Complément 2 : Méthode de Newton

1) Motivation :

- $\min_{x \in \mathbb{R}^n} J \Rightarrow \nabla J(x) = 0$ si x mini. local

$x \in \mathbb{R}^n \to F(x) = \nabla J(x) \in \mathbb{R}^n \to$ recherche de zéro de F

- $\min_{Cx=f} J$: x mini. local $+ (CQ)_x \Rightarrow \exists \lambda \in \mathbb{R}^p$ $\nabla J(x) + C^T\lambda = 0$, $Cx = f$

  - Cas $J(x) = \frac{1}{2}(x, Ax) - (b, x)$, A SDP : $\nabla J = Ax - b$

  Trouver (x, λ) t.q. $\begin{bmatrix} A & C^T \\ C & 0 \end{bmatrix} \begin{pmatrix} x \\ \lambda \end{pmatrix} = \begin{pmatrix} 0 \\ f \end{pmatrix}$ dans $\mathbb{R}^n \times \mathbb{R}^p$

  - Cas général : $\min J(x)$, $\varphi(x) = 0$

$$\begin{cases} \nabla J(x) + \sum_{j=1}^{p} \lambda_j \nabla \varphi_j(x) = 0 & \mathbb{R}^n \quad (\text{si } (\nabla \varphi_j) \text{ libre}) \\ \varphi(x) = 0 & \mathbb{R}^p \end{cases}$$

$$F(x, \lambda) = \begin{cases} \nabla J(x) + \sum_j \lambda_j \nabla \varphi_j(x) \\ \varphi(x) \end{cases}$$ trouver un zéro ds $\mathbb{R}^{n+p}$

- $\min_{Cx \leq f} J$ (J quadratique) : $\exists \lambda \in \mathbb{R}^p, (\lambda \geq 0)$

$$Ax - b + C^T\lambda = 0$$
$$\lambda \geq 0, \ (\lambda, Cx - f) = 0$$
$$Cx - f \leq 0$$

$$\Longleftrightarrow F(x, \lambda) = \begin{cases} Ax - b + C^T\lambda = 0 \\ \min(\lambda, -(Cx - f)) = 0 \end{cases}$$

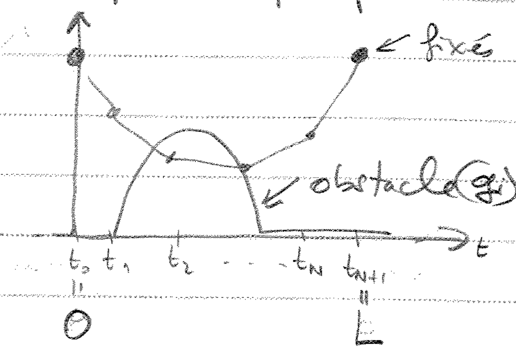
• Cas $\min_{x \geq g} J(x)$ $\quad J(x) = \frac{1}{2}(x, Ax) - (b, x)$: pb obstacle

↓
modèle de fil élastique pesant

$x_i = u(t_i)$ hauteurs

$g_i = \varphi(t_i)$ hauteur obstacle

$A = \frac{1}{h^2}\begin{bmatrix} 2 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{bmatrix}$ $\quad h = \frac{L}{N+1}$



A : SDP, $u \in \mathbb{R}^N$, $K = \{x \in \mathbb{R}^N / x \geq g\}$.

Prop On a équivalence entre

(i) u est le minimum de J sur K

(ii) $u \geq g$ et $(Au - b, v - u) \geq 0 \quad \forall v \in K$

(iii) $Au - b \geq 0$, $u \geq g$, $(Au - b, u - g) = 0$

(iv) $\min(Au - b, u - g) = 0$ dans $\mathbb{R}^N$

$\Rightarrow$ recherche $F(x) := \min(Ax - b, x - g) = 0$ dans $\mathbb{R}^N$

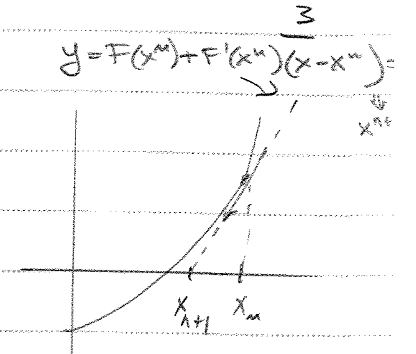
Rem / GPF $\quad x^{n+1} = P_{\{x \geq g\}}(x^n - \rho(Ax^n - b)) = \max(g, x^n - \rho(Ax^n - b))$

$\Leftrightarrow$ recherche d'un point fixe de $f_1(x) = \max(g, x - \rho(Ax - b))$

$\Leftrightarrow$ x zéro de : $\quad x - \max(g, x - \rho(Ax - b))$

$= \min(x - g, \rho(Ax - b)) \quad \forall \rho > 0$

2) Méthode de Newton (cas régulier)



Th (point fixe)

Algo (Newton) :
- $x^0 \in \mathbb{R}^N$ départ
- pour $n \geq 0$:
  $x^{n+1} = x^n - F'(x^n)^{-1} \cdot F(x^n)$.

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$E = \mathbb{R}^N$; $F : E \to E$, C^2.
Soit $a \in E$ tq $F(a) = 0$ et $F'(a)$ inversible.
Alors $\exists r > 0$,
(i) $\exists !$ zéro de F dans $B_F(a,r)$ (c'est a)
(ii) $\forall x^0 \in B_F(a;r)$ $x^n \to a$
(iii) Convergence quadratique :
$\exists C, \exists n_0, \forall n \geq n_0, \quad \|x^{n+1} - a\| \leq C \|x^n - a\|^2$

Preuve : $g(x) = x - F'(x)^{-1} \cdot F(x)$: $g(a) = a$.

g est C^1 et

$$g(x+h) = x + h - \underbrace{(F'(x) + F''(x) \cdot h)^{-1}} \cdot (F(x) + F'(x) \cdot h + o(h))$$

$$(F'(x)[I + F'(x)^{-1} \cdot F''(x) \cdot h])^{-1}$$

$$= (F'(x)^{-1} - F'(x)^{-1}(F''(x) \cdot h) \cdot F'(x)^{-1})$$

$$= g(x) + \cancel{h} - \underbrace{[\cancel{h} - F'(x)^{-1}(F''(x) \cdot h) F'(x)^{-1} \cdot F(x)]}_{= \, g'(x) \cdot h} + o(h)$$

donc

$$\|g'(x) \cdot h\| \leq \|F'(x)^{-1}\|^2 \cdot \|F''(x)\| \cdot \|h\| \times \|F(x)\| \leq \tilde{M}_1^2 M_1 \cdot M_2 \|x - a\| \cdot \|h\|$$

$\tilde{M}_1 = \sup_{B_F(a,r_0)} \|F'(x)^{-1}\|$

où $M_2 = \sup_{B_F(a,r_0)} \|F''(x)\|$

$\left(\|F(x) - F(a)\| \leq \|F'(\theta)\| \cdot \|x - a\| \leq M_1 \cdot \|x - a\| \right)$

donc $\|g'(x)\| \leq M_1 \tilde{M}_1^2 M_2 \, r$ dans $B_F(a,r)$: on prend $r = \min\left(r_0, \frac{1/2}{M_1 M_2 \tilde{M}_1^2}\right)$

$\forall x \in B_F(a, r_1)$: $\|g'(x)\| \leq 1/2$: g contractante de $B_F(a,r)$ complet $\Rightarrow \exists !$ pt fixe quadr : $0 = F(x^n) + F'(x^n)(x^{n+1} - x^n)$.

$$0 = F(a) = F(x^n) + F'(x^n)(a - x^n) + \overbrace{\mathcal{O}(\|a - x^n\|^2)}^{\varepsilon_n}$$

$F \in C^2(\mathcal{V}(a))$

$$\Rightarrow \quad F'(x^n)(x^{n+1} - a) = \varepsilon_n .$$

$$\|x^{n+1} - a\| \le \|F'(x^n)^{-1}\| \, \|\varepsilon_n\| \le M_1 \|\varepsilon_n\| \le M_1 . C . \|x^n - a\|^2$$

Rem. $\quad C\|x^n - a\| \le (C\|x^{n-1} - a\|)^2 \cdots \le (C\|x^0 - a\|)^{\underbrace{2 \times \cdots \times 2}_{n \text{ fois}} = 2^n}$

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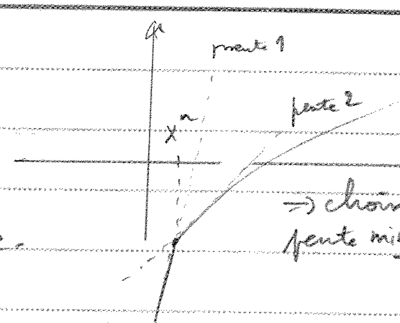
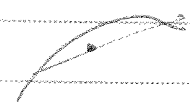
3) Newton "Semi-Smooth"



$\Rightarrow$ choisir la pente minimale

$F(x) = \min(Ax - b, x - g)$ concave.

Lemme : Si $F(x) = \min_{\alpha \in A} L_\alpha(x)$ où $L_\alpha(x)$ concave, Alors F est concave.



D $\quad L_\alpha(\theta x + (1-\theta) y) \geqslant \theta L_\alpha(x) + (1-\theta) L_\alpha(y)$

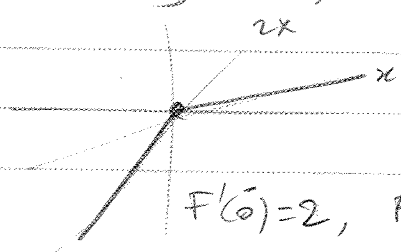
$\min_\alpha$: $\quad F(\theta x + (1-\theta)y) \geqslant \min_{\alpha \in A} \left(\theta L_\alpha(x) + (1-\theta) L_\alpha(y) \right)$

$$\geqslant \min_{\alpha \in A} \left(\underset{\geqslant 0}{\theta} L_\alpha(x) \right) + \min_{\beta \in A} \left(\underset{\geqslant 0}{(1-\theta)} L_\beta(y) \right)$$

$$= \theta F(x) + (1-\theta) F(y) \qquad \square$$

Remarque F n'est plus dérivable au point x t.q. $\exists i, \quad (Ax - b)_i = (x - g)_i \quad , A_{ii} \neq 1$

Ex : $F(x) = \min(2x, x)$



$F'(0^-) = 2, \quad F'(0^+) = 1$

Algorithme : $\begin{cases} x^0 \in \mathbb{R}^N, \\ x^{k+1} = x^k - F'(x^k)^{-1} \cdot F(x^k) \end{cases}$

où $F(x^k)_i = \begin{cases} (Ax^k - b)_i & \text{si } (Ax^k - b)_i \leq (x^k - g)_i \\ (x^k - g)_i & \text{sinon} \end{cases}$

$F'(x^k)_{ij} = "\dfrac{\partial F(x^k)_i}{\partial x_j}" := \begin{cases} A_{ij} & \text{si } (Ax^k - b)_i \leq (x^k - g)_i \\ I_{ij} & \text{sinon} \end{cases}$

Réécriture avec des contrôles : $F(x)_i = \min(\underbrace{(Ax - b)_i}_{\alpha_i = 0}, \underbrace{(x - g)_i}_{\alpha_i = 1})$

$\alpha \in \{0,1\}^N$: $B_{ij}(\alpha) = \begin{cases} A_{ij} & \text{si } \alpha_i = 0 \\ I_{ij} & \text{si } \alpha_i = 1 \end{cases}$

$b_{ij}(\alpha) = \begin{cases} b_i & \text{si } \alpha_i = 0 \\ g_i & \text{si } \alpha_i = 1 \end{cases}$

alors $F(x)_i = \min(\underset{\substack{(B(\alpha)x - b(\alpha))_i \\ \text{pour } \alpha_i = 0}}{(Ax - b)_i}, \underset{\substack{(B(\alpha)x - b(\alpha))_i \\ \text{pour } \alpha_i = 1}}{(x - g)_i})$

$= \min_{\alpha_i \in \{0,1\}} (B(\alpha)x - b(\alpha))_i$

$[(\alpha_j)_{j \neq i}$ quelconques$]$

donc $\boxed{F(x) = \min_{\alpha \in \{0,1\}^N} B(\alpha)x - b(\alpha)}$ composante par composante.

Algorithme de Howard : $x^0 \in \mathbb{R}^N$, puis $k \geq 0$:

$\begin{cases} \alpha_i^{k+1} := \operatorname{argmin}_\alpha (B(\alpha)x^k - b(\alpha))_i \\ x^{k+1} : \text{ tq } B(\alpha^{k+1})x^{k+1} - b(\alpha^{k+1}) = 0 \end{cases}$

Prop: | x^n : itérés de Newton. y^n : itérés de Howard
| $x^0 = y^0 \Rightarrow \forall k \geq 0,\ x^k = y^k$

D. • $F(x^k) = \min_\alpha \left(B(\alpha) x^k - b(\alpha) \right) \overset{\alpha = \alpha^{k+1}}{=} B(\alpha^{k+1}) x^k - b(\alpha^{k+1})$

$= B(\alpha^{k+1}) y^{k+1}$

$\Rightarrow y^{k+1} = x^k - B(\alpha^{k+1})^{-1} . F(x^k)$

• $B(\alpha^{k+1})_{ij} = \begin{cases} A_{ij} & \text{si } \alpha_i^{k+1} = 0 \iff (Ax^k - b)_i \leq (x^k - g)_i \\ I_{ij} & \text{sinon} \end{cases} \quad = F'(x^k)$ sinon

$\Rightarrow y^{k+1} = x^{k+1}$. $\square$

Définition: 1) M est dite Monotone si $\forall x \in \mathbb{R}^N$, $Ax \geq 0 \Rightarrow x \geq 0$.

2) M est à diagonale dominante (DD) si $\exists \delta > 0, |A_{ii}| \geq \delta + \sum_{j \neq i} |A_{ij}|$.

3) M est dite M-matrice si | $M_{ii} \geq 0$, $M_{ij} \leq 0\ \forall i \neq j$
| M : D.D.

Lemme 1 A : M-matrice $\Rightarrow$ A monotone

$\left(i / x_i = \inf x_j \quad 0 \leq (Ax)_i = A_{ii} x_i - \sum_{j \neq i} |A_{ij}| x_j \leq \left(A_{ii} - \sum_{j \neq i} |A_{ij}| \right) x_i \leq \delta x_i \right)$

EX: • $A = \begin{bmatrix} 2 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{bmatrix}$ est monotone. • $(A + \varepsilon I)$: M-matrice, $\varepsilon > 0$

Lemme 2 : A : M-matrice $\Rightarrow \forall \alpha$, $B(\alpha)$ monotone

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Théorème: (On suppose que $\forall \alpha \in \mathcal{A}$, $B(\alpha)$ monotone. Alors.

$F(x) = \min_{\alpha \in \{0,1\}^N} \left(B(\alpha) x - b(\alpha) \right)$

| (i) $\exists ! x \in \mathbb{R}^N$, $F(x) = 0$
| (ii) $x^k \nearrow$ et $\lim x^k = x$
| (iii) $\exists k \leq 2^N$, $x^k = x$.

Th 2: | (pb obstacle): A. M-matrice. Alors $\exists k \leq N+1$, $x^k = x$

$F(x) = \min(Ax - b, x - g)$

Preuve Th 1 :

* Unicité de x : Notons $\alpha^x := \underset{\alpha}{\mathrm{argmin}}\, B(\alpha)x - b(\alpha)$

Si $F(x) \le F(y)$ alors $B(\alpha^x)x - \cancel{b(\alpha^x)} \le B(\alpha^y)y - b(\alpha^y)$
$\le B(\alpha^x)y - \cancel{b(\alpha^x)}$

$\Rightarrow B(\alpha^x)(y-x) \geq 0$

$\Rightarrow y - x \geq 0$

$\Rightarrow x \le y$

Donc $F(x) = F(y) \Rightarrow x = y$: F injective

* $x^k \nearrow$: $B(\alpha^{k+1})x^{k+1} - \cancel{b(\alpha^{k+1})} = 0 = B(\alpha^k)x^k - b(\alpha^k)$
$\geq B(\alpha^{k+1})x^k - \cancel{b(\alpha^{k+1})}$

$\Rightarrow x^{k+1} \geq x^k$ ↑ min pour α^k

* $x^k = B(\alpha^k)^{-1} b(\alpha^k)$ fini $\Rightarrow$ borné ; $(x^k_i)_k \nearrow$ bornée donc CV
notons $x = \lim x^k$.

$$F(x^k) = B(\alpha^{k+1})\underset{x^{k+1}+\varepsilon_k}{\underset{\parallel}{x^k}} - b(\alpha^{k+1}) = 0 + \varepsilon_k \xrightarrow[k\to\infty]{} 0$$

et F est lipschitz donc continue : $F(x) = 0$: x solution
$\left(\|F(y) - F(x)\| \le (\max_\alpha \|B(\alpha)\|) \, \|y - x\| \right)$.

* $\alpha^k \in \{0,1\}^N = \mathcal{A}$ de cardinal fini 2^N $\{\alpha^1, \dots, \alpha^{2^N}\}$
donc $\exists p < q$, $p \le 2^N$, $\alpha^p = \alpha^q$. $x^p = x^q$

$x^p \le x^{p+1} \dots \le x^q \Rightarrow x^p = x^{p+1} \dots = x^q$

donc $F(x^p) = B(\alpha^{p+1})\underset{x^{p+1}}{\underset{\parallel}{x^p}} - b(\alpha^{p+1}) = 0$: x^p solution. □

* Th2 : $\boxed{L\ \alpha^k_i = 0 \Rightarrow \alpha^{k+1}_i = 0}$ $\alpha^k_i = 0 \Rightarrow (Ax^k - b)_i = (B(\alpha^k)x^k)_i = 0$ or
$(x^k - g)_i \geq 0$ car $x^k \geq x^1 \geq g$.

donc $\alpha^{k+1}_i \overset{def}{=} 0$.

donc $\alpha^k = \begin{pmatrix} \alpha^k_1 \\ \vdots \\ \alpha^k_N \end{pmatrix}$ suite $\searrow$ $\alpha^1 \geq \alpha^2 \quad \geq \alpha^N \geq \alpha^{N+1}$ $s^k = \sum_{i=1}^{N} \alpha^k_i \in [0, N]$

$\exists p < q$ $1 \le p \le N+1$, $\alpha^p = \alpha^q$: x^p solution □

$x^1 \geq g$ $\underset{pr}{\Rightarrow}$ $0 = (Ax^1 - b)_i \le (x^1 - g)_i$ $\in \{0,1\}^N$
$0 = (x^1 - g)_i$