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What does it mean to do mathematics within a topos ?

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Abstract

The purpose of this short text is to explain some peculiarities and pitfalls of the interpretation of mathematical statements proved within a topos as actual properties of this topos. This interpretation is performed through the so-called "Kripke-Joyal semantics", and due to the fact that statements whose root is an additive connector ($\vee$ or $\exists$) have a non obvious translation, the real (external) meaning of what we prove within a topos may not correspond to what we could expect. Some pitfalls are quite subtle.

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1 Preliminaries.

It's not necessary to give the precise definition of a topos in order to fullfil the goals of the above abstract. First of all, we have to know that a topos (either a Lawvere-Tierney elementary topos or a Grothendieck topos) is some kind of mathematical universe. A topos has a so-called "internal language", which allows to state facts in a "set theoretical style", which just means that we are allowed to "work with elements". Hence, the internal language looks almost like the usual mathematical language, and this is why it is possible to "do mathematics within a topos".

A topos is a category with special properties which make it look almost like the category of sets (the category of sets is denoted **Set**). Any topos has a special object, denoted Ω, which plays the role of the "set of truth values". In **Set**, this object is just a set with two elements (denoted $\top$ ("true") and $\bot$ ("false")), but in general the description of Ω is not as simple as this, as we shall see on some examples.

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Nevertheless, (mathematical) statements of the internal language, which are of course supposed to represent truth values, are interpreted as arrows targeting Ω. The source of such an arrow depends on the "context" within which the statement (let's denote it E) is interpreted. The "context" within which E is interpreted is a sequence of typed declarations of variables, hence something like $(x_1 \in X_1) \dots (x_n \in X_n)$, where the "types" X_i are just objects of the topos, so that, since objects in a category have no notion of element, the use of the membership symbol $\in$ is purely conventional. Of course, any variable occuring free in E must be declared in the context, otherwise E is not interpretable in this context.

If Γ is a context as shown above, it is itself interpreted as an object of the topos that we denote $\overline{\Gamma}$. Intuitively, the "elements" of $\overline{\Gamma}$ are all possible tuples of values we may assign to the variables declared in Γ. Summarizing, a statement E, when interpreted in the context Γ gives rise to an arrow

$$\overline{\Gamma} \xrightarrow{\lfloor E \rfloor_\Gamma} \Omega$$

Now, the object Ω in any topos comes equiped with an arrow $\top : \mathbf{1} \to \Omega$ which represents the truth value "true" (where $\mathbf{1}$, the terminal object of the topos, plays the role of a singleton, i.e. of a one element set). By definition, saying that our statement E of the internal language is "true in the context Γ" is just saying that the arrow $\lfloor E \rfloor_\Gamma$ factorizes through $\top$:

$$\begin{array}{ccc} & \nearrow & \mathbf{1} \\ & & \downarrow \top \\ \overline{\Gamma} & \xrightarrow[\lfloor E \rfloor_\Gamma]{} & \Omega \end{array}$$

What we expect of course, is that proving a statement E of the internal language of a topos $\mathcal{T}$ within a given context Γ will ensure that E is true in this context. This is actually what happens, provided that all proof methods we are using are "valid" for the given topos. The rules are the following :

- All intuitionnistic methods of proof are valid in all toposes.
- Using the principle of excluded middle is valid in all "boolean toposes".
- Using the axiom of choice is valid in all toposes all objects of whose are internally projective.([2])

At this point we know what it means for a statement of the internal language of a topos $\mathcal{T}$ to be true in a given context, and we know what kind of methods we are allowed to use in order to prove that such a statement is true.

The internal language of a topos allows to use the name of an arrow of the topos as a unary operator. For example, if $f : X \to Y$ is an arrow of the topos $\mathcal{T}$, the statement

$$\forall_{y \in Y} \; \exists_{x \in X} \; y = f(x)$$

is a correct statement of the internal language of $\mathcal{T}$. Notice that this statement is intuitively expressing the fact that "f is a surjective map". However, saying that f is surjective is a priori meaningless since the objects X and Y of the topos $\mathcal{T}$ may have no notion of element. This single example shows that interpreting the meaning of a statement of the internal language is not as obvious as one may have expected.

2 Example 1 : Subsets of a singleton.

Here we assume that we are working with a boolean topos $\mathcal{T}$, i.e. a topos for which the use of the principle of excluded middle is a valid method of proof. Any topos has a terminal object $\mathbf{1}$ which is

[2] Which is the case for example if they are all projective.

to be though of as a singleton. In the topos **Set**, $\mathbf{1}$ is actually an ordinary singleton, and it has two subsets, namely the empty set $\emptyset$ and the singleton $\mathbf{1}$ itself. This fact can be expressed by the statement

$$\forall_{A\in\mathcal{P}(\mathbf{1})}\ A=\emptyset \vee A=\mathbf{1}$$

where $\mathcal{P}$ denotes the power object functor.([3]) This is a correct statement in the internal language of any topos.([4]) Furthermore, this statement is provable using the methods of proof which are valid in a boolean topos. Indeed, we first declare $A\in\mathcal{P}(\mathbf{1})$. Then, the statement $*\in A \vee * \notin A$ is an instance of the excluded middle.([5]) With this hypothesis, we can perform a proof by disjunction of the cases. In the first case, $*\in A$, we have $A=\mathbf{1}$ because the statement $\forall_{x\in\mathbf{1}}\ x=*$ is true in any topos. In the second case, $*\notin A$ we have $A=\emptyset$, because the statement $\forall_{x\in A}\ x=*$ is true.

So, we may think that we have proved that in any boolean topos, $\mathbf{1}$ has exactly two subobjects. Actually, this is false. For example, in the product topos $\mathbf{Set}\times\mathbf{Set}$ (which is boolean), $\mathbf{1}$ is the pair $(\mathbf{1},\mathbf{1})$, which has four subobjects which are $(\emptyset,\emptyset)$, $(\mathbf{1},\emptyset)$, $(\emptyset,\mathbf{1})$ and $(\mathbf{1},\mathbf{1})$. Hence, the internal statement $\forall_{A\in\mathcal{P}(\mathbf{1})}\ A=\emptyset\vee A=\mathbf{1}$, which is true in $\mathbf{Set}\times\mathbf{Set}$, does not mean that $\mathbf{1}$ has exactly two subobjects.([6])

Nevertheless, since the statement $\forall_{A\in\mathcal{P}(\mathbf{1})}\ A=\emptyset\vee A=\mathbf{1}$ is true in any boolean topos, Kripke-Joyal semantics says something about its meaning. Here is how it works. This statement is to be interpreted in the empty context (no declaration at all). Kripke-Joyal semantics for the universal quantifier says that it is true if and only if the statement $A=\emptyset\vee A=\mathbf{1}$ is true in the context $(A\in\mathcal{P}(\mathbf{1}))$. Now, Kripke-Joyal semantics does not say that if a statement of the form $E\vee F$ is true in some context, then either E or F is true in the same context. To each statement interpretable in the context Γ, we can associate a subobject of $\overline{\Gamma}$, that we denote by $D(E)_\Gamma$ and call the "domain of validity of E in the context Γ" (more simply the "domain" of E). Formally, it is defined by the fact that the square

$$\begin{array}{ccc} D(E)_\Gamma & \longrightarrow & \mathbf{1} \\ \downarrow & & \downarrow{\scriptstyle\top} \\ \overline{\Gamma} & \xrightarrow[\lfloor E\rfloor_\Gamma]{} & \Omega \end{array}$$

is cartesian, but intuitively, it may be though of as the "set of all tuples of values assignable to the variables declared in Γ for which the statement E is true". Saying that E is true in the context Γ is then the same as saying that $D(E)_\Gamma=\overline{\Gamma}$. Now, what Kripke-Joyal semantics says about the true statement $E\vee F$ in the context Γ is just that $D(E)_\Gamma\cup D(F)_\Gamma=\overline{\Gamma}$. This is of course weaker than saying that either E or F is true in the context Γ.

So, in the case of our example, Kripke-Joyal semantics says that the domains of $A=\emptyset$ and $A=\mathbf{1}$ are covering $\overline{(A\in\mathcal{P}(\mathbf{1}))}$ (which is in fact $\mathcal{P}(\mathbf{1})$), not that one of them is $\mathcal{P}(\mathbf{1})$. In order to finish with this example, we need to point out another (more classical) subtlety. Indeed, we remark that in the case of the topos **Set**, $\mathcal{P}(\mathbf{1})$ is just a set with two elements, say $\{\bot,\top\}$, so that it has four subsets $\emptyset$, $\{\bot\}$, $\{\top\}$ and $\{\bot,\top\}$. It is possible to compute directly the arrows $\lfloor A=\emptyset\rfloor_{(A\in\mathcal{P}(\mathbf{1}))}$ and $\lfloor A=\mathbf{1}\rfloor_{(A\in\mathcal{P}(\mathbf{1}))}$, which are both from $\mathcal{P}(\mathbf{1})$ to Ω, in other words, from $\mathcal{P}(\mathbf{1})$ to itself. What we find is that $\lfloor A=\mathbf{1}\rfloor_{(A\in\mathcal{P}(\mathbf{1}))}$ is the identity of $\mathcal{P}(\mathbf{1})$ and that $\lfloor A=\emptyset\rfloor_{(A\in\mathcal{P}(\mathbf{1}))}$ is the non trivial permutation of $\mathcal{P}(\mathbf{1})$, so that the domains of validity of $A=\emptyset$ and $A=\mathbf{1}$ are respectively $\{\bot\}$ and $\{\top\}$. These two subsets are covering $\mathcal{P}(\mathbf{1})$ but none of them is $\mathcal{P}(\mathbf{1})$. Hence, even in the case of **Set**, Kripke-Joyal semantics does not say that A is empty or that A is $\mathbf{1}$.

But this is normal, because A has been declared, not defined. In other words, we don't know who is A, and this is why we cannot prove either that it is empty or that is is equal to $\mathbf{1}$. In order

[3] $\mathcal{P}$ may be considered as part of the definition of a topos.

[4] Modulo some conventions, for example the fact that we may denote $\{x\in\mathbf{1}\mid\top\}$ by $\mathbf{1}$.

[5] The term $*$ of type $\mathbf{1}$ is part of the definition of the internal language.

[6] The only thing we can deduce in general for a boolean topos is that the set of subobjects of any object is a boolean algebra.

to get such a conclusion, we have to take an actual subobject A of $\mathbf{1}$ and consider the statement $A = \emptyset \vee A = \mathbf{1}$, which is now interpretable in the empty context, and which is of course true, since it is a specialization of $\forall_{A\in\mathcal{P}(\mathbf{1})}\ A = \emptyset \vee A = \mathbf{1}$. In this new situation, Kripke-Joyal semantics says that $D(A = \emptyset)_\bullet \cup D(A = \mathbf{1})_\bullet = \overline{\bullet} = \mathbf{1}$, where $\bullet$ represents the empty context. Now, in the case of **Set**, this equality means that either $D(A = \emptyset)_\bullet = \mathbf{1}$ or $D(A = \mathbf{1})_\bullet = \mathbf{1}$, so that our subobject A is either $\emptyset$ or $\mathbf{1}$. In the case of $\mathbf{Set} \times \mathbf{Set}$, if we take for example $A = (\emptyset, \mathbf{1})$ we will have $D(A = \emptyset)_\bullet = (\mathbf{1}, \emptyset)$ and $D(A = \mathbf{1})_\bullet = (\emptyset, \mathbf{1})$. These two subobjects are covering $\mathbf{1}$ but none of them is $\mathbf{1}$, so that, depite the fact that the statement $A = \emptyset \vee A = \mathbf{1}$ is true, Kripke-Joyal semantics will not say that A is either empty or $\mathbf{1}$.

As a conclusion, just remember that the fact that a statement of the form $E \vee F$ is true in the empty context will entail that either E or F is true if the topos is two-valued (i.e. if $\mathbf{1}$ has just two subobjects), but that this may fail otherwise.

3 Example 2 : Cantor-Bernstein theorem.

The conclusions of example 1 are rather negative as they show that a simple true statement may in most cases not mean much about the topos itself. This second example is more positive, and shows on the contrary that some expected properties hold.

Cantor-Bernstein theorem says that if we have two sets X and Y, and two injective maps $f : X \to Y$ and $g : Y \to X$, then there is a bijective map between X and Y. It is known that this theorem may be proved using only intuitionnistic principles and the excluded middle. Hence, Cantor-Bernstein theorem is true in any boolean topos. But what does it mean for the topos itself ?

Of course, the theorem as it is stated does not mean anything from the external point of view, because "injective" and "bijective" have no meaning in general in a category. However, categorists have generalized these notions. The concept which generalizes that of bijective map is that of "isomorphism" (the fact of having an inverse arrow), and the concept which generalizes that of injective map is that of "monomorphism" (the fact of being left cancellable with respect to the composition). Hence, it is meaningful to ask if, in a boolean topos, it is always true that if we have two objetcs X and Y, two monomorphisms $f : X \to Y$ and $g : Y \to X$, then we have an isomorphism between X and Y.

Furthermore, the answer to this question is "yes". Indeed, it is a consequence of Kripke-Joyal semantics, that an arrow $f : X \to Y$ in a topos is a monomorphism if and only if the statement which in the internal language of this topos says that f is injective is true. Similarly, an arrow f of a topos is an isomorphism if and only if the statement which in the internal language of this topos says that f is bijective is true. Furthermore, all toposes satisfy the "principle of description" (see for example [5]) which asserts that if you prove (in the empty context) a statement of the form $\forall_{x\in X}\ \exists!_{y\in Y}\ \varphi(x, y)$, then there is a unique arrow $f : X \to Y$ such that $\forall_{x\in X}\ \varphi(x, f(x))$. This allows the construction of arrows of the topos from within the internal language. Hence, if $f : X \to Y$ and $g : Y \to X$ are two monomorphisms, the two statements of the internal language saying that they are injective are true, and since the topos is boolean, the conclusion of the Cantor-Bernstein theorem can be drawn. Now, because of the above remarks, this conclusion gives the wanted isomorphism.

There is a simple situation in which we can put this at work. Consider a group G and the category whose objects are G-sets, i.e. sets on which G acts, and whose arrows are G-equivariant maps. This category is a topos, actually a topos of presheaves, and it is boolean. Hence, the Cantor-Bernstein theorem, as restated above using the notions of monomorphism and isomorphism, automatically applies to this topos. But it is also easy to check that in this particular case, monomorphism means injective and isomorphism means bijective. In other words, the usual Cantor-Bernstein theorem is valid for G sets and G-equivariant maps.([7])

[7]This may of course also be proved directly without any topos theory, and more generally, there is an algorithm

4 Example 3 : The finite and the infinite.

Richard Dedekind has suggested that infinite sets are sets which may be embeded within themselves as a non full subset. In other words, a set X is "infinite" if and only there is an injective map $f : X \to X$ which is not surjective. One may also define a "finite" set as a set X such that any injective map from X to itself is bijective. It is clear that with these two definitions a set cannot be at the same time finite and infinite. Intuitionnistically, this entails that a finite set is not infinite and that an infinite set is not finite, but this does not entail that a non finite set is infinite nor that a non infinite set is finite. Furthermore, in intuitionnistic mathematics, there are several non equivalent definitions of the notion of finite set, sometimes represented by distinct denominations. For example, one may define a finite set as one which is in bijective correspondance with a set of the form $\{p \in \mathbb{N} \mid p < n\}$, and define a set "of finite type" as a set X such that any injective map from X to itself is a bijection. These to concepts are generally not equivalent, and it is easy to exhibit toposes where they are distinct. For example, in the topos of graphs, the set $\{p \in \mathbb{N} \mid p < n\}$ is a graph with n vertices and for each vertex an edge from this vertex to itself (a "loop"). On the other hand, it is immediate that a graph which is finite in the usual sens (i.e. which has a finite set of vertices and a finite set of edges) is of finite type in the above sens, because in the case of graphs (more generally in the case of presheaves) "monomorphism/isomorphism" means "injective/bijective" on all sorts of elements, i.e. on vertices and on edges.([8]) Such a graph is in general not isomorphic to any graph with n vertices and just one loop at each vertex.

Nevertheless, we should be more careful and distinguish between internal and external versions of theses concepts. For example, the external form of the previous definition of a set of finite type, should be that of an objet X such that any monomorphism from X to itself is an isomorphism. However, because Kripke-Joyal semantics tells us that the internal notion of "injective" is equivalent to the external notion of "monomorphism", and similarly for "bijective" and "isomorphism", the external and internal forms of the notion of "being of finite type" are actually equivalent.

Of course, a presheaf with a finite number of elements of each sort is of finite type. But the converse is not true as we shall see now. There are presheaves with an infinite number of elements (of at least one sort) which are of finite type, i.e. which satisfy Dedekind's definition of "finite".

Within the internal language of any topos (i.e. using only intuitionnistic proof methods) one can prove that "any injective map from Ω to itself is an involution".([9]) As a consequence, Ω is always of finite type. We now exhibit a topos of presheaves (with only one sort) in which Ω has an infinity of elements.

A "discrete dynamical system" is just a set X with a map $\varphi : X \to X$. Each element of X is to be though of as a "state" of the system, and the map φ (the "dynamics" of the system) assigns to each state its successor state. In this topos, Ω is as follows :

$$\circlearrowleft 0 \leftarrow\!\shortmid 1 \leftarrow\!\shortmid 2 \leftarrow\!\shortmid 3 \leftarrow\!\shortmid \cdots \qquad\qquad \infty \circlearrowright$$

It has one element for each natural integer plus an extra element denoted ∞. The arrows are representing the transitions between states, i.e. the map φ. So, this "set", which is finite according to Dedekind, has an infinity of elements !([10])

The reader may be interested in an intuitive explanation of the above Ω, explanation which actually has a variant for any presheaf topos. Consider a "set" X with a "subset" $A \subset X$.([11]), and consider an

translating any proof using internal reasoning and Kripke-Joyal semantics into an "ordinary" (external) proof. However using internal reasoning may shorten the proof quite a lot.

[8] A presheaf ζ is a (contravariant) functor from a small category $\mathcal{C}$ to **Set**. Hence for any object X of $\mathcal{C}$, $\zeta(X)$ is a set, and the elements of this set are said to be "of sort X".

[9] This theorem is due to Higgs.

[10] Notice however that there are only two arrows from $\mathbf{1}$ to Ω in this topos, which is thus bivalued (but not boolean). But there are similar toposes with an infinity of arrows from $\mathbf{1}$ to Ω, which is still "of finite type".

[11] Actually, a presheaf and a subpresheaf, and in this particular case a dynamical system and a dynamical subsystem.

element $x \in X$,([12]) and ask the question "does x belong to A ?". The answer to this question is not just "yes" or "no". It has a lot of shades. Indeed, in the case of discrete dynamical systems, even if x is not in A, it may enter A after some time, in other words $\varphi^k(x)$ may be in A for some $k \in \mathbb{N}$. It may also never enter A. There is one element in Ω for each such situation ($0 =$ already in A, $1 =$ in A after one unit of time, etc..., and $\infty =$ never in A in the future), and the dynamics of Ω just tells us how the answer to our question varies over time. Also notice that the element 0 represents the truth value "true", and that ∞ represents the truth value "false". All other values are "shades of truth".

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[12] Which is meaningful in a presheaf, provided that we allow different "types" (or "sorts") of elements.