-1$. Again we do not need to shrink the covering if it has been chosen as above (Stein, contractible), so by successive approximations we get $a \sim 0$.

(iii) If b_{ij} is a cocycle with coefficients in $\mathrm{Aut}_m\mathcal{A}$ ($m \leq 0$) its symbol $\sigma(b_{ij})$ is a cocycle with coefficients in $gr_m \mathrm{Aut}\,\mathcal{E}$ but in general this does not give rise to a map $\mathrm{gr}\,H^1(PX, \mathrm{Aut}\,\mathcal{A}) \to H^1(PX, \mathrm{gr}\,\mathrm{Aut}\,\mathcal{E})$ nor the other way round, at best an ill-defined "noncommutative spectral sequence".

However if $H^2(PX, \mathrm{gr}\,\mathrm{Aut}\,\mathcal{E} = 0)$, the same argument as above shows that if a cochain $a_{ij} \in H^0(U_i \cap U_j, \mathrm{Aut}\,\mathcal{A})$ is a cocycle mod. $(\mathrm{Aut}\,\mathcal{A})_m, m < 0$, i.e. $a_{ij} a_{jk} a_{ki} \in (\mathrm{Aut}\,\mathcal{A})_m$ then $\sigma_m(a_{ij} a_{jk} a_{ki})$ is a coboundary $\sum \sigma_m(b_{jk})$ with coefficients in $\mathcal{O}(m)$, and again by successive approximations there exists a cocycle a'_{ij} with coefficients in $\mathrm{Aut}\,\mathcal{A}$ equal to a_{ij} mod. $\mathcal{A}_m$.

In particular, by successive approximations, we see that any cocycle a with coefficients in $\mathrm{gr}_m \mathrm{Aut}\,\mathcal{E}$ ($m \leq 0$) is the symbol of a cocycle $b \in H^1(PX, \mathrm{Aut}_m\mathcal{A})$

which is well defined mod. $H^1(PX, \mathrm{Aut}_{m-1}\mathcal{A})$ and vanishes if a is a coboundary. Thus our map is well defined and onto (if $H^0(PX, \mathrm{Aut}\,\mathcal{A}) \neq 0$ it may not be injective because two cocycles of degree m with coefficients in $\mathrm{Aut}\,\mathcal{A}$ can then be equivalent although their symbols are not).

4 $\mathcal{E}$-Algebras on T^*X, $\dim X \geq 2$

4.1 General Results.

We first point out the following results (which will also be useful in section 5) :

Lemma 2 (Global automorphisms of $\mathcal{E}$) *If $\mathcal{A}$ is an $\mathcal{E}$-algebra $\mathcal{A}$ on X and $n = \dim X \geq 2$ the symbol map*

$$H^0(PX, \mathrm{Aut}\,\mathcal{A}) \simeq H^0(PX, \omega) \simeq H^0(X, \mathcal{O}_X/\mathbf{C}).$$

is injective. It is bijective if $\mathcal{A} = \mathcal{E}$.

Proof : If $n \geq 2, \mathcal{A}$ and $\mathrm{Aut}\,\mathcal{A}$ have no global section of degree < 0 so the symbol map $u \to \sigma(u)$ is injective. More generally if $\mathcal{A}, \mathcal{A}'$ are two $\mathcal{E}$-algebras and u, v two isomorphisms $\mathcal{A} \to \mathcal{A}'$ the difference symbol $\sigma(u^{-1}v) \in H^0(PX, \omega)$ is well defined and completely determines v (given u) (note that we have $\sigma(u^{-1}v) = \sigma(vu^{-1})$).

On the other hand if $\mathcal{A} = \mathcal{E}$ (more generally if $\mathcal{A}$ comes from a $\mathcal{D}$-algebra) the symbol map is onto because, by Proposition 5, $H^0(PX, \omega) \simeq H^0(X, \mathcal{O}/\mathbf{C}) \simeq \mathrm{Aut}\,\mathcal{D}$, and this obviously lifts to $\mathrm{Aut}\,\mathcal{E}$.

If X is a ball of $\mathbf{C}^n$ or more generally a Stein contractible domain, we have $H^1(PX, \omega) = 0$ (Lemma 1) so $H^1(PX, \mathcal{E}_-^\times) \to H^1(PX, \mathrm{Aut}\,\mathcal{E})$ is onto, i.e. any $\mathcal{E}$-algebra can be defined by a cocycle with coefficients in$\mathcal{E}_-^\times$.

Now let $\mathcal{A}, \mathcal{A}'$ be two algebras defined by cocycles $a = (a_{ij}), a' = (a'_{ij}) \in \mathcal{E}_-^\times$ and let $u : \mathcal{A}' \to \mathcal{A}$ be an isomorphism, i.e. a family $(u_i) \in \mathrm{Aut}\,\mathcal{E}$ such that $u_i a'_{ij} = a_{ij} u_j$. Then the symbols $\sigma(u_i)$ patch together since $\sigma(a_{ij}) = \sigma(a'_{ij}) = 0$, and the resulting symbol $\sigma_{aa'}(u)$ is well defined. It only depends on the classes of a, a' in $H^1(PX, \mathcal{E}_-^\times)$ (however it does depend on $a, a' \in H^1(PX, \mathcal{E}_-^\times)$ and not just on their images in $H^1(PX, \mathrm{Aut}\,\mathcal{E})$: any other representatives are of the form $\alpha \cdot a, \alpha' \cdot a'$ with $\alpha, \alpha' \in H^0(PX, \omega)$ for the action of $H^0(PX, \omega)$ of Proposition 4, and we get $\sigma_{\alpha\cdot a, \alpha'\cdot a'}(u)) = \sigma(u) + \alpha - \alpha'$.

If $X \subset \mathbf{C}^n, n \geq 2$, is a Stein contractible domain, the exponent of $\sigma_{aa'}(u)$ vanishes : $\sigma_{aa'}(u) \in H^0(X, \mathcal{O}/\mathbf{C})$, and again $\sigma_{aa'}(u)$ completely determines u.

Let now $\mathcal{A} \in \mathrm{Alg}_{\mathcal{E}}$. There exists a covering $X = \bigcup X_i$ where all finite intersections are isomorphic to Stein contractible domains of $\mathbf{C}^n$. Then $\mathcal{A}_i = \mathcal{A}_{|X_i}$ can be defined by a cocycle (a_i) with coefficients in $\mathcal{E}_-^\times$; this being so the patching isomorphisms u_{ij} all have exponent 0 and are determined by their symbols $\sigma_{a_i a_j}(u_{ij})$ (for fixed $\mathcal{A}_i$). In particular we have proved :

Proposition 8 *If* $\dim X \geq 2$ *any* $\mathcal{E}$*-algebra* $\mathcal{A}$ *has exponent* 0 *(the image of* $\sigma(\mathcal{A}) \in H^1(PX, \omega)$ *in* $H^1(PX, \mathbf{C})$ *by the exponent map is zero), so* $\mathcal{A}$ *can be defined by a cocycle with coefficients in* $\operatorname{Int} \mathcal{E}_0 = \mathcal{E}_0^\times / \mathbf{C}^\times$ *(*$\mathcal{E}_0$ *is the sheaf of pseudo-differential operators of degree* ≤ 0*).*

4.2 The case $\dim X \geq 3$

If X is a ball and $\dim X \geq 3$ we have $H^1(PX, \mathcal{O}(-k)) = 0$ for all $k > 0$, i.e. $H^1(PX, \operatorname{gr} \mathcal{E}_-^\times) = 0$ (this is also true if X is a Stein manifold).

It follows that we have $H^1(PX, \mathcal{E}_-^\times) = 0$, and more generally for any $\mathcal{E}$-algebra $\mathcal{A}$ we have $H^1(PX, \mathcal{A}_-^\times) = 0$.

Hence if $\mathcal{A}$ is an $\mathcal{E}$-algebra, it is built by patching together models of $\mathcal{E}$ over a covering X_i of X, where the patching cocycle belongs to $H^1(X, \mathcal{O}_X/\mathbf{C})$.

Moreover if $\mathcal{A}, \mathcal{B}$ are two such algebras, any isomorphism $\mathcal{B} \to \mathcal{A}$ comes from a $\varphi \in H^0(X, \mathcal{O}_X/\mathbf{C})$ i.e. comes locally from an inner automorphism $P \to \varphi\, P\, \varphi^{-1}$. Summing up we have proved :

Theorem 1 *If* $\dim X \geq 3$ *the functor which to a* $\mathcal{D}$*-algebra associates the corresponding* $\mathcal{E}$*-algebra is an equivalence.*

This result is closely related to the result of [8] on microlocally free $\mathcal{D}$-modules in dimension ≥ 3.

4.3 The case $\dim X = 2$

If $\dim X = 2$ what was said above remains true, in particular any symbol $\alpha \in H^1(PX, \omega)$ is the symbol of an $\mathcal{E}$-algebra (in fact of a $\mathcal{D}$-algebra). However the picture changes considerably because $H^1(PX, \mathcal{E}_-^\times)$ is usually very large. The following examples show what can happen, and also how, in global situations on compact manifolds, things can nevertheless at least partially cancel out.

Example 1. Let X be the unit ball of $\mathbf{C}^2$ (or more generally a Stein contractible manifold). [7]

Then $H^1(X, \omega) = H^1(X, \mathcal{O}/\mathbf{C}) = 0$ so $H^1(PX, \operatorname{Aut} \mathcal{E})$ is the quotient of $H^1(PX, \mathcal{E}_-^\times)$ by the action of $H^0(X, \operatorname{Aut} \mathcal{E}) = H^0(X, \mathcal{O}/\mathbf{C})$.

Now PX is the union of the two Stein subcones $U_i = \{\xi_i \neq 0\}$ $(i = 1, 2)$ so a cocycle is represented by just one section $a_{12} \in \mathcal{E}_-^\times(U_1 \cap U_2)$. It is elementary that any $a \in H^1(PX, \mathcal{E}_-^\times)$ has a unique normalized representative of the form

$$a_{12} = \sum_{p,q<0} a_{pq}(x) \xi_1^p \xi_2^q \tag{16}$$

i.e. with no holomorphic term in ξ_1 or ξ_2 (this is obvious for the additive cohomology $H^1(PX, \operatorname{gr} \mathcal{E}_{-1})$ and follows by successive approximation for $\mathcal{E}_-^\times$).

[7] what is used is $H^1(X, \mathcal{O}/\mathbf{C}) = 0$ and the fact that T^*X is a trivial holomorphic vector bundle.

So $H^1(PX, \operatorname{Aut}\mathcal{E})$ is the set of conjugate classes of normalized symbols a_{12} as above, with $a_{12} \sim \varphi(x) a_{12} \varphi(x)^{-1}$ for φ a nonvanishing function on X. This set is still very large ; on the other hand such algebras tend to have very few global sections or automorphisms.

The analysis of these algebras is closely related to that of "microlocally" free $\mathcal{D}$-modules in dimension 2, made by M. Carette [7].

For global compact manifolds, some things may cancel out.

Example 2. Let $X = P_2(\mathbf{C})$ be the complex projective plane : then PX is isomorphic to the incidence manifold $\{x \cdot \xi = 0\} \subset X \times X^*$ (X^* the dual projective space). T^*X itself is the quotient of the incidence cone $\Gamma = \{x.\xi = 0\} \subset \mathbf{C} - \{0\} \times \mathbf{C}$ by the group action $(x, \xi) \sim (\lambda x, \frac{1}{\lambda}\xi)$. The sheaf $\mathcal{O}_{PX}(n)$ of homogeneous functions of degree n on T^*X identifies with the sheaf of restrictions to Γ of functions $f(x, \xi)$ such that $f(\lambda x, \xi) = f(x, \lambda\xi) = \lambda^n f(x, \xi)$ i.e. $\mathcal{O}_{PX}(n) = \mathcal{O}_X(n) \otimes \mathcal{O}_{X^*}(n)$ (where exceptionally here $\mathcal{O}_X(n)$ denotes the canonical sheaf of the projective space). It follows easily that $H^1(PX, \operatorname{gr}\mathcal{E}_-^\times) = 0$ so $H^1(PX, \mathcal{A}_-^\times) = 0$ for any $\mathcal{E}$-algebra $\mathcal{A}$.

The symbol map $H^1(X, \operatorname{Aut}\mathcal{E}) \to H^1(X, \omega) = H^1(X, \mathcal{O}/\mathbf{C})$ is one to one, and again, as in dimension ≥ 3, the correspondence $\mathcal{D}$-algebras $\to$ $\mathcal{E}$-algebras is an equivalence.

Note that in this case we have $H^1(PX, \mathcal{O}/\mathbf{C}) \simeq H^{20} + H^{11} \simeq H^{11} = \mathbf{C}$, and $\mathcal{E}$-algebras $\sim$ $\mathcal{D}$-algebras are parameterized by $H^{11} = \mathbf{C}$.

Example 3 Let X be a holomorphic complex torus of dimension 2 (a torus $\mathbf{C}^2/\Gamma$ with $\Gamma \simeq \mathbf{Z}^4$ acting by translations).

The group of automorphisms of $\mathcal{E}$ or $\mathcal{D}$ is

$$H^0(PX, \operatorname{Aut}\mathcal{E}) = H^0(X, \mathcal{O}/\mathbf{C}) = H^{10} = \mathbf{C}^2 \tag{17}$$

and any automorphism comes from an inner automorphism of $\mathcal{E}$ and $\mathcal{D}$ on $\mathbf{C}^2$ of the form :

$$P = P(x, d) \to e^{a.x} P e^{-a.x} \qquad (x \to x,\ d \to d - a). \tag{18}$$

Any $\mathcal{E}$- or $\mathcal{D}$-algebra on X lifts as the trivial algebra $\mathcal{E}_{\mathbf{C}^2}$ on the universal cover $\mathbf{C}^2$, and is the quotient of $\mathcal{E}_{\mathbf{C}^2}$ by a group of isomorphisms over the translation group of periods Γ.

By Proposition 6 we have

$$H^1(PX, \omega) = H^1(X, \mathcal{O}/\mathbf{C}) = H^{20} + H^{11} = \mathbf{C}^5.$$

More precisely an element $\alpha \in H^{20} + H^{11}$ is represented by a harmonic form

$$\alpha = a\, dz_1 dz_2 + \sum a_{ij} d\overline{z_i} dz_j. \tag{19}$$

There is a unique corresponding $\mathcal{D}$-algebra, which is isomorphic to the quotient of $\mathcal{D}_{\mathbf{C}^2}$ by the lifting $\mu \to U_\mu$ of the group Γ of periods (acting by translations) :

$$U_\mu : \begin{array}{l} x \to x + \mu \\ d \to d + p(\mu, \overline{\mu}) \end{array} \tag{20}$$

where $p = (p_1, p_2)$ is a linear map $\mathbf{C}^4 = \mathbf{C}^2 \times \overline{\mathbf{C}^2} \to \mathbf{C}^2$ such that $dp(z, \overline{z}).dz = dp_1 dz_1 + dp_2 dz_2 = \alpha$, where z, resp. $\overline{z}$ denotes the variable in $\mathbf{C}^2$ resp. $\overline{\mathbf{C}^2}$, and we use the notations of differential calculus. Such a map p splits into holomorphic and antiholomorphic parts : $p = p'(z) + p''(\overline{z})$. They form an 8-dimensional space, but it is classical that maps which differ by a symmetric holomorphic map ($dp.dz = 0$) define isomorphic algebras.

Remark 5 Cocycles coming from H^{11} are related to holomorphic line bundles on X : if L is a line bundle, $\mathcal{D}_L$ the sheaf of differential operators on L, the corresponding cocycle is the image in $H^1(X, \mathcal{O}/\mathbf{C})$ of the multiplicative cocycle with coefficients in $\mathcal{O}^\times$ defining L ; the corresponding harmonic form is an integral form in H^{11}, and such forms generate H^{11} if X is algebraic.

The cocycle associated to $dz_1 dz_2 \in H^{20}$ corresponds to the group of isomorphisms

$$U_\mu : z \to z + \mu, \quad d_1 \to d_1, \quad d_2 \to d_2 + \mu_1 \tag{21}$$

This corresponds to the 1-form $p(x) \cdot dx = x_1 dx_2$ (which could be replaced by any holomorphic primitive of $dx_1 dx_2$). It never appears in a context of line bundles.

We may now classify $\mathcal{E}$-algebras. The map $H^1(PX, \operatorname{Aut}\mathcal{E}) \to H^1(PX, \omega)$ is onto, and for $\alpha \in H^1(PX, \omega) \simeq H^{20} + H^{11}$ the fiber $\sigma^{-1}(\alpha)$ is the image of $H^1(PX, \mathcal{A}_-^\times)$ for $\mathcal{A}$ the unique $\mathcal{D}$-algebra as above with this symbol.

Let us examine $H^1(PX, \mathcal{A}_-^\times)$: by (18) two elements of $H^1(PX, \mathcal{A}_-^\times)$ give the same $\mathcal{E}$-algebra iff there is a translation $\xi \to \xi + a$ which transforms one to the other. An $a \in H^1(PX, \mathcal{A}_-^\times)$ lifts to an element $\tilde{a} \in H^1(\mathbf{C}^2, \mathcal{E}_-^\times)$ invariant by the U_μ, so the normalized representative (16) is invariant:

$$a(x, \xi) = a(x + \mu, \xi + p'(\mu) + p''(\overline{\mu})) \tag{22}$$

where as above $p' : \mathbf{C}^2 \to \mathbf{C}^2$, resp. $p'' : \overline{\mathbf{C}^2} \to \mathbf{C}^2$ denote the holomorphic and antiholomorphic parts of p, which correspond to the H^{20}, H^{11} components of the symbol α. Equivalently the symbol $b(x, \xi) = a(x, \xi - p'(x))$ satisfies

$$b(x, \xi) = b(x + \mu, \xi + p''(\overline{\mu})) = \sum \frac{1}{\gamma!} \partial_\xi^\gamma b(x + \mu, \xi)(p''(\overline{\mu}))^\gamma. \tag{23}$$

If $p'' = 0$ this means that b does not depend on x (it is periodic hence constant). If $p'' \neq 0$, the periodicity condition implies $b = 0$: for if b is of degree $n \leq -1$, its leading term is periodic in x hence independent of x : $b_n = b_n(\xi)$; the next term satisfies

$$b_{n-1}(x,\xi) - b_{n-1}(x+\mu,\xi) = b'_n(\xi)\cdot p''(\overline{\mu}) \tag{24}$$

so it is linear in x : $b_{n-1} = \beta(\xi) + \gamma(\xi)\cdot x$ with $p''(\overline{\mu}) = -\gamma(\xi).\mu$. Since p'' is antiholomorphic this implies $b'_n = 0$ so $b_n = 0$ since its degree is negative.

Summing up we have proved :

Proposition 9 *If X is a torus ($\mathbf{C}^2/\Gamma, \Gamma \simeq \mathbf{Z}^4$), we have*

$$H^1(PX,\omega) \simeq H^1(X,\mathcal{O}/\mathbf{C}) \simeq H^{20}(X) + H^{11}(X) \simeq \mathbf{C}^5. \tag{25}$$

Any symbol $\alpha \in H^1(PX,\omega)$ is the symbol of a unique $\mathcal{D}$-algebra on X.

If the H^{11} component of α is $\neq 0$ there is no other $\mathcal{E}$-algebra with this symbol.

If $\alpha \in H^{20}$ the $\mathcal{E}$-algebras with symbol α can be defined by a mormalized cocycle

$$b(\xi) = \sum_{p,q<0} b_{pq}\xi_1^p\xi_2^q \quad (b_{pq}\in\mathbf{C}) \tag{26}$$

whose coefficients are translation invariant (independant of x). Two such cocycles b, b' define the same $\mathcal{E}$-algebra iff $b'(\xi) \simeq b(\xi+a)$ for some constant vector a.

(this is an asymptotic relation between symbols : $b(\xi+a) = \sum \frac{1}{\alpha!}\partial_\xi^\alpha b\, a^\alpha$)

5 $\mathcal{E}$-Algebras over Curves ($\dim X = 1$)

We now describe $\mathcal{E}$-algebras, and compare them to $\mathcal{D}$-algebras, when X is a curve ($\dim X = 1$). In this case $PX = X$. The general method is the same but as we will see the classification is strikingly different depending on whether X is an open curve, or a closed curve of genus $g = 0, 1$ or ≥ 2.

5.1 Open curves

If X is an open curve, the exponent map $H^1(X,\omega) \to H^1(X,\mathbf{C})$ is an isomorphism (Proposition 6). Also X is Stein, so $H^j(X,\mathcal{O}(n)) = 0$ for $j > 0$ and for all $n, j \geq 1$, so $H^j(X, \operatorname{gr}\mathcal{E}_-^\times) = 0$ for $j = 1, 2$, and $H^1(X,\mathcal{E}_-^\times) = 0$ (Proposition 7). Finally we have

$$H^1(X;\operatorname{Aut}\mathcal{E}) \simeq H^1(X,\omega) \simeq H^1(X,\mathbf{C}). \tag{27}$$

Typically if (s_{ij}) is a cocycle with coefficients in $\mathbf{C}$, the corresponding algebra is defined by a cocycle with symbol $(\operatorname{Int}\xi)^{s_{ij}}$, ξ a global nonvanishing vector field.

These algebras have many sections because we have $H^1(X, \mathcal{E}_-^\times) = 0$ so by Proposition 7 the map $H^0(X, \operatorname{gr} \mathcal{E}) \simeq \mathcal{O}(X)[\xi, \xi^{-1}] \to \operatorname{gr} H^0(X, \mathcal{A})$ is one to one. They also have many automorphisms, because the sequence $0 \to H^0(X, \mathcal{A}_-^\times) \to H^0(X, \operatorname{Aut} \mathcal{A}) \to H^0(X, \omega) \to 0$ is exact.

$\mathcal{D}$-algebras are classified by $H^1(X, \mathcal{O}/\mathbf{C}) = 0$ and all give isomorphic $\mathcal{E}$-algebras. All non trivial $\mathcal{E}$-algebras come from the exponent map. [8]

5.2 Curves of genus $g \geq 2$

Note that in any case $\mathcal{O}_{PX}(1)$ identifies with the sheaf of sections of TX (vector fields) and the dual $\mathcal{O}_{PX}(-1)$ identifies with the sheaf of sections of T^*X. If X is of genus ≥ 2, we have $H^1(X, O(-k)) = 0$ if $k > 1$, but $H^1(X, O(-1)) = \mathbb{C}$.[9]

Proposition 10 *If X is a closed curve of genus $g > 1$, $\mathcal{E}$-algebras on X are classified by $H^1(X, \operatorname{Aut} \mathcal{E}) = \mathbb{C} \oplus \mathbb{C}^{2g}$. $\mathcal{D}$-algebras are classified by $H^1(X, \mathcal{O}/\mathbf{C}) = H^1(X, \mathbf{C}) = \mathbf{C}$ and give isomorphic $\mathcal{E}$-algebras.*

We have a split exact sequence of sheafs of groups

$$0 \to \mathcal{E}_{-1} \to \operatorname{Int} \mathcal{E} \to \mathcal{O}_{\mathrm{X}}^\times / \mathcal{C}^\times \to 0$$

It follows that we have $H^1(X, \operatorname{Int} \mathcal{E}) = \mathbb{C} \oplus \mathbb{C}$: the second factor comes from $H^1(X, \mathcal{O}_X^\times / \mathbb{C}^\times) \sim H^1(X, O(-1)) = \mathbb{C}$; it classifies twisted $\mathcal{D}$-algebras.

The first factor is the image of $H^1(X, \mathcal{E}_{-1}) \sim H^1(X, O(-1)) = \mathbb{C}$ ($\mathcal{E}_{-k}$ denotes the multiplicative group of elements $(1 + a), \deg \mathrm{a} \leq -\mathrm{k}$; the graded sheaf associated to $\mathcal{E}_{-1}$ is $\bigoplus_{k \leq -1} O(k)$, and $O(k)$ is cohomologically trivial for $k < -1$).

Consider the cohomology exact sequence (where the four last terms are commutative groups, even though $\operatorname{Int} \mathcal{E}$ and $\operatorname{Aut} \mathcal{E}$ are not):

$$0 \to H^0(X, \operatorname{Int} \mathcal{E}) \to \mathrm{H}^0(\mathrm{X}, \operatorname{Aut} \mathcal{E}) \to \mathrm{H}^0(\mathrm{X}, \mathbb{C}) \to$$

$$\to H^1(X, \operatorname{Int} \mathcal{E}) \to \mathrm{H}^1(\mathrm{X}, \operatorname{Aut} \mathcal{E}) \to \mathrm{H}^1(\mathrm{X}, \mathbb{C}) \to 0.$$

The second factor $\mathbb{C}^{2g}$ in prop.10 lifts $H^1(X, \mathbb{C})$; it classifies "exotic" algebras as for open curves. Such an algebra can be defined by a cocycle $\operatorname{Ad} \xi^{\mathrm{s_{ij}}}$ where ξ is a nonvanishing vector field over X minus one point, subordinate to a covering $X = \bigcup X_j$ where all X_j except one avoid the point.

In the long exact sequence, the unit $1 \in H^0(X, \mathbb{C})$ maps to the cocycle $(\xi_i \xi_j{}^{-1}) \in H^1(X, \mathcal{O}_X^\times / \mathbb{C}^\times) \subset H^1(X, \operatorname{Int} \mathcal{E})$, where (ξ_i) is a family on nonvanishing vector fields over a covering (X_i) of X. This is is not zero since the Chern class of TX is not zero ; it is obviously killed by the map $H^1(X, \operatorname{Int} \mathcal{E}) \to \mathrm{H}^1(\mathrm{X}, \operatorname{Aut} \mathcal{E})$ (the $\mathcal{E}$-algebra defined by a $\mathcal{D}$ algebra is always trivial). In prop.

[8] the fact that such "exotic" algebras exist is related to the fact that coherent $\mathcal{D}$-modules do not always possess global good filtrations.

[9] the original manuscript contained an error, corrected by P.Polesello.

10 the first factor, range of the map $H^1(X, \mathrm{Int}\,\mathcal{E}) \to \mathrm{H}^1(\mathrm{X}, \mathrm{Aut}\,\mathcal{E})$, is isomorphic to $H^1(X, \mathcal{E}_{-1}) \sim \mathbb{C}$.

Here again $\mathcal{E}$-algebras on X have many sections of negative degree and many automorphisms.

5.3 Curves of genus 1

This is the most complicated of the cases examined here. Let X be a closed curve of genus 1 : $X = \mathbf{C}/\Gamma$ where the group of periods $\Gamma \simeq \mathbf{Z}^2$ acts by translations.

We denote ξ the symbol of the constant vector field $\partial/\partial x$ on $\mathbf{C}$.

Since TX is trivial, ω is split : $\omega = \mathcal{O}/\mathbf{C} + \mathbf{C}$. Also, for all n, we have $H^0(\mathcal{O}(n)) = H^{00} \simeq H^1(X, \mathcal{O}(n)) = H^{01} \simeq \mathbf{C}$, $H^2(\mathcal{O}(n)) = 0$.

We denote

$$\mathcal{G}\ , \qquad \text{resp. } \mathcal{G}_- \subset \mathcal{G} \tag{28}$$

the group of automorphisms of $\mathcal{E}$ of the form $\mathrm{Int}\,\xi^s \mathrm{Int}\,(1 + a(\xi^{-1}))$, resp. the sub-group $s = 0$: this is the commutant of ξ, it is a constant subsheaf of $\mathrm{Aut}\,\mathcal{E}$.

For any $\alpha \in \mathbf{C}$ we set $\xi_a = e^{ax}\xi$. This is only defined up to a multiplicative constant $e^{a\mu}$, $\mu \in \Gamma$, but the inner automorphism

$$\mathrm{Int}\,(e^{ax}\xi)$$

is well defined, as well as the corresponding commutator sheaf

$$\mathcal{G}_{a-} \subset \mathcal{G}_a \tag{29}$$

which is a locally constant subsheaf of $\mathrm{Aut}\,\mathcal{E}$.

Proposition 11 *We have*

$$H^0(X, \omega) = H^0(X, \mathcal{O}/\mathbf{C}) + H^0(X, \mathbf{C}) = H^{10} + H^{00} \simeq \mathbf{C}^2$$

$$H^1(X, \omega) = H^1(X, \mathcal{O}/\mathbf{C}) + H^1(X, \mathbf{C}) = H^{11} + (H^{10} + H^{01}) = \mathbf{C}^3.$$

For the commutative locally constant sheaf $\mathcal{G}_{a-}$ *we have* [10]

$$H^j(\mathcal{G}_-) = H^j(X, \mathbf{C}) \otimes \mathcal{G}_- \text{ if } a = 0, \quad 0 \text{ if } a \neq 0.$$

We have $\mathrm{gr}\,\mathcal{E}_- = \bigoplus_{n<0} \mathcal{O}(n)\xi^n$ *and with an obvious notation*

$$H^0(X, \mathrm{gr}\,\mathcal{E}_-^\times) = \mathrm{gr}\,\mathcal{G} \simeq \xi^{-1}\mathbf{C}[\xi^{-1}]$$

$$H^1(X, \mathrm{gr}\,\mathcal{E}_-^\times) \simeq H^{10} \otimes \xi^{-1}\mathbf{C}[\xi^{-1}]$$

$$H^2(X, \mathrm{gr}\,\mathcal{E}_-^\times) = 0.$$

[10] the cohomology of the locally constant sheaf generated by $e^{ax}\xi$ or $e^{nax}\xi^n$ vanishes if $na \neq 0$, because $e^{an\mu}$ cannot be identically 1 for $\mu \in \Gamma$, so $H^*(X, \mathrm{gr}\,\mathcal{G}_{a-}) = 0$.

Theorem 2 *If X is of genus 1, the symbol map $\mathrm{Alg}_{\mathcal{E}} \to H^1(X,\omega)$ is onto. We will denote $\sigma(\mathcal{A}) = \alpha = (\alpha^{11}, \alpha^{10}, \alpha^{01}) \in H^{11} \times H^{10} \times H^{01}$ the symbol of an $\mathcal{E}$-algebra $\mathcal{A}$. Then*

(i) Algebras such that $\alpha^{11} = 0$ are characterized by the fact that they possess a global section of degree $\neq 0$, or an automorphism of symbol $\frac{d\xi}{\xi} = \sigma(\mathrm{Int}\,\xi)$. For such an algebra the set of global sections is $\mathbf{C}((\xi^{-1}))$ and except for $\mathcal{E}$ the group of automorphisms is $\mathcal{G}$.

$\mathcal{E}$ is distinguished by the fact that its symbol map $H^0(X, \mathrm{Aut}\,\mathcal{E}) \to H^0(X,\omega)$ is onto.

(ii) If $\alpha^{11} \neq 0$, $\mathcal{A}$ has no section of degree $\neq 0$ ($H^0(X,\mathcal{A}) = \mathbf{C}$), and $\mathcal{A}$ is completely determined by its symbol, in other words the image of $H^1(X,\mathcal{A})$ in $H^1(X, \mathrm{Aut}\,\mathcal{A})$ is reduced to a single point.

For such an algebra the group of automorphisms is a one parameter group with symbol $\mathbf{C}(a\,dx + b\frac{d\xi}{\xi})$ for some $(a,b) \neq 0$, except in the case $\alpha^{01} = 0, \alpha^{11}, \alpha^{10} \neq 0$ where there is no automorphism other than Id.

(iii) Among these, $\mathcal{E}$-algebras associated to a nontrivial $\mathcal{D}$-algebra are those for which $\sigma(\mathcal{A}) = \alpha^{11} \in H^{11}$ ($\alpha^{10} = \alpha^{01} = 0$). They are characterized by the fact that their group of automorphisms is a one parameter group with symbol $\sigma(\mathrm{Int}\ e^{tx})$, ($t \in \mathbf{C}$).

Thus for a torus X of genus 1, $\mathcal{D}$-algebras which give isomorphic $\mathcal{E}$-algebras are already isomorphic as $\mathcal{D}$-algebras, and $\mathcal{E}$-automorphisms are the same as $\mathcal{D}$-automorphisms, except for the canonical algebra $\mathcal{E}$.

Let $\mathcal{A}$ be an $\mathcal{E}$-algebra. The symbol map $\mathrm{Alg}_{\mathcal{E}} \to H^1(X,\omega)$ is onto because $H^2(X, \mathrm{gr}\,\mathcal{E}_-^{\times}) = 0$ so (Proposition 7) any cocycle with coefficients in ω is the symbol of an $\mathcal{E}$-algebra.

Next note that any star algebra on X lifts as the trivial $\mathcal{E}$-algebra $\mathcal{E}_{\mathbf{C}}$ on $\mathbf{C}$ with an action of the group Γ over the translation group :

$$\mu \to \mathcal{T}_\mu = T_\mu U_\mu \tag{30}$$

where T_μ is the translation ($x \to x + \mu, \xi \to \xi$) and the U_μ are automorphisms of $\mathcal{E}_{\mathbf{C}}$, subjected to the cocycle condition expressing that $\mu \to T_\mu U_\mu$ is a group homomorphism.

Here are typical examples (models) :

Example 4 Let $\mu \in \Gamma \to U_\mu = \alpha(\xi) \in \mathcal{G}$ be an additive map. This defines such a cocycle, because the U_μ commute with translations, hence an $\mathcal{E}$-algebra, obviously of the first type. since ξ is invariant. Typically the period group

$$\mu \to U_\mu = (\mathrm{Int}\,\xi)^{\alpha^{10}\mu + \alpha^{01}\overline{\mu}}$$

defines such an $\mathcal{E}$-algebra with symbol $\alpha^{10}+\alpha^{01}$ ($\alpha^{11}=0$).

Example 5 Let $\alpha(\mu)=\alpha^{10}\mu+\alpha^{01}\overline{\mu}$ be an additive map $\Gamma\to\mathbf{C}$ and $a\in\mathbf{C}$. Then the automorphisms $\mathrm{Int}\,(\exp\,\alpha(\mu)(ax+\mathrm{Log}\,\xi))$ commute and also commute with translations (because the commutator $[\xi, ax+\mathrm{Log}\,\xi]=a$ is a constant so $\exp s(ax+\mathrm{Log}\,\xi)$ commutes with translations, mod. constant factors which give trivial inner automorphisms). So the group homomorphism

$$\mu\to U_\mu=\exp\,\alpha(\mu)(ax+\mathrm{Log}\,\xi)$$

defines an $\mathcal{E}$-algebra, whose symbol is $a\alpha^{01},\alpha^{10},\alpha^{01}\in H^{11}\times H^{10}\times H^{01}$. If we identify the H^{pq} with spaces of differential forms, the symbol of $\mathcal{A}$ writes

$$\sigma(\mathcal{A})=(a\,d\alpha(x,\overline{x})d\overline{x}\ ,\ d\alpha(x,\overline{x})).$$

Such an algebra admits the automorphisms $\mathrm{Int}\,\exp s(ax+\mathrm{Log}\,\xi), s\in\mathbf{C}$.

The only symbols we have missed are those for which

$$a=\alpha^{11}\neq 0,\ b=\alpha^{10}\neq 0,\ \alpha^{01}=0$$

As model for this case we can take the algebra defined by

$$U_\mu=(\mathrm{Int}\,\xi)^{b\mu}\ \mathrm{Int}\,(e^{a\overline{\mu}\tilde{x}})$$

with $\tilde{x}=x(1+\frac{b}{\xi})^{-1}$ so that $\sigma(\tilde{x})=x$, $[\xi+b\mathrm{Log}\,\xi\,,\,\tilde{x}]=1:$ with this choice the $T_\mu\,(\mathrm{Int}\,\xi)^{b\mu}$ (symbolically $\exp\mu(\xi+b\mathrm{Log}\,\xi)$) commute with the $\mathrm{Int}\,e^{a\overline{\mu}\tilde{x}}$, so again the U_μ define an $\mathcal{E}$-algebra with symbol $\alpha^{11}=a\ ,\ \alpha^{10}=b\ ,\ \alpha^{01}=0$.

We now prove Theorem 2.

(i) First suppose that $\mathcal{A}$ has a nonzero section s of degree $\neq 0$. Then $\sigma(s)=c\xi^k$ for some constant $c\neq 0$ and integer $k\neq 0$; $c^{-1}s$ has a unique k-th root with symbol ξ (this is true locally because it works for pseudo-differential calculus; the roots with symbol ξ are unique and patch together into a global section). Similarly if a is an automorphism with symbol $\sigma(\mathrm{Int}\,\xi)$, locally there exists a unique section s with symbol ξ such that $a=\mathrm{Int}\,s$ ($a=\mathrm{Int}\,b$ determines b locally up to a constant, and $\sigma(b)=\xi$ fixes the constant so again these patch into a global section with symbol ξ).

If $\mathcal{A}$ has a section s with symbol ξ, then clearly all global sections of $\mathcal{A}$ are of the form $\sum_{k\leq k_0}c_k s^k$, and $H^0(X,\mathrm{Aut}\,\mathcal{A})$ contains the group $\mathcal{G}$ (formula (28)).

Furthermore, again by elementary pseudo-differential calculus, any two sections, resp. automorphisms of $\mathcal{E}$ of symbol ξ are locally conjugate, so $(\mathcal{A},a)$ is locally isomorphic to $(\mathcal{E},\mathrm{Int}\,e^{\xi})$, and $\mathcal{A}$ can be defined by a cocycle with coefficients in $\mathcal{G}$, the commutator of $\mathrm{Int}\,\xi$ (formula (28)). Hence $\alpha=\sigma(\mathcal{A})$ belongs to the image of $H^1(X,\mathcal{G})$ in $H^1(X,\omega)$ i.e. $\alpha\in H^1(X,\mathbf{C})$ and $\alpha^{11}=0$.

Conversely let $\alpha \in H^1(X, \mathbf{C})$ ($\alpha^{11} = 0$). Example 4 gives an algebra $\mathcal{A}$ with symbol α which has a section of symbol ξ.

Now any other algebra $\mathcal{A}'$ with symbol α is defined by a cocycle $(a_{ij}) \in H^1(X, \mathcal{A}_-^\times)$. We know that there is a surjective map from $H^1(X, \mathrm{gr}\,\mathcal{A}_-^\times)$ to $\mathrm{gr}\,H^1(X, \mathcal{A}_-^\times)$ and also that the map $H^1(X, \mathcal{G}) \to H^1(X, \mathcal{E}_-^\times) = H^1(X, \mathcal{A}_-^\times)$ is surjective. It follows that the embedding $\mathcal{G}_- \to \mathcal{A}_-^\times$ gives a surjective map $H^1(X, \mathcal{G}) \to H^1(X, \mathcal{A}_-^\times)$ (the symbol map (gr) is onto, and surjectivity follows by successive approximations). Thus any $\mathcal{E}$-algebra with symbol α can be defined from $\mathcal{A}$ by a cocycle with coefficients in $\mathcal{G}_-$, or from $\mathcal{E}$ with coefficients in $\mathcal{G}$; in particular it has a section with symbol ξ.

Lemma 3 *If such an algebra $\mathcal{A}$ ($\alpha^{11} = 0$) is not trivial, it has no other global automorphism than those of $\mathcal{G}$.*

Proof : If $\mathcal{A}$ has two automorphisms a, b with independent symbols, we may suppose that these symbols are $\sigma(\mathrm{Int}\,\xi), \sigma(\mathrm{Int}\,e^x)$. So $\mathcal{A}$ has a section α with symbol ξ such that $a = \mathrm{Int}\,\alpha$. The section $\mathrm{Log}\,b = \beta$ is locally well defined up to an additive constant, so the section $\gamma = [\alpha, \beta]$ is globally defined and commutes with α (as any global section of degree < 0).

Now the symbol of γ is 1, so γ is invertible, and replacing β by $\beta\gamma^{-1}$, we see that we can suppose $[\alpha, \beta] = 1$ (or equivalently $b^{-1}\xi b = \xi + 1$). It follows again, by successive approximations, that $\mathcal{A}$ equipped with two such automorphisms is locally isomorphic to $\mathcal{E}$ equipped with $\mathrm{Int}\,\xi$ and $\mathrm{Int}\,e^x$; but the only automorphisms which commute with both are obviously trivial (the leading term is constant because it commutes with x and ξ), so $\mathcal{A}$ is isomorphic to $\mathcal{E}$.

(ii) Suppose now $\alpha^{11} \neq 0$. Then any any section is a constant (of degree 0) and there is no global section of degree $\neq 0$. Let us choose an algebra $\mathcal{A}$ with symbol α (one of the models above). Here again since $H^2(X, \mathrm{gr}\,\mathcal{E}_-^\times) = 0$ the graded map $H^1(X, \mathrm{gr}\,\mathcal{A}_-^\times) \to \mathrm{gr}\,H^1(X, \mathcal{A}_-^\times)$ is surjective, and any cochain with coefficients in $\mathcal{A}_-^\times$ which is a cocycle mod. $\mathcal{A}_n (n < 0)$ is equivalent mod. $\mathcal{A}_n$ to a cocycle.

Lemma 4 *We have $H^1(X, \mathcal{A}_-^\times) = \mathbf{C}$.*

Proof : Let $\mathcal{A}$ be defined by a cocycle U_{ij} relative to some Stein covering $X = \bigcup X_i$. We can choose $U_{ij} = \mathrm{Int}\,e^{\alpha_{ij}x}\mathrm{Int}\,\xi^{s_{ij}}$ mod. lower order terms, with constant α_{ij}, s_{ij}.

Then $\alpha^{11} = \sigma(\mathcal{A})^{11} \in H^{11}$ corresponds to the (01) part of the cocycle α_{ij}, and does not vanish. Let $a = (a_i)$ be the "constant" 1-cochain $a_i = \xi^n$ ($n \leq 0$) on X_i. Its coboundary with coefficients in $\mathcal{A}_-^\times$ (computed in $\mathcal{A}(X_i)$) is of degree $n - 1$; more precisely on X_i we have

$$a_i - \alpha_{ij}(a_j) = \xi^n - a_{ij}(\xi^n) = \xi^n - (\xi - \alpha_{ij})^n + \cdots = n\alpha_{ij}\xi^{n-1} + \ldots \tag{31}$$

where the ... are lower order terms, because $\mathrm{Int}\,\xi$ commutes with ξ and the rest only contributes to terms of degree $\leq n-2$.

It follows by successive approximations that any cocycle of degree < -1 is equivalent to 0, and the same computation shows that two cocycles with the same leading term are equivalent. Since the symbol map $H^1(X, \mathcal{A}_-^\times) \to H^1(X, \mathcal{O}(-1)) = H^{01}$ is onto, we have $H^1(X, \mathcal{A}_-^\times) = \mathbf{C}$: any cocycle $\beta_{ij}\xi^{-1}$ is the symbol of a unique element of $H^1(X, \mathcal{A}_-^\times)$.

Lemma 5 *The map $H^1(X, \mathcal{A}_-^\times) \to H^1(X, \mathrm{Aut}\,\mathcal{A})$ is constant.*

Proof : The fibers of this map are the orbits of the action of $H^0(X, \omega)$ (Proposition 4). We will prove that this is transitive.

For this action $\sigma(Int\xi)$ acts by $u = (u_{ij}) \to u^s = u^s_{ij}$ with $\mathrm{Int}\,u^s_{ij} = \mathrm{Int}\,\xi^s_i\,\mathrm{Int}\,u_{ij}\,\mathrm{Int}\,\xi^s_j$ (mod. coboundary equivalence), where $\xi_i \in \mathcal{A}(X_i)$ has symbol ξ, and multiplication is the multiplication of $\mathcal{A}$. Now in the local frame on X_i we have $\xi_j = U_{ij}(\xi) = x - \alpha_{ij} + \ldots$ so for leading terms we get

$$\sigma(u^s_{ij}) = \sigma(u_{ij})\,\frac{\xi}{\xi - \alpha_{ij}} = \sigma(u)\,(1 + \alpha_{ij}\xi^{-1}) \tag{32}$$

or with additive notation $\sigma(u^s) = \sigma(u) + s\alpha^{11}$.

This proves Lemma 4, and the other assertions of Theorem 2 are immediate consequences.

5.4 The projective line

Let X be the projective line ($X = P_1(\mathbf{C})$). It is the union of the two open sets $X_0 = \{z \neq \infty\}, X_\infty = \{z \neq 0\}$, and since these are Stein, contractible ($\simeq \mathbf{C}$), $\mathcal{E}$ or $\mathcal{D}$-algebras are classified by cocycles reduced to one function on $X_0 \cap X_\infty$.

$\mathcal{D}$-algebras are classified by $H^1(X, \mathcal{O}/\mathbf{C}) = H^2(X, \mathbf{C}) = \mathbf{C}$. The $\mathcal{D}$-algebra $\mathcal{D}_s$ ($s \in \mathbf{C} = H^1(X, \mathcal{O}/\mathbf{C})$) is defined by the cocycle $(\mathrm{Int}\ z)^s$.

Let us introduce homogeneous coordinates $x, y\,(z = \frac{x}{y})$. We make use of the sheaf of homogeneous differential operators $\mathcal{D}^{hom}$ on $\mathbf{C}^2$, i.e. differential operators of x and y which commute with the generator of homotheties $\rho = x\partial_x + y\partial_y$; this algebra is generated by ρ and the operators

$$e = x\partial_y, \quad h = x\partial_x - y\partial_y, \quad f = y\partial_x \tag{33}$$

which satisfy the relations

$$[h, e] = 2e, \quad [h; f] = -2f, \quad [e, f] = h, \quad h^2 + 2(ef + fe) = \rho(\rho + 2). \tag{34}$$

$\mathcal{D}_s$ is isomorphic to the quotient sheaf $\mathcal{D}^{hom}/(\rho + s)$ and can be thought of as the sheaf of differential operators on the virtual sheaf $\mathcal{O}(s)$ on X of homogeneous functions of degree s of x, y (which really only exists s when s is an integer).

We now turn to $\mathcal{E}$-algebras.

Lemma 6 *If $X = P_1$ is the projective line then*

(i) $H^0(X, \mathrm{gr}\,\mathrm{Aut}\,\mathcal{E}) = 0$ hence $H^0(X, \mathrm{Aut}\,\mathcal{A}) = 0$ for any $\mathcal{E}$-algebra $\mathcal{A}$.

(ii) $H^1(X, \omega) = 0$ and $H^1(X, \mathcal{E}_-^\times) \to H^1(X, \mathrm{Aut}\,\mathcal{E})$ is one to one.

Proof : In homogeneous coordinates as above, $\mathcal{O}(n)$ is the sheaf of homogeneous functions of degree $2n$ of x, y and has no global section if $n < 0$, hence $H^0(X, \mathrm{gr}\,\mathcal{E}_-^\times) = H^0(X, \mathcal{E}_-^\times) = 0$.

We have proved above (Proposition 5, 6) that for the projective line we have $H^0(X, \omega) = H^1(X, \omega) = 0$, hence the lemma.

Note that any $q \in H^1(X, \mathrm{Aut}\,\mathcal{E})$ has a unique "normalized" representative :

$$q_{0\infty} = \sum_{0>p>2q} a_{pq} z^p \zeta^q \in \widehat{\mathcal{O}}(X_0 \cap X_\infty). \tag{35}$$

This is because the two vector fields $\partial_0 = \partial_z, \partial_\infty = \partial_{1/z}$ are globally holomorphic and elliptic on X_0, resp. X_∞ and their symbols are $\zeta, -z^2\zeta$, so any cocycle can uniquely be reduced to the form above, as for the additive cohomology group $H^1(X, \mathrm{gr}\,\mathcal{E}_-^\times)$.

To compare $\mathcal{D}$-algebras and $\mathcal{E}$-algebras it is convenient to use the following intermediate exact sequence ; let $\mathrm{Int}\,\mathcal{E}_0 \simeq \mathcal{E}_0^\times / \mathbf{C}^\times$ be the group of inner automorphisms of $\mathcal{E}_0$; we have an exact sequence :

$$0 \to \mathrm{Int}\,\mathcal{E}_0 \to \mathrm{Aut}\,\mathcal{E} \to \mathbf{C} \to 0$$

hence a surjection

$$H^1(X, \mathrm{Int}\,\mathcal{E}_0) \to H^1(X, \mathrm{Aut}\,\mathcal{E}) \tag{36}$$

whose fibers are the orbits of the action of $\mathbf{C} = H^0(X, \mathbf{C})$ on $H^1(X, \mathrm{Int}\,\mathcal{E})$ ($q_{0\infty} \to (\mathrm{Int}\,\partial_0)^s\, q_{0\infty}\, (\mathrm{Int}\,\partial_\infty)^{-s}$, cf. Proposition 4).

Lemma 7 *We have the following relation :*

$$(\mathrm{Int}\,z)^{-s-2} = (\mathrm{Int}\,\partial_0)^{s+1} (\mathrm{Int}\,z)^s (\mathrm{Int}\,\partial_\infty)^{-s-1}. \tag{37}$$

Proof : If $s = k$ is a positive integer we have

$$z^{-k-2}(z^2\partial)^{k+1} = \partial^{k+1} z^k.$$

Indeed both are ordinary differential operators of order $k+1$, with leading term $z^s \partial^{s+1}$, which kill all monomials z^j, $0 \geq j \geq -k$.

Identity (37) for arbitrary s follows, because it is polynomial in s mod. $(\mathrm{Aut}\,\mathcal{E})_m$, for any $m < 0$.

It follows that $\mathcal{D}_s$ resp. $\mathcal{D}_{-s-2}$ give isomorphic $\mathcal{E}$-algebras, although they are not isomorphic $\mathcal{D}$-algebras. This is the only case where two $\mathcal{D}$-algebras on

$X = P_1$ give isomorphic $\mathcal{E}$-algebras: the algebra of global sections is obviously an invariant of an $\mathcal{E}$-algebra, and in this the global sections e, h, f (with the notations above) are well defined (up to an additive constant by their symbols, and the commutation relations fix these constants). It follows that $s(s+2) = h^2 + 2(ef + fe)$ is an invariant of the $\mathcal{E}$-algebra coming from $\mathcal{D}_s$.

Note that $\mathcal{D}$-algebras form a one-parameter family, so there are many $\mathcal{E}$-algebras which do not come from an $\mathcal{E}$-algebra.

As last remark we turn to the following problem: does there exist a global symbolic calculus, i.e. is the underlying sheaf of a given $\mathcal{E}$-algebra isomorphic to $\widehat{\mathcal{O}}$? This is always true for real $\mathcal{E}$-algebras, where one can patch global symbols using a partition of the unity.

Let us examine what happens on $X = P_1(\mathbf{C})$. There is a canonical 2-covering of $T^*X - \{0\}$ by $\mathbf{C}^2 - \{0\}$: $(u, v) \to (z = u/v, \zeta = \frac{1}{2}v^2)$. If $\mathcal{A}$ is a $\mathcal{E}$-algebra on $\Sigma = T^*X$ its pull-back on $\Sigma' = \mathbf{C}^2 - \{0\}$ is a star-algebra for the canonical Poisson bracket ($\{v, u\} = 1$), equipped with an involution above the symmetry $(u, v) \to (-u, -v)$ (note that on Σ', u and v are of degree $\frac{1}{2}$). If $\mathcal{A}$ has a global symbolic calculus, its pull-back defines a star-product on $\widehat{\mathcal{O}}(\Sigma')$.

Now on $\widehat{\mathcal{O}}(\Sigma')$ there is (up to isomorphism) only one star-algebra law for the canonical Poisson bracket, generated by u, v with the relation $[v, u] = 1$. Up to isomorphism this is given by the representation $\sum a_{pq} u^p v^q \to \sum a_{pq} u^p \partial_u^q$. For this law there are many global sections (i.e. all polynomials of u, v) : the global sections e, h, f are necessarily

$$e = -\frac{1}{2}u^2, \quad h = 2u * v + \frac{1}{2}, \quad f = \frac{1}{2}v^2 \tag{38}$$

because their respective symbols are

$$\sigma(e) = -z^2\zeta \sim -\frac{1}{2}u^2 \quad \sigma(h) = 2z\zeta \sim u\,v, \quad \sigma(f) = \zeta \sim \frac{1}{2}v^2$$

these determine e, h, f up to additive constants, and the commutation relations (34) determine the constants as above.

For these constants we get

$$h^2 + 2(ef + fe) = -\frac{3}{4} \quad \text{so} \quad s = -\frac{1}{2}. \quad \text{or} \quad s = -\frac{3}{2} \tag{39}$$

We have proved :

Proposition 12 *The only $\mathcal{D}$-algebras on P_1 for which there is a global total symbolic calculus are $\mathcal{D}_{-1/2}$ and $\mathcal{D}_{-3/2}$. In particular there is no global total symbolic calculus for $\mathcal{D}$.*

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