

Mahler measure of curves and surfaces

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April 16, 2008

Introduced by Mahler in 1962,
the logarithmic Mahler measure of a polynomial P is

$$m(P) := \frac{1}{(2\pi i)^n} \int_{\mathbb{T}^n} \log |P(x_1, \dots, x_n)| \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n}$$

and its Mahler measure

$$M(P) = \exp(m(P))$$

where

$$\mathbb{T}^n = \{(x_1, \dots, x_n) \in \mathbb{C}^n / |x_1| = \dots = |x_n| = 1\}.$$

- $n = 1$

By Jensen's formula, if $P \in \mathbb{Z}[X]$ is monic, then

$$M(P) = \prod_{P(\alpha)=0} \max(|\alpha|, 1).$$

So it is related to **Lehmer's question (1933)**

Does there exist $P \in \mathbb{Z}[X]$, monic, non cyclotomic, satisfying

$$1 < M(P) < M(P_0) = 1.1762 \dots ?$$

The polynomial

$$P_0(X) = X^{10} + X^9 - X^7 - X^6 - X^5 - X^4 - X^3 + X + 1$$

is the Lehmer polynomial, in fact a Salem polynomial.

Lehmer's problem is still open.

A partial answer by Smyth (1971)

$$M(P) \geq 1.32 \dots$$

if P is non reciprocal.

- $n > 1$,
- $m(x_0 + x_1) = 0$ (by Jensen's formula)
-

$$m(x_0 + x_1 + x_2) = \frac{3\sqrt{3}}{4\pi} L(\chi_{-3}, 2) = L'(\chi_{-3}, -1) \text{ Smyth (1980)}$$

- $$m(x_0 + x_1 + x_2 + x_3) = \frac{7}{2\pi^2} \zeta(3) \text{ Smyth (1980)}$$

These are the first explicit Mahler measures.

Now, many explicit Mahler measures from **Boyd, Smyth, Villegas, Lalin, Rogers, ...**

But I want to point out two results.

For $n = 2$, there is **Deninger's** guess (1996)

$$m\left(X + \frac{1}{X} + Y + \frac{1}{Y} + 1\right) \stackrel{?}{=} \frac{15}{4\pi^2} L(E, 2) = L'(E, 0)$$

where E is the elliptic curve of conductor 15 defined by the Laurent polynomial,

and also a pioneer result by **Cassaigne-Maillot** (1997)

$$m(a+bx+cy) = \begin{cases} \frac{1}{\pi} (D(\frac{|a|}{|b|} e^{i\gamma}) + \alpha \log |a| + \beta \log |b| + \gamma \log |c|) & \text{if } \Delta \\ \max\{\log |a|, \log |b|, \log |c|\} & \text{if not } \Delta \end{cases}$$

Δ means that $|a|, |b|, |c|$ are the sides of a triangle with angles α, β, γ
 D is the **Bloch-Wigner dilogarithm**

The Bloch-Wigner dilogarithm

In Maillot's result we notice the Bloch-Wigner dilogarithms, so perhaps a link with the **Bloch group of number fields** and **zeta functions of number fields**.

The Bloch-Wigner dilogarithm of a complex number z is defined as

$$D(z) := \Im Li_2(z) + \log |z| \arg(1 - z)$$

where Li_2 is the ordinary dilogarithm. It is a univalued, real analytic function in $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$, continuous in $\mathbb{P}^1(\mathbb{C})$.

Properties

- $$D\left(\frac{1}{z}\right) = D(1 - z) = D(\bar{z}) = -D(z)$$

- the distribution relation

$$D(z^n) = n \sum_{k=0}^{n-1} D(\zeta^k z)$$

where ζ denotes a primitive n -th root of unity

Properties of the Bloch-Wigner dilogarithm

- the five-term relation

$$D(x) + D(y) + D(1 - xy) + D\left(\frac{1-x}{1-xy}\right) + D\left(\frac{1-y}{1-xy}\right) = 0.$$

Moreover, if the wedge differential η is defined by

$$\eta(x, y) = \log(|x|)d \arg y - \log(|y|)d \arg x,$$

the derivative of D verifies

$$dD(x) = \eta(x, 1 - x).$$

Notice that the differential η is

- multiplicative in each variable
- antisymmetric
- and if $\alpha \neq \beta$ satisfies the **Tate's relation**

$$\eta(t-\alpha, t-\beta) = \eta\left(\frac{t-\alpha}{t-\beta}, 1 - \frac{t-\alpha}{t-\beta}\right) + \eta(t-\alpha, \alpha-\beta) + \eta(\beta-\alpha, t-\beta).$$

The Bloch group

Let F be a field and define the abelian groups

$$\mathcal{C}(F) \subset \mathcal{A}(F) \subset \mathbb{Z}[\mathbb{P}_F^1]$$

where $\mathcal{A}(F) = \ker \beta$ if

$$\beta : \mathbb{Z}[\mathbb{P}_F^1] \rightarrow \Lambda^2 F^\times$$

is defined by

$$\beta(0) = \beta(1) = \beta(\infty) = 0$$

$$\beta(x) = (x) \wedge (1 - x)$$

and

$$\mathcal{C}(F) := \langle [x] + [y] + [1 - xy] + \left[\frac{1-x}{1-xy}\right] + \left[\frac{1-y}{1-xy}\right] \rangle$$

is generated by the five-term relation.

The Bloch group (continued)

The **Bloch group** is now defined by the exact sequence

$$0 \rightarrow \mathcal{B}_2(F) \rightarrow \mathbb{Z}[\mathbb{P}_F^1]/\mathcal{C}(F) \xrightarrow{\beta} \Lambda^2 F^\times$$

The class of x in $\mathcal{B}_2(F)$, $[x]$, behaves like a Bloch-Wigner dilogarithm.

The complex

$$\mathcal{B}_F(2) \otimes \mathbb{Q} : \mathcal{B}_2(F)_{\mathbb{Q}} \xrightarrow{\delta_1^2} (\Lambda^2 F^\times)_{\mathbb{Q}}$$

has a cohomology related to K -theory by Matsumoto's theorem

$$H^2(\mathcal{B}_F(2)) \simeq K_2(F).$$

These things are related to results by **Zagier** on $\zeta_F(2)$.

Theorem

(Zagier) Let F be a number field, $[F : \mathbb{Q}] = n_+ + n_-$. Then the Bloch group modulo torsion satisfies

①

$$\mathcal{B}(F)/\text{tors} \simeq \mathbb{Z}^{n_-}$$

② *If ξ_1, \dots, ξ_{n_-} is a basis of $\mathcal{B}(F)/\text{tors}$ and $\sigma_1, \dots, \sigma_{n_-}$ the complex embeddings of F into \mathbb{C} , then*

$$\det(D(\sigma_i(\xi_j)))_{i,j=1,\dots,n_-} \frac{\pi^{2n_+}}{\sqrt{|D_F|}} = r\zeta_F(2).$$

Maillot (2003) using a result of Darboux (1875): the Mahler measure of P which is the integration of a differential form on a variety, when P is non reciprocal, is in fact an integration on a smaller variety and the expression of the Mahler measure is encoded in the cohomology of the smaller variety. This insight can be realized for curves of genus 0 through Vandervelde's theorem

Elementary case of Maillot's insight

Theorem

Vandervelde Let $P \in \mathbb{C}[x, y]$ such $P = 0$ parametrized by $x = f(t) = \lambda_1 \prod ((t - \alpha_r)^{l_r})$ and $y = g(t) = \lambda_2 \prod (t - \beta_s)^{m_s}$. Let $S = \{(x, y) \in \mathbb{C}^2 / |x| = 1 \quad |y| \geq 1\}$ and $\gamma_1, \dots, \gamma_n$ be the paths which map to S under $t \mapsto (f(t), g(t))$, oriented via $f(\gamma_j) \subset \mathbb{T}^1$. If u_j and v_j denote the initial and terminal points of γ_j and $P(x, y)$ has leading coefficient $\lambda(x)$ as a polynomial in y , then

$$2\pi m(P) = 2\pi m(\lambda(x)) + \sum_{j=1}^n \left(\sum_{r,s} l_r m_s \left[D\left(\frac{u_j - \alpha_r}{\beta_s - \alpha_r}\right) - D\left(\frac{v_j - \alpha_r}{\beta_s - \alpha_r}\right) \right] \right. \\ \left. + \sum_r l_r \log |\tilde{g}(\alpha_r)| \operatorname{wind}(\gamma_j, \alpha_r) - \sum_m l_s \log |\tilde{f}(\beta_s)| \operatorname{wind}(\gamma_j, \beta_s) \right),$$

where

$$\operatorname{wind}(\gamma, \beta) = \int_{\gamma} \frac{dz}{z - \beta}.$$

Theorem

Bertin (2008)

$$\begin{aligned}m((x+1)^2y + x^2 + x + 1) &= \frac{1}{3}L'(\chi_{-7}, -1) = \frac{1}{3} \frac{7^{3/2}}{4\pi} L(\chi_{-7}, 2) \\ &= \frac{7\sqrt{7}}{2\pi^3} \zeta_{\mathbb{Q}(\sqrt{-7})}(2)\end{aligned}$$

$$\begin{aligned}m((x^2 + x + 1)y + x^2 + 1) &= \frac{1}{12}L'(\chi_{-15}, -1) = \frac{1}{12} \frac{15^{3/2}}{4\pi} L(\chi_{-15}, 2) \\ &= \frac{15\sqrt{15}}{8\pi^3} \zeta_{\mathbb{Q}(\sqrt{-15})}(2).\end{aligned}$$

Proof.

- By Vandervelde, only one path γ_1 with initial point $u_1 = \frac{-3+\sqrt{-7}}{4} = \alpha$ and terminal point $\bar{\alpha}$.

-

$$2\pi m(P) = 2D\left(\frac{-3+\sqrt{-7}}{4}j^2\right) + 2D\left(\frac{-3+\sqrt{-7}}{4}j\right) + 4D\left(\frac{3+\sqrt{-7}}{4}\right).$$

- By distribution formula for $n = 3$ and $n = 2$,

$$\pi m(P) = D(\alpha) - D(\alpha^2) + \frac{1}{3}D(\alpha^3).$$

- Easy to prove

$$\xi = [\alpha] - [\alpha^2] + \frac{1}{3}[\alpha^3] \in \mathcal{B}_2(\mathbb{Q}(\sqrt{-7})).$$



Proof.

- Thus, by Zagier

$$\pi^2 D(\xi) = r' \sqrt{7} \zeta_F(2)$$

for $r' \in \mathbb{Q}^\times$, that is $m(P) = rd_7$. We must prove now that $r = \frac{1}{3}$.

- From five-term relation, distribution relations and the equation $1 + \alpha + \alpha^2 = -\frac{\alpha}{2}$,

$$D(\xi) = -2D(-\alpha) - \frac{1}{3}D\left(\frac{\alpha}{2}\right).$$

- Thus

$$D(\xi) = \frac{4}{3}D\left(\frac{1 + \sqrt{-7}}{2}\right) + \frac{2}{3}D\left(\frac{-1 + \sqrt{-7}}{4}\right)$$



There is a similar proof for the second relation but we have to use a result by **Gangl**, namely

$$\pi d_{15} = \frac{15\sqrt{15}}{4} L(\chi_{-15}, 2) = 6D\left(\frac{a+3}{4}\right) + 8D(a-1) + 8D\left(\frac{a}{2}\right)$$

where $a = \frac{1+\sqrt{-15}}{2}$. His proof uses the triangulation into ideal tetrahedra of the orbifold quotient $M = \Gamma \backslash \mathbb{H}^3$ with $\Gamma = \text{PSL}_2(\mathcal{O}_K)$ and $K = \mathbb{Q}(\sqrt{-15})$.

If there was a similar formula for $\mathbb{Q}(\sqrt{-11})$, we could prove

$$m((x+1)^2(x^2+x+1)y + (x^2-x+1)^2) \stackrel{?}{=} \frac{2}{3} L'(\chi_{-11}, -1).$$

Results by **Boyd** (mostly experimental), **Villegas** (Deninger-type when E is CM), **Brunault** (modular case)

- confirming Beilinson's conjecture
- expressing the Mahler measure in terms of elliptic dilogarithm D^E
- related to elliptic regulators when the polynomial is "tempered"

So perhaps the level 1 link to regulators in Algebraic Geometry (**Lewis**, **Kerr**)

- For the notion of “tempered” and the link with the $K_2(E)$, **CCGLS**’s paper is crucial (**Cooper, Culler, Gillet, Long, Shalen** Plane curves associated to character varieties of 3-manifolds).
- The fact that an elliptic curve is a Calabi-Yau variety is also fundamental.
- Maillot’s insight for P non reciprocal but using Deninger-type results for reciprocal ones is also very important.

These are my motivations for studying $m(P)$ for P reciprocal defining a $K3$ -surface.

Basic facts on K3-surfaces

Our results concern polynomials of the family

$$P_k = x + \frac{1}{x} + y + \frac{1}{y} + z + \frac{1}{z} - k$$

defining K3-surfaces Y_k . **What's a K3-surface?**

- A double covering branched along a plane sextic for example defines a K3-surface X .

In our case

$$(2z + x + \frac{1}{x} + y + \frac{1}{y} - k)^2 = (x + \frac{1}{x} + y + \frac{1}{y} - k)^2 - 4$$

- There is a unique holomorphic 2-form ω on X up to a scalar.
- $H_2(X, \mathbb{Z})$ is a free group of rank 22.

- With the intersection pairing, $H_2(X, \mathbb{Z})$ is a lattice and

$$H_2(X, \mathbb{Z}) \simeq U_2^3 \perp (-E_8)^2 := \mathcal{L}$$

\mathcal{L} is the $K3$ -lattice, U_2 the hyperbolic lattice of rank 2, E_8 the unimodular lattice of rank 8.

- $$Pic(X) \subset H^2(X, \mathbb{Z}) \simeq Hom(H_2(X, \mathbb{Z}), \mathbb{Z})$$

where $Pic(X)$ is the group of divisors modulo linear equivalence, parametrized by the algebraic cycles.

- $$Pic(X) \simeq \mathbb{Z}^{\rho(X)}$$

$\rho(X) :=$ Picard number of X

$$1 \leq \rho(X) \leq 20$$



$$T(X) := (\text{Pic}(X))^\perp$$

is the transcendental lattice of dimension $22 - \rho(X)$

- If $\{\gamma_1, \dots, \gamma_{22}\}$ is a \mathbb{Z} -basis of $H_2(X, \mathbb{Z})$ and ω the holomorphic 2-form,

$$\int_{\gamma_i} \omega$$

is called a period of X and

$$\int_{\gamma} \omega = 0 \text{ for } \gamma \in \text{Pic}(X).$$

- If $\{X_z\}$ is a family of $K3$ surfaces, $z \in \mathbb{P}^1$ with generic Picard number ρ and ω_z the corresponding holomorphic 2-form, then the periods of X_z satisfy a Picard-Fuchs differential equation of order $k = 22 - \rho$. For our family $k = 3$.

- In fact, by Morrison, a \mathcal{M} -polarized $K3$ -surface, with Picard number 19 has a Shioda-Inose structure, that means

$$\begin{array}{ccc}
 X & & A = E \times E / C_N \\
 & \searrow & \swarrow \\
 & Y = Kum(A / \pm 1) &
 \end{array}$$

- If the Picard number $\rho = 20$, then the elliptic curve is CM.

Theorem

(B. 2005) Let $k = t + \frac{1}{t}$ and

$$t = \left(\frac{\eta(\tau)\eta(6\tau)}{\eta(2\tau)\eta(3\tau)} \right)^6, \quad \eta(\tau) = e^{\frac{\pi i \tau}{12}} \prod_{n \geq 1} (1 - e^{2\pi i n \tau}), \quad q = \exp 2\pi i \tau$$

$$m(P_k) = \frac{\Im \tau}{8\pi^3} \left\{ \sum_{m, \kappa} \left(-4(2\Re \frac{1}{(m\tau + \kappa)^3(m\bar{\tau} + \kappa)} + \frac{1}{(m\tau + \kappa)^2(m\bar{\tau} + \kappa)^2}) \right. \right. \\ \left. \left. + 16(2\Re \frac{1}{(2m\tau + \kappa)^3(2m\bar{\tau} + \kappa)} + \frac{1}{(2m\tau + \kappa)^2(2m\bar{\tau} + \kappa)^2}) \right. \right. \\ \left. \left. - 36(2\Re \frac{1}{(3m\tau + \kappa)^3(3m\bar{\tau} + \kappa)} + \frac{1}{(3m\tau + \kappa)^2(3m\bar{\tau} + \kappa)^2}) \right. \right. \\ \left. \left. + 144(2\Re \frac{1}{(6m\tau + \kappa)^3(6m\bar{\tau} + \kappa)} + \frac{1}{(6m\tau + \kappa)^2(6m\bar{\tau} + \kappa)^2}) \right) \right\}$$

Sketch of proof

Let

$$P_k = x + \frac{1}{x} + y + \frac{1}{y} + z + \frac{1}{z} - k$$

defining the family (X_k) of $K3$ -surfaces.

- For $k \in \mathbb{P}^1 \setminus \{\infty, \pm 2, \pm 6\}$, $\rho \geq 19$.
- The family is \mathcal{M}_k -polarized with

$$\mathcal{M}_k \simeq U_2 \perp (-E_8)^2 \perp \langle -12 \rangle$$

- Its transcendental lattice satisfies

$$T_k \simeq U_2 \perp \langle 12 \rangle$$

- The Picard-Fuchs differential equation is

$$(k^2 - 4)(k^2 - 36)y'''' + 6k(k^2 - 20)y''' + (7k^2 - 48)y'' + ky' = 0$$

- The family is modular in the following sense
if $k = t + \frac{1}{t}$, $\tau \in \mathcal{H}$ and τ as in the theorem

$$t\left(\frac{a\tau + b}{c\tau + d}\right) = t(\tau) \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(6, 2)^* \subset \Gamma_0(12)^* + 12$$

where

$$\Gamma_1(6) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid a \equiv d \equiv 1 \pmod{6} \quad c \equiv 0 \pmod{6} \right\}$$

$$\Gamma_1(6, 2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(6) \mid c \equiv 6b \pmod{12} \right\}$$

and

$$\Gamma_1(6, 2)^* = \langle \Gamma_1(6, 2), w_6 \rangle$$

- The P-F equation has a basis of solutions $G(\tau)$, $\tau G(\tau)$, $\tau^2 G(\tau)$ with

$$G(\tau) = \eta(\tau)\eta(2\tau)\eta(3\tau)\eta(6\tau)$$

satisfying

$$G(\tau) = F(t(\tau)), \quad F(t) = \sum_{n \geq 0} v_n t^{2n+1}, \quad v_n = \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}^2$$

- $\frac{dm(P_k)}{dk}$ is a period, hence satisfies the P-F equation

$$\frac{dm(P_k)}{dk} = G(\tau)$$

$$dm(P_k) = -G(\tau) \frac{dt}{t} \frac{1-t^2}{t}$$

is a weight 4 modular form for $\Gamma_1(6, 2)^*$

- so can be expressed as a combination of $E_4(n\tau)$ for $n = 1, 2, 3, 6$

- By integration you get

$$m(P_k) = \Re\left(-\pi i\tau + \sum_{n \geq 1} \left(\sum_{d|n} d^3\right) \left(4 \frac{q^n}{n} - 8 \frac{q^{2n}}{n} + 12 \frac{q^{3n}}{n} - 24 \frac{6n}{n}\right)\right)$$

- Then using a Fourier development one deduces the expression of the Mahler measure in terms of an Eisenstein-Kronecker series

For some values of k , the corresponding τ is imaginary quadratic.
 For example

k	0	2	3	6	10
τ	$\frac{-3+\sqrt{-3}}{6}$	$\frac{-2+\sqrt{-2}}{6}$	$\frac{-3+\sqrt{-15}}{12}$	$\frac{\sqrt{-6}}{6}$	$\frac{\sqrt{-2}}{2}$

For these quadratic τ called “singular moduli”, the corresponding K3-surface is singular, that means its Picard number is $\rho = 20$ and the elliptic curve E of the Shioda-Inose is CM

So, an expression of the Mahler measure in terms of Hecke L-series (arithmetic aspect) and perhaps in terms of the L-series of the hypersurface K3 (geometric aspect).

Mahler measure and Hecke L-series

$K = \mathbb{Q}(\sqrt{d})$ an imaginary quadratic field, O_K its ring of integers, D its discriminant

Definition A Hecke “Größencharacter” of weight k , $k \geq 2$, and conductor Λ , Λ being an ideal of O_K , is an homomorphism ϕ

$$\phi : I(\Lambda) \rightarrow \mathbb{C}^*$$

such that

$$\phi(\alpha O_K) = \alpha^{k-1} \quad \text{if } \alpha \equiv 1(\Lambda)$$

The corresponding Hecke L -series is defined by

$$L(\phi, s) := \sum_P \frac{\phi(P)}{N(P)^s} = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}$$

the summation being on the prime ideals $P \subset O_K$ prime to Λ .

Replacing O_K by an order R , you get a similar definition.

(B. 2007)

$$m(P_0) = d_3$$

$$m(P_2) = \frac{16\sqrt{2}}{\pi^3} L_{\mathbb{Q}(\sqrt{-2})}(\phi, 3)$$

$$m(P_3) = \frac{15\sqrt{15}}{2\pi^3} L_{\mathbb{Q}(\sqrt{-15})}(\phi_1, 3)$$

where $\phi_1(P) = -\omega$ if $P = (2, \omega)$ and $\omega = \frac{1+\sqrt{-15}}{2}$, P being a representative of the second ideal class of the number field $\mathbb{Q}(\sqrt{-15})$ of class number 2.

$$m(P_6) = \frac{24\sqrt{6}}{\pi^3} L_{\mathbb{Q}(\sqrt{-6})}(\phi_2, 3)$$

where $\phi_2(P) = -2$ if $P = (2, \sqrt{-6})$, P being a representative of the second ideal class of the number field $\mathbb{Q}(\sqrt{-6})$ of class number 2.

$$m(P_{10}) = 2d_3 + 3\frac{16\sqrt{2}}{\pi^3} L_{\mathbb{Q}(\sqrt{-2})}(\phi, 3)$$

where

$$d_3 := \frac{3\sqrt{3}}{4\pi} L(\chi_{-3}, 2) = \frac{2\sqrt{3}}{\pi^3} \sum'_{m,k} \frac{1}{(m^2 + 3k^2)^2}$$

and under the following relations, **proved by Zagier**,

$$\begin{aligned} \frac{9\sqrt{2}}{\pi^3} \left(\sum' \frac{-1}{(9m^2 + 2k^2)^2} + \frac{1}{(k^2 + 18m^2)^2} \right) &= 2d_3, \\ \sum' \frac{k^2 - 18m^2}{(k^2 + 18m^2)^3} + \sum' \frac{9m^2 - 2k^2}{(9m^2 + 2k^2)^3} &= \\ &= \frac{10}{9} \sum' \frac{m^2 - 2k^2}{(m^2 + 2k^2)^3}. \end{aligned}$$

Theorem

(B. 2008) Let Y_k the K3 hypersurface associated to the polynomial P_k , $L(Y_k, s)$ its L-series and T_Y its transcendental lattice. Then,



$$L(Y_2, s) = L(Y_{10}, s) = L(f, s)$$

where $f(z) = \eta(z)^2 \eta(2z) \eta(4z) \eta(8z)^2 \in S_3(\Gamma_0(8), \epsilon_8)$, ϵ_8 being some Dirichlet character modulo 8,



$$m(P_2) = 4 \frac{|\det T_{Y_2}|^{3/2}}{4\pi^3} L(Y_2, 3)$$

- Under the previous relations

$$m(P_{10}) = 2d_3 + \frac{1}{9} \frac{|\det T_{Y_{10}}|^{3/2}}{2\pi^3} L(Y_{10}, 3),$$

Some ingredients in the proof

Let

$$P_k = x^2yz + xy^2z + xyz^2 + t^2(xy + xz + yz) - kxyzt.$$

- Y_k is the desingularization of the set of zeroes of P_k .
- With some fibration, Y_k is an elliptic surface with singular fibers of type I_n .
- Use Shioda's theorems on elliptic surfaces to compute the determinant of $NS(Y_k)$, in particular, the formula

$$\rho_k = r_k + 2 + \sum_{\nu} (m_{\nu,k} - 1)$$

where r_k is the rank of $MW(Y_k)$.

Some ingredients in the proof (continued)

- If $k = 2$, since $\rho_2 = 20$ and fibers are of type $I_{12}, I_6, I_2, I_2, I_1, I_1$, $r_2 = 0$ (easy case).
- So,

$$|\det NS(Y_2)| = \frac{\prod m_{\nu,2}}{\text{Torsion}^2} = 8$$

- If $k = 10$, since $\rho_2 = 20$ and fibers are of type $I_{12}, I_3, I_3, I_2, I_2, I_1, I_1$, $r_{10} = 1$ (difficult case).
 - So, have to guess an infinite section,
 - have to use Néron's desingularization.

Some ingredients in the proof (continued)

- The value of $\det NS(Y_{10})$ gives the CM-field of the elliptic curve in the Shioda-Inose structure.
- Have to count the number of points of the reduction of Y_k modulo q ($q = p^r$).
- In case Y_k modular this allows to determine which modular form gives the equality

$$L(Y_k, s) = L(f, s).$$

- Compare to the expression of the Mahler measure and conclude.

- We have

$$\det NS(Y_2) = -8$$

$$\det NS(Y_{10}) = -72$$

so the underlying elliptic curves E_2 and E_{10} are both CM on $\mathbb{Q}(\sqrt{-2})$.

- Since

$$L(Y_2, s) = L(Y_{10}, s) = L(f, s),$$

by Tate's conjecture, Y_2 and Y_{10} are related by an algebraic correspondance.