

# Mahler measure of K3-hypersurfaces

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Introduced by Mahler in 1962,  
the logarithmic Mahler measure of a polynomial  $P$  is

$$m(P) := \frac{1}{(2\pi i)^n} \int_{\mathbb{T}^n} \log |P(x_1, \dots, x_n)| \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n}$$

and its Mahler measure

$$M(P) = \exp(m(P))$$

where

$$\mathbb{T}^n = \{(x_1, \dots, x_n) \in \mathbb{C}^n / |x_1| = \dots = |x_n| = 1\}.$$

- $n = 1$

By Jensen's formula, if  $P \in \mathbb{Z}[X]$  is monic, then

$$M(P) = \prod_{P(\alpha)=0} \max(|\alpha|, 1).$$

So it is related to **Lehmer's question (1933)**

**Does there exist  $P \in \mathbb{Z}[X]$ , monic, non cyclotomic, satisfying**

$$1 < M(P) < M(P_0) = 1.1762 \dots ?$$

The polynomial

$$P_0(X) = X^{10} + X^9 - X^7 - X^6 - X^5 - X^4 - X^3 + X + 1$$

is the Lehmer polynomial, in fact a Salem polynomial.

Lehmer's problem is still open.

A partial answer by Smyth (1971)

$$M(P) \geq 1.32 \dots$$

if  $P$  is non reciprocal.

- $n > 1$  Interesting Mahler measures are related to **Calabi-Yau varieties** that is elliptic curves ( $n = 2$ ), K3-surfaces ( $n = 3$ ), threefolds ( $n = 4$ ),  $\dots$

The story can be explained with polynomials

$$x_0 + x_1 + x_2 + \dots + x_n.$$

- $m(x_0 + x_1) = 0$  (by Jensen's formula)

- 

$$m(x_0 + x_1 + x_2) = \frac{3\sqrt{3}}{4\pi} L(\chi_{-3}, 2) = L'(\chi_{-3}, -1) \quad \text{Smyth (1980)}$$

- 

$$m(x_0 + x_1 + x_2 + x_3) = \frac{7}{2\pi^2} \zeta(3) \quad \text{Smyth (1980)}$$

These are the first explicit Mahler measures.



$m(x_0 + x_1 + x_2 + x_3 + x_4) \stackrel{?}{=} ** L(f, 3)$  conjectured by Villegas (2004)

$f$  cusp form of weight 3 and conductor 15

$L(f, 3)$  is also the L-series of the K3 surface defined by

$$\begin{aligned}x_0 + x_1 + x_2 + x_3 + x_4 &= 0 \\ \frac{1}{x_0} + \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{1}{x_4} &= 0\end{aligned}$$

## How such a conjecture possible?

Because of deep insights of two people.

- Deninger (1996) who conjectured

$$m\left(x + \frac{1}{x} + y + \frac{1}{y} + 1\right) \stackrel{?}{=} \frac{15}{4\pi^2} L(E, 2) = L'(E, 0)$$

$E$  elliptic curve of conductor 15 defined by the polynomial

Here the polynomial is **reciprocal**.

- Maillot (2003) using a result of Darboux (1875): the Mahler measure of  $P$  which is the integration of a differential form on a variety, when  $P$  is **non reciprocal**, is in fact an integration on a smaller variety and the expression of the Mahler measure is encoded in the cohomology of the smaller variety.

- $n = 2$  The smaller variety is defined by

$$x_0 + x_1 + x_2 = 0$$

$$\frac{1}{x_0} + \frac{1}{x_1} + \frac{1}{x_2} = 0 \Leftrightarrow x_1^2 + x_2^2 + x_1x_2 = 0$$

It is a curve of genus 0. So  $m(x_0 + x_1 + x_2)$  is expressed as a Dirichlet L-series.

- $n = 3$  The smaller variety is defined by

$$x_0 + x_1 + x_2 + x_3 = 0$$

$$\frac{1}{x_0} + \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} = 0 \Leftrightarrow (x_1 + x_2)(x_1 + x_3)(x_2 + x_3) = 0$$

It is the intersection of 3 planes. Thus Smyth's result.

- $n = 4$  (Villegas's Conjecture) The smaller variety is defined by

$$\begin{aligned}x_0 + x_1 + x_2 + x_3 + x_4 &= 0 \\ \frac{1}{x_0} + \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{1}{x_4} &= 0\end{aligned}$$

It is the modular  $K3$ -surface studied by Peters, Top, van der Vlugt defined by a reciprocal polynomial. Its  $L$ -series is related to  $f$ .

- $n = 5$  (Villegas's Conjecture again)

$$m(x_0 + x_1 + x_2 + x_3 + x_4 + x_5) = ** L(g, 4)$$

$g$  cusp form of weight 4 and conductor 6 related to  $L$ -series of the Barth-Nieto quintic.

Briefly, to guess the Mahler measure of a non reciprocal polynomial we need results on reciprocal ones.

In particular, it is very important to collect many examples of Mahler measures of  $K3$ -hypersurfaces.

Notice that Maillot's insight predicts only the type of formula expected. Also Deninger's guess comes from Beilinson's Conjectures.

# Basic facts on K3-surfaces

Our results concern polynomials of the family

$$P_k = x + \frac{1}{x} + y + \frac{1}{y} + z + \frac{1}{z} - k$$

defining K3-surfaces  $Y_k$ . **What's a K3-surface?**

- A double covering branched along a plane sextic for example defines a K3-surface  $X$ .

In our case

$$(2z + x + \frac{1}{x} + y + \frac{1}{y} - k)^2 = (x + \frac{1}{x} + y + \frac{1}{y} - k)^2 - 4$$

- There is a unique holomorphic 2-form  $\omega$  on  $X$  up to a scalar.
- $H_2(X, \mathbb{Z})$  is a free group of rank 22.

- With the intersection pairing,  $H_2(X, \mathbb{Z})$  is a lattice and

$$H_2(X, \mathbb{Z}) \simeq U_2^3 \perp (-E_8)^2 := \mathcal{L}$$

$\mathcal{L}$  is the  $K3$ -lattice,  $U_2$  the hyperbolic lattice of rank 2,  $E_8$  the unimodular lattice of rank 8.

- $$\text{Pic}(X) \subset H_2(X, \mathbb{Z}) \simeq \text{Hom}(H_2(X, \mathbb{Z}), \mathbb{Z})$$

where  $\text{Pic}(X)$  is the group of divisors modulo linear equivalence, parametrized by the algebraic cycles.

- $$\text{Pic}(X) \simeq \mathbb{Z}^{\rho(X)}$$

$\rho(X) :=$  Picard number of  $X$

$$1 \leq \rho(X) \leq 20$$



$$T(X) := (\text{Pic}(X))^\perp$$

is the transcendental lattice of dimension  $22 - \rho(X)$

- If  $\{\gamma_1, \dots, \gamma_{22}\}$  is a  $\mathbb{Z}$ -basis of  $H_2(X, \mathbb{Z})$  and  $\omega$  the holomorphic 2-form,

$$\int_{\gamma_i} \omega$$

is called a period of  $X$  and

$$\int_{\gamma} \omega = 0 \text{ for } \gamma \in \text{Pic}(X).$$

- If  $\{X_z\}$  is a family of  $K3$  surfaces,  $z \in \mathbb{P}^1$  with generic Picard number  $\rho$  and  $\omega_z$  the corresponding holomorphic 2-form, then the periods of  $X_z$  satisfy a Picard-Fuchs differential equation of order  $k = 22 - \rho$ . For our family  $k = 3$ .

- In fact, by Morrison, a  $\mathcal{M}$ -polarized K3-surface, with Picard number 19 has a Shioda-Inose structure, that means

$$\begin{array}{ccc}
 X & & A = E \times E / C_N \\
 & \searrow & \swarrow \\
 & Y = \text{Kum}(A / \pm 1) &
 \end{array}$$

- If the Picard number  $\rho = 20$ , then the elliptic curve is CM.

## Theorem

(B. 2005) Let  $k = t + \frac{1}{t}$  and

$$t = \left( \frac{\eta(\tau)\eta(6\tau)}{\eta(2\tau)\eta(3\tau)} \right)^6, \quad \eta(\tau) = e^{\frac{\pi i \tau}{12}} \prod_{n \geq 1} (1 - e^{2\pi i n \tau}), \quad q = \exp 2\pi i \tau$$

$$m(P_k) = \frac{\Im \tau}{8\pi^3} \left\{ \sum_{m, \kappa} \left( -4(2\Re \frac{1}{(m\tau + \kappa)^3(m\bar{\tau} + \kappa)} + \frac{1}{(m\tau + \kappa)^2(m\bar{\tau} + \kappa)^2}) \right. \right. \\ \left. \left. + 16(2\Re \frac{1}{(2m\tau + \kappa)^3(2m\bar{\tau} + \kappa)} + \frac{1}{(2m\tau + \kappa)^2(2m\bar{\tau} + \kappa)^2}) \right. \right. \\ \left. \left. - 36(2\Re \frac{1}{(3m\tau + \kappa)^3(3m\bar{\tau} + \kappa)} + \frac{1}{(3m\tau + \kappa)^2(3m\bar{\tau} + \kappa)^2}) \right. \right. \\ \left. \left. + 144(2\Re \frac{1}{(6m\tau + \kappa)^3(6m\bar{\tau} + \kappa)} + \frac{1}{(6m\tau + \kappa)^2(6m\bar{\tau} + \kappa)^2}) \right) \right\}$$

# Sketch of proof

Let

$$P_k = x + \frac{1}{x} + y + \frac{1}{y} + z + \frac{1}{z} - k$$

defining the family  $(X_k)$  of  $K3$ -surfaces.

- For  $k \in \mathbb{P}^1 \setminus \{\infty, \pm 2, \pm 6\}$ ,  $\rho \geq 19$ .
- The family is  $\mathcal{M}_k$ -polarized with

$$\mathcal{M}_k \simeq U_2 \perp (-E_8)^2 \perp \langle -12 \rangle$$

- Its transcendental lattice satisfies

$$T_k \simeq U_2 \perp \langle 12 \rangle$$

- The Picard-Fuchs differential equation is

$$(k^2 - 4)(k^2 - 36)y'''' + 6k(k^2 - 20)y''' + (7k^2 - 48)y'' + ky' = 0$$

- The family is modular in the following sense  
if  $k = t + \frac{1}{t}$ ,  $\tau \in \mathcal{H}$  and  $\tau$  as in the theorem

$$t\left(\frac{a\tau + b}{c\tau + d}\right) = t(\tau) \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(6, 2)^* \subset \Gamma_0(12)^* + 12$$

where

$$\Gamma_1(6) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid a \equiv d \equiv 1 \pmod{6} \quad c \equiv 0 \pmod{6} \right\}$$

$$\Gamma_1(6, 2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(6) \mid c \equiv 6b \pmod{12} \right\}$$

and

$$\Gamma_1(6, 2)^* = \langle \Gamma_1(6, 2), w_6 \rangle$$

- The P-F equation has a basis of solutions  $G(\tau)$ ,  $\tau G(\tau)$ ,  $\tau^2 G(\tau)$  with

$$G(\tau) = \eta(\tau)\eta(2\tau)\eta(3\tau)\eta(6\tau)$$

satisfying

$$G(\tau) = F(t(\tau)), \quad F(t) = \sum_{n \geq 0} v_n t^{2n+1}, \quad v_n = \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}^2$$

- $\frac{dm(P_k)}{dk}$  is a period, hence satisfies the P-F equation

$$\frac{dm(P_k)}{dk} = G(\tau)$$

$$dm(P_k) = -G(\tau) \frac{dt}{t} \frac{1-t^2}{t}$$

is a weight 4 modular form for  $\Gamma_1(6, 2)^*$

- so can be expressed as a combination of  $E_4(n\tau)$  for  $n = 1, 2, 3, 6$

- By integration you get

$$m(P_k) = \Re(-\pi i \tau + \sum_{n \geq 1} (\sum_{d|n} d^3) (4 \frac{q^n}{n} - 8 \frac{q^{2n}}{n} + 12 \frac{q^{3n}}{n} - 24 \frac{6n}{n}))$$

- Then using a Fourier development one deduces the expression of the Mahler measure in terms of an Eisenstein-Kronecker series

For some values of  $k$ , the corresponding  $\tau$  is imaginary quadratic.  
 For example

$k$	0	2	3	6	10
$\tau$	$\frac{-3+\sqrt{-3}}{6}$	$\frac{-2+\sqrt{-2}}{6}$	$\frac{-3+\sqrt{-15}}{12}$	$\frac{\sqrt{-6}}{6}$	$\frac{\sqrt{-2}}{2}$

For these quadratic  $\tau$  called “singular moduli”, the corresponding K3-surface is singular, that means its Picard number is  $\rho = 20$  and the elliptic curve  $E$  of the Shioda-Inose is CM

So, an expression of the Mahler measure in terms of Hecke L-series (arithmetic aspect) and perhaps in terms of the L-series of the hypersurface K3 (geometric aspect).

# Mahler measure and Hecke L-series

$K = \mathbb{Q}(\sqrt{d})$  an imaginary quadratic field,  $O_K$  its ring of integers,  $D$  its discriminant

**Definition** A Hecke “Größencharacter” of weight  $k$ ,  $k \geq 2$ , and conductor  $\Lambda$ ,  $\Lambda$  being an ideal of  $O_K$ , is an homomorphism  $\phi$

$$\phi : I(\Lambda) \rightarrow \mathbb{C}^*$$

such that

$$\phi(\alpha O_K) = \alpha^{k-1} \quad \text{if } \alpha \equiv 1(\Lambda)$$

The corresponding Hecke  $L$ -series is defined by

$$L(\phi, s) := \sum_P \frac{\phi(P)}{N(P)^s} = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}$$

the summation being on the prime ideals  $P \subset O_K$  prime to  $\Lambda$ .

Replacing  $O_K$  by an order  $R$ , you get a similar definition.

$$m(P_0) = d_3$$

$$m(P_2) = \frac{16\sqrt{2}}{\pi^3} L_{\mathbb{Q}(\sqrt{-2})}(\phi, 3)$$

$$m(P_3) = \frac{15\sqrt{15}}{2\pi^3} L_{\mathbb{Q}(\sqrt{-15})}(\phi_1, 3)$$

where  $\phi_1(P) = -\omega$  if  $P = (2, \omega)$  and  $\omega = \frac{1+\sqrt{-15}}{2}$ ,  $P$  being a representative of the second ideal class of the number field  $\mathbb{Q}(\sqrt{-15})$  of class number 2.

$$m(P_6) = \frac{24\sqrt{6}}{\pi^3} L_{\mathbb{Q}(\sqrt{-6})}(\phi_2, 3)$$

where  $\phi_2(P) = -2$  if  $P = (2, \sqrt{-6})$ ,  $P$  being a representative of the second ideal class of the number field  $\mathbb{Q}(\sqrt{-6})$  of class number 2.

$$m(P_{10}) \stackrel{?}{=} 2d_3 + 3 \frac{16\sqrt{2}}{\pi^3} L_{\mathbb{Q}(\sqrt{-2})}(\phi, 3)$$

where

$$d_3 := \frac{3\sqrt{3}}{4\pi} L(\chi_{-3}, 2) = \frac{2\sqrt{3}}{\pi^3} \sum'_{m,k} \frac{1}{(m^2 + 3k^2)^2}$$

and under the following relations,

$$\begin{aligned} \frac{9\sqrt{2}}{\pi^3} \left( \sum' \frac{-1}{(9m^2 + 2k^2)^2} + \frac{1}{(k^2 + 18m^2)^2} \right) &\stackrel{?}{=} 2d_3, \\ \sum' \frac{k^2 - 18m^2}{(k^2 + 18m^2)^3} + \sum' \frac{9m^2 - 2k^2}{(9m^2 + 2k^2)^3} & \\ &\stackrel{?}{=} \\ \frac{10}{9} \sum' \frac{m^2 - 2k^2}{(m^2 + 2k^2)^3}, & \end{aligned}$$

## Theorem

Let  $Y_k$  the K3 hypersurface associated to the polynomial  $P_k$ ,  $L(Y_k, s)$  its L-series and  $T_Y$  its transcendental lattice. Then,



$$L(Y_2, s) = L(Y_{10}, s) = L(f, s)$$

where  $f(z) = \eta(z)^2 \eta(2z) \eta(4z) \eta(8z)^2 \in S_3(\Gamma_0(8), \epsilon_8)$ ,  $\epsilon_8$  being some Dirichlet character modulo 8,



$$m(P_2) = 4 \frac{|\det T_{Y_2}|^{3/2}}{4\pi^3} L(Y_2, 3)$$

- Under the previous relations

$$m(P_{10}) = 2d_3 + \frac{1}{9} \frac{|\det T_{Y_{10}}|^{3/2}}{2\pi^3} L(Y_{10}, 3),$$

# Some ingredients in the proof

Let

$$P_k = x^2yz + xy^2z + xyz^2 + t^2(xy + xz + yz) - kxyzt.$$

- $Y_k$  is the desingularization of the set of zeroes of  $P_k$ .
- With some fibration,  $Y_k$  is an elliptic surface with singular fibers of type  $I_n$ .
- Use Shioda's theorems on elliptic surfaces to compute the determinant of  $NS(Y_k)$ , in particular, the formula

$$\rho_k = r_k + 2 + \sum_{\nu} (m_{\nu,k} - 1)$$

where  $r_k$  is the rank of  $MW(Y_k)$ .

## Some ingredients in the proof (continued)

- If  $k = 2$ , since  $\rho_2 = 20$  and fibers are of type  $I_{12}, I_6, I_2, I_2, I_1, I_1$ ,  $r_2 = 0$  (easy case).
- So,

$$|\det NS(Y_2)| = \frac{\prod m_{\nu,2}}{\text{Torsion}^2} = 8$$

- If  $k = 10$ , since  $\rho_2 = 20$  and fibers are of type  $I_{12}, I_3, I_3, I_2, I_2, I_1, I_1$ ,  $r_{10} = 1$  (difficult case).
  - So, have to guess an infinite section,
  - have to use Néron's desingularization.

## Some ingredients in the proof (continued)

- The value of  $\det NS(Y_{10})$  gives the CM-field of the elliptic curve in the Shioda-Inose structure.
- Have to count the number of points of the reduction of  $Y_k$  modulo  $q$  ( $q = p^r$ ).
- In case  $Y_k$  modular this allows to determine which modular form gives the equality

$$L(Y_k, s) = L(f, s).$$

- Compare to the expression of the Mahler measure and conclude.

- We have

$$\det NS(Y_2) = -8$$

$$\det NS(Y_{10}) = -72$$

so the underlying elliptic curves  $E_2$  and  $E_{10}$  are both CM on  $\mathbb{Q}(\sqrt{-2})$ .

- Since

$$L(Y_2, s) = L(Y_{10}, s) = L(f, s),$$

by Tate's conjecture,  $Y_2$  and  $Y_{10}$  are related by an algebraic correspondance.